Stability and Stabilization for Singular Markov
Switched Systems with Time Delays

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Abstract

The stability and stabilization of linear continuous time singular Markov switched systems with time delay are investigated. Based on the linear matrix inequality method (LMIs), the robust stability of uncertain linear stochastic delay system is guaranteed irrespective of the value of the time delay. Then the proposed theory can be extended to discuss the robust stabilization of uncertain stochastic differential delay systems.

Keywords: Markov switched systems; stabilization; time delay

1 Introduction

In practice, many industrial systems are subject to abrupt changes, such as component and/or interconnection failure or random communication delays in automobile vehicles. Such systems usually can be modeled by the class of Markov jump systems. Markov jump system has attracted considerable attention and many problems have been tackled and solved. The main focus is to study the stability of jump linear systems with Markov switching parameters [1-3]. LMIs conditions based on switched Lyapunov functions have been given to test the stability and the stabilization of linear continuous-time systems in [4]. The singular system represents a variety of practical systems like electrical circuits, mechanical systems, robotics, etc [5]. During the past decades, considerable attention has been devoted to the analysis and synthesis of linear singular systems [6]-[7]. However, to the best of our knowledge, the stability of

\[ \text{Keywords: Markov switched systems; stabilization; time delay} \]
linear continuous time singular systems with time delay and Markov switchings has received little attention so far. This paper aims to deal with the stability of linear continuous-time singular Markov switched systems with time delay in the state vector, and LMI conditions are given. The stability is assured, independently of the size of the time-delays (which can be unknown), by means of a Lyapunov-Krasovskii functional with Markov switched matrices. Owing to some extra matrix variables, the stability conditions can be extended to cope with the design of a stabilizing state feedback control law.

2 Problem statement

Let us consider a dynamical singular system:

\[
\begin{align*}
E dx(t) &= [(A_{\sigma(t)} + \Delta A_{\sigma(t)}(t))x(t) + (A_{d\sigma(t)} + \Delta A_{d\sigma(t)}(t))(x(t) - \tau) \\
&+ (B_{\sigma(t)} + \Delta B_{\sigma(t)}(t))u(t)]dt + C_{\sigma(t)}d\omega(t) \\
x(0) &= x_0, \quad x(t) = \phi(t), \quad \forall t \in [-\tau, 0]
\end{align*}
\]

\{\sigma(t), t \geq 0\} is the continuous time Markov process taking values in a finite set \(S = \{1, 2, \cdots, s\}\). \(A_i, A_{di}, B_i\) and \(C_i\) are known constant matrices, \(\Delta A_i(t), \Delta A_{di}(t)\) and \(\Delta B_i(t)\) are unknown matrices with appropriate dimensions which represent the system uncertainties. \(\omega(t)\) is a scalar Brownian motion. \(\phi(t)\) is any given initial state. \(E\) is a known singular matrix with \(\text{rank}(E) = n_E < n\) and \(\tau\) denotes the time-delay in the state. In this note, for every \(i \in S\), we assume that the uncertainties can be described as follows:

\[
\begin{bmatrix}
\Delta A_i(t) \\
\Delta A_{di}(t) \\
\Delta B_i(t)
\end{bmatrix} = M_i F_i(t) \begin{bmatrix}
N_{1i} \\
N_{2i} \\
N_{3i}
\end{bmatrix}
\]

(2)

where \(M_i\) and \(N_{ji}, j = 1, 2, 3\) are known constant real matrices, \(F_i(t) \in \mathbb{R}^{k \times l}\) is an unknown matrix function with Lebesgue measurable elements and satisfies

\[
F_i^T(t)F_i(t) \leq I.
\]

Definition 2.1.

(i) System (1) with is said to be stochastically stable if there exists a constant matrix such that the following inequality holds for any pair of initial conditions \((x_0, \sigma(0))\)

\[
E \left[ \int_0^\infty x^T(t)x(t)dt \mid x_0, \sigma(0) \right] \leq x_0^T G x_0
\]
(ii) System (1) with is said to be robust stochastically stable if it is stochastically stable for all admissible uncertainties.

