Nonlinear Lagrange Dual for Multi-Objective Programming Problems

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Abstract

In this paper nonlinear Lagrange dual for a nonlinear multi-objective programming problem is formulated and it is proved that pareto optimal solution of the primal multi-objective programming problem and optimal solution of its nonlinear Lagrange dual are equal.

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1 Introduction

Consider a nonlinear optimization problem:

\[(P): \quad \inf f_0(x) \quad \text{Subject to} \quad f_i(x) \leq 0\]

\[x \in M \subset \mathbb{R}^n, f_0, f_i : \mathbb{R}^n \to \mathbb{R}, i = 1, 2, ..., m.\]

Its Lagrange Dual is

\[(LD): \quad \sup_{u > 0} \inf_{x \in M} \left[ L(x, u) \right]\]

where \(L(x, u) = f_0(x) + \sum_{i=1}^{m} u_i f_i(x), \) \(u = (u_1, u_2, ..., u_m), u_i \in \mathbb{R}.\) Here the Lagrange dual function, \(L(x, u),\) a linear function of dual variables \(u_1, u_2, ..., u_m.\) \((LD)\) becomes a strong dual of\((P)\) under convexity or generalized convexity assumption for the functions \(f_0\) and \(f_i.\) The concept of nonlinear Lagrange dual was introduced by Goh and Yang[1, 2]). After these two outstanding papers, the concept of nonlinear Lagrange dual has been applied in Discrete Optimization, Bounded Integer programming, Variational Inequality problem,
Semidefinite Programming etc. In this paper the nonlinear Lagrange dual for a Multi-objective nonlinear programming problem is formulated and the duality results are proved without using convexity or generalized convexity assumptions. A general Multi-objective programming problem is

\[(MP) \quad \inf_{x \in M} f(x)\]

\[M \subseteq \mathbb{R}^n, \quad f : X \to \mathbb{R}^m, \quad f(x) = (f_1(x), f_2(x), \ldots, f_m(x)).\]

The image space of \(f\) is assumed to be a partially ordered set in a natural way (i.e. \(\mathbb{R}^n_+\) is the ordering cone). The following preliminaries are required in sequel to prove the duality results.

**Definition 1.1** Edgeworth Pareto optimality:

\(x^* \in M\) is called an Edgeworth Pareto optimal point (or an efficient solution of \((MP)\)) if \(f(x^*)\) is a minimal element of the image set \(f(M)\) with respect to the natural partial ordering. i.e there is no \(x \in M\) satisfying

\[f_i(x) \leq f_i(x^*), \quad \forall i \quad \text{and} \quad f(x) \neq f(x^*)\]

The following relationship between vector optimization and approximation theory is due to Jahn[6].

**Theorem 1.1** Let \(S\) be a non empty subset of a real topological linear space \(Y\) partially ordered by a closed pointed convex cone \(C\) with a non empty interior \(\text{int}(C)\). Moreover, let an element \(\hat{y} \in Y\) be given with the property \(S \subset \{\hat{y}\} + \text{int}(C)\). An element \(\bar{y} \in S\) is a minimal element of \(S\) iff there is an element \(a \in \text{int}(C)\) so that

\[\|\bar{y} - \hat{y}\|_a < \|y - \hat{y}\|_a \quad \forall y \in S - \{\bar{y}\}\]

where \(\|y\|_a = \max_i \{|y_i|_a\}, \quad y \in Y\).

2 Existence of solution of \(MP\) using nonlinear Lagrange dual

Assume that each objective function \(f_i(x)\) of \(MP\) is bounded by a lower bound \(\hat{y}_i\) for \(i = 1, 2, \ldots, m\) and \(x^*\) be the Edgeworth Pareto optimal solution of \((MP)\). So \(f_i(x) - \hat{y}_i \geq 0, \forall i\). Consider \(S = f(M)\). It follows from the consequence of Theorem 1.1 that \(x^*\) is the pareto optimal solution of \((MP)\) iff there are positive real numbers \(a_1, a_2, \ldots, a_m\) such that

\[\max_i \left\{ \frac{f_i(x^*) - \hat{y}_i}{a_i} \right\} < \max_i \left\{ \frac{f_i(x) - \hat{y}_i}{a_i} \right\}, \quad f(x) \neq f(x^*), \quad \forall x \in M \quad (1)\]
Let \( \lambda(x) = \max_i \left\{ \frac{f_i(x) - \hat{y}_i}{\alpha_i} \right\} \), \( f(x) \neq f(x^*) \). Then \( \lambda(x) \geq 0 \), \( f_i(x) - \hat{y}_i < \alpha_i \lambda(x) \) \( \forall i \) and \( \forall x \in M \). Hence \( x^* \) is the optimal solution of the following problem

\[
(MP') : \quad \inf_x \lambda(x)
\]

subject to \( f_i(x) - \hat{y}_i < \lambda(x) \alpha_i, \ i = 1, 2, \ldots, m, \ x \in M \)