**Definition 2.2.**

(i) System (1) is said to be stochastically stabilizable if there exists a control

\[ u(t) = K_{\sigma(t)}x(t) \quad (3) \]

with \( K_i \), when \( \sigma(t) = i \), a constant matrix such that the closed-loop system is stochastically stable.

(ii) robust stochastically stabilizable if there exists a control of the form (3) such that the closed-loop system is stochastically stable for all admissible uncertainties.

**Lemma 2.1.** Let \( H, F \) and \( G \) be real matrices of appropriate dimensions with \( F^TF \leq I \), then, for any scalar \( \epsilon > 0 \), one has the following

\[ HFG + G^TF^TH^T \leq \epsilon HH^T + \epsilon^{-1}G^TG. \]

**Lemma 2.2.** For any matrices \( H \in R^{n \times n} \) and \( F \in R^{n \times n} \) with \( F > 0 \), we have

\[ HF^{-1}H^T \geq H + H^T - F. \]

### 3 Main results

The purpose here is to introduce delay-independent conditions for robust stability of (1) via the Lyapunov-Krasovskii functional approach.

**Theorem 3.1.** Given \( \tau \), system (1) with \( u(t) \equiv 0 \) is robustly stable, if there exist a set of nonsingular matrices \( P = (P(1), \cdots, P(s)) \) and a symmetric positive-definite matrix \( Q \) such that the following coupled matrix inequalities hold for every \( i \in S \)

\[
\begin{align*}
E^TP(i) &= P(i)E \geq 0, \\
\Gamma_i &= \begin{bmatrix}
J & P(i)A_{di} \\
A_{di}^TP(i) & [Q - \varepsilon_2N_{2i}^TN_{2i}]
\end{bmatrix} < 0
\end{align*}
\]

where \( J = A_{i}^TP(i) + P(i)A_{i} + Q + \varepsilon^{-1}P(i)M_{i}M_{i}^TP(i) + \varepsilon_1N_{1i}^TN_{1i} + \sum_{j=1}^{s} \pi_{ij}E^TP(j) \), \( \varepsilon^{-1} = \varepsilon_1^{-1} + \varepsilon_2^{-1} \)
Proof. Let us denote the following Lyapunov-Krasovskii function
\[ V(x) = x^T(t)E^TP(i)x(t) + \int_{t-\tau}^{t} x^T(\theta)Qx(\theta)d\theta. \]

Using Lemma 1, one has
\[ x^T(t)\Delta A_i^T(t)P(i)x(t) + x^T(t)P(i)\Delta A_i(t)x(t) \leq \varepsilon_1 x^T(t)N_{1i}^T N_{1i}x(t) + \varepsilon_1^{-1} x^T(t)P(i)M_iM_i^T P(i)x(t), \]
and
\[ x^T(t)P(i)\Delta A_{di}(t)x(t) - x^T(t-\tau)\Delta A_{di}(t)P(i)x(t) \leq \varepsilon_2 x^T(t-\tau)N_{2i}^T N_{2i}x(t-\tau) + \varepsilon_2^{-1} x^T(t)P(i)M_iM_i^T P(i)x(t). \]

Then we have
\[ \psi V \leq \Phi^T \Gamma_i \Phi < 0 \] (4)

Denote \( \Xi_i(\tau) = \begin{bmatrix} E^TP(i) & 0 \\ 0 & \tau Q \end{bmatrix} \). From (4), we can obtain
\[ \frac{\psi V}{V} \leq \frac{\lambda_{\max}(\Gamma_i) \| \Phi \|^2}{x^T(t)E^TP(i)x(t) + \int_{t-\tau}^{t} x^T(\theta)Qx(\theta)d\theta} \leq -\alpha_i V, \]
where \( \alpha_i = \frac{\lambda_{\max}(\Gamma_i)}{\lambda_{\max}(\Xi_i(\tau))} > 0, (i = 1, 2, \ldots s) \) are positive constants. From the Dynkin formula and Grownwall-Bellman Lemma, one has
\[ E \{ V(x(t)) \} \leq e^{-\alpha t} V(x_0, \sigma(0)). \] (5)

Now let \( \alpha = \max\{\alpha_i, i = 1, 2, \ldots s\} \), we deduce from (5)
\[ E \{ x^T(t)P(i)x(t) \} \leq E \{ V(x(t)) \} \leq e^{-\alpha t} V(x_0, \sigma(0)). \]

This implies
\[ E \left\{ \int_0^T x^T(t)P(i)x(t)dt \right\} \leq -\frac{1}{\alpha} V(x_0, \sigma(0)) \left[ e^{-\alpha T} - 1 \right]. \]

Denote \( p = \min \{ \lambda_{\max}(P(i)) \}, \quad h = \max \{ \lambda_{\max}(P(i)) + \tau \lambda_{\max}(Q)I \}, i = 1, 2, \ldots, s. \) As \( T \to \infty \), we obtain
\[ \lim_{T \to \infty} E \left\{ \int_0^T x^T(t)P(i)x(t)dt \right\} \leq x_0^T \left\{ \frac{h}{ap} \right\} I x_0 = x_0^T G x_0, \]
where \( G = \left\{ \frac{h}{ap} \right\} I. \) According definiton 1, this completes the proof. \( \square \)
In following section, we will present the problem of robust stabilization for uncertain stochastic time delay systems.

**Theorem 3.2.** Given \( \tau \), system (1) is robustly stabilizable if there exist a set of nonsingular matrixes \( P = (P(1), \cdots, P(s)) \), \( X = (X(1), \cdots, X(s)) \) and symmetric positive-definite matrixes \( Q, G \) such that the following coupled LMI holds for every \( i \in S \)

\[
\begin{bmatrix}
J_3 & X_i^T N_{i1}^T (N_{3i}Y_i)^T & X_i^T A_{di} & X_i^T C_i^T & \Xi_i(X)
\end{bmatrix}
\begin{bmatrix}
N_{i1}X_i & -\varepsilon_1 I & 0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
N_{3i}Y_i & -\varepsilon_3 I & 0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
X_i & 0 & 0 & -G & 0 & 0
\end{bmatrix}
\begin{bmatrix}
A_{di} & 0 & 0 & 0 & -X_i
\end{bmatrix}
\begin{bmatrix}
\Xi_i^T(X) & 0 & 0 & 0 & 0 & -\Psi_i(X)
\end{bmatrix} < 0
\]

where \( J_3 = X_i^T A_{di}^T + A_{di}X_i + B_i Y_i + (B_i Y_i)^T + \varepsilon_1 M_i M_i^T + \pi_{ii} X_i^T E^T, G = Q^{-1}, G_1 = Q - \varepsilon_2 N_{2i}^T N_{2i}, \varepsilon = \varepsilon_1 + \varepsilon_3 + \varepsilon_2^{-1}, \Xi_i(X) = [\sqrt{\pi_{i1}} X_i^T, \cdots, \sqrt{\pi_{ii-1}} X_i^T, \sqrt{\pi_{ii+1}} X_i^T, \cdots, \sqrt{\pi_{is}} X_i^T], \Psi_i(X) = \text{diag} [X_1 + X_i^T, X_{i-1} + X_{i-1}^T - W_i, X_{i-1}^T - W_{i-1}, \cdots, X_s + X_s^T - W_s]. \)

Then, a set of suitable stabilizing controllers are given by \( u_i(t) = Y_i X_i^{-1} x(t) \), where \( Y_i = K_i X_i (i = 1, \cdots, s) \).

**Proof.** With the control law \( u_i(t) = K_i x(t) \), where the matrix \( K_i \in R^{n \times n} \) is to be found, (1) becomes

\[
Edx(t) = [(A_{\sigma(t)} + \Delta A_{\sigma(t)}(t)) x(t) + (A_{d\sigma(t)} + \Delta A_{d\sigma(t)}(t)) x(t - \tau) + (B_{\sigma(t)} + \Delta B_{\sigma(t)}(t)) K_i x(t)] dt + C_{\sigma(t)} d\omega(t).
\]

The Lyapunov-Krasovskii function can be constructed as

\[
V(x) = x^T(t) E^T P(i) x(t) + \int_{t-\tau}^t x^T(\theta) Q x(\theta) d\theta.
\]

Define \( \Phi \) as in Theorem 3.1., the above inequality can be rewritten as

\[
\psi V \leq \Phi^T \Lambda_i \Phi < 0,
\]

where

\[
\Lambda_i = \begin{bmatrix}
J_1 & P^T(i) A_{di} \\
A_{di}^T P(i) & G_1
\end{bmatrix}, G_1 = Q - \varepsilon_2 N_{2i}^T N_{2i},
\]

\[
W_i > E \Lambda_i X(i) = X^T(i) E^T \geq 0,
\]

where \( J_3 = X_i^T A_{di}^T + A_{di}X_i + B_i Y_i + (B_i Y_i)^T + \varepsilon_1 M_i M_i^T + \pi_{ii} X_i^T E^T, G = Q^{-1}, G_1 = Q - \varepsilon_2 N_{2i}^T N_{2i}, \varepsilon = \varepsilon_1 + \varepsilon_3 + \varepsilon_2^{-1}, \Xi_i(X) = [\sqrt{\pi_{i1}} X_i^T, \cdots, \sqrt{\pi_{ii-1}} X_i^T, \sqrt{\pi_{ii+1}} X_i^T, \cdots, \sqrt{\pi_{is}} X_i^T], \Psi_i(X) = \text{diag} [X_1 + X_i^T, X_{i-1} + X_{i-1}^T - W_i, X_{i-1}^T - W_{i-1}, \cdots, X_s + X_s^T - W_s]. \)
\[ J_1 = A_i^T P(i) + P(i) A_i + P(i) B_i K_i + K_i^T B_i^T P(i) + \sum_{j=1}^{s} \pi_{ij} E_i^T P(j) + C_i^T P(i) C_i + \varepsilon_i^{-1} N_i N_i + Q + \varepsilon P(i) M_i M_i^T P(i). \]

From (6), we can obtain \( \Lambda_i < 0 \). By the Schur complement, \( \Lambda_i \) can also be transformed into

\[ \begin{bmatrix}
    J_2 & N_i^T & (N_i K_i)^T & P(i) A_{d_i} & C_i^T \\
    N_i & -\varepsilon_1 I & 0 & 0 & 0 \\
    N_i K_i & 0 & -\varepsilon_3 I & 0 & 0 \\
    A_{d_i}^T P(i) & 0 & 0 & -G_1 & 0 \\
    C_i & 0 & 0 & 0 & -P^{-1}(i)
\end{bmatrix} < 0, \quad (7) \]

where \( J_2 = A_i^T P(i) + P(i) A_i + P(i) B_i K_i + K_i^T B_i^T P(i) + \sum_{j=1}^{s} \pi_{ij} E_i^T P(j) + Q \) is nonlinear in the design parameters \( P(i) \) and \( K_i \). To put it into the LMI form, let \( X_i = P^{-1}(i) \), \( Y_i = K_i X_i \). And pre- and post multiplying the (7) by \( \text{diag}(P^{-1}(i), I, I, I, I) \) and \( \text{diag}(P^{-1}(i), I, I, I, I) \), respectively, then it can be easily to obtain the results.

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**References**


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