Let \( \lambda^* = \lambda(x^*) \) and \( \alpha_1, \alpha_2, \ldots, \alpha_m \) are some positive real numbers. Consider \( E = \{ u \in \mathbb{R}^{m+1} | u = (u_0, u_1, u_2, \ldots, u_m), \ u_0 = 1, \alpha_i > 0, \ i = 1, 2, \ldots, m \} \). We define \( F_i(x) = f_i(x) - \lambda(x) \alpha_i \) for \( i = 1, 2, \ldots, m \) and \( F_0(x) = \lambda(x) \). For \( u \in E \), let

\[
\phi(u) = \begin{cases} 
\inf_{x \in M} \max_{0 \leq i \leq m} \left\{ \frac{F_i(x)}{u_i} \right\} & \text{if } u_i \geq \frac{\alpha_i \hat{y}_i}{\lambda^*}, \ i = 1, 2, \ldots, m \\
-\infty & \text{otherwise}
\end{cases}
\]

Now we define nonlinear Lagrange dual of \( MP \) by

\[
(NLD) : \quad \sup_{u \in E} \phi(u)
\]

**Theorem 2.1** Solution of \( NLD \) exists if \( \lambda(x) < \min\{a_1 \hat{y}_1, a_2 \hat{y}_2, \ldots, a_m \hat{y}_m \} \), \( \forall x \in M \).

Proof: It is obvious that the dual function \( \phi(u) \) can not be maximized if we take \( -\infty \). So consider \( u_i \geq \frac{\alpha_i \hat{y}_i}{\lambda^*} \) for each \( i \). Since \( \lambda(x) < a_i \hat{y}_i, \forall (i) \), for \( u^1 \leq u^2 \)

\[
(u^j = (u^1_1, u^1_2, \ldots, u^1_m), j = 1, 2)
\]

\[
\frac{F_i(x)}{u^1_i} \geq \frac{F_i(x)}{u^2_i}
\]

that is

\[
\inf_{x \in M} \max_{0 \leq i \leq m} \left\{ \frac{F_i(x)}{u^1_i} \right\} \geq \inf_{x \in M} \max_{0 \leq i \leq m} \left\{ \frac{F_i(x)}{u^2_i} \right\}
\]

This implies that \( \inf_{x \in M, \lambda > 0} \max_{0 \leq i \leq m} \left\{ \frac{F_i(x)}{u_i} \right\} \) is a non increasing function. Hence the dual function must be maximized at the lowest possible value of \( u \).

**Theorem 2.2** (Strong duality) If \( x^* \) is a pareto optimal solution of \( MP \), \( \lambda(x) < \min\{a_1 \hat{y}_1, a_2 \hat{y}_2, \ldots, a_m \hat{y}_m \} \), \( \forall x \in M \) and \( u^* = (u^1_1, u^1_2, \ldots, u^1_m) \) is an optimal solution of \( NLD \) then \( \lambda(x^*) = \phi(u^*) \).

Proof:
Since \( \lambda(x) < \min\{a_1 \hat{y}_1, a_2 \hat{y}_2, \ldots, a_m \hat{y}_m \} \), by Theorem 2.1, solution of \( NLD \) exists. Let \( u^* \) be the optimal solution of \( NLD \). Let \( x \) be a feasible solution of \( MP \). Suppose each objective function \( f_i(x) \) of \( MP \) is bounded by a lower
bound $\hat{y}_i$. Then $MP$ is equivalent to $MP'$. Hence $x$ is a feasible solution of $MP'$ and $\frac{F_i(x)}{a_i\hat{y}_i} \leq 1$ for $i = 1, 2, ..., m$ and $F_0(x^*) \leq F_0(x)$. Now

$$\phi(u^*) = \inf_{x \in M} \max \{F_0(x), \frac{F_1(x)}{u_1^*}, \frac{F_2(x)}{u_2^*}, ..., \frac{F_m(x)}{u_m^*}\}$$

$$= \inf_{x \in M} \max \{F_0(x), \frac{F_1(x)}{a_1\hat{y}_1}F_0(x), \frac{F_2(x)}{a_2\hat{y}_2}F_0(x), ..., \frac{F_m(x)}{a_m\hat{y}_m}F_0(x)\}$$

$$= \inf_{x \in M} F_0(x) = \lambda(x^*)$$

If at least one objective function $f_i(x)$ of $MP$ is not bounded by a lower bound $\hat{y}_i$ then there exists at least one $k$ such that $\frac{F_k(x)}{a_k\hat{y}_k} > 1$. So

$$\max_{0 \leq i \leq m} \{\frac{F_i(x)}{u_i^*}\} \geq F_0(x^*).$$

That is $F_0(x^*) \geq \phi(u^*)$. Combining both possibilities we have

$$\lambda(x^*) \leq \phi(u^*) \quad (2)$$

If $u_i < \frac{a_i\hat{y}_i}{\lambda^*}$ for at least one $i$ then $\lambda(x) \geq \phi(u)$ is true. Suppose $u_i \geq \frac{a_i\hat{y}_i}{\lambda^*}, \forall i$. Then

$$\phi(u) = \inf_{x \in M} \max \{F_0(x), \frac{F_1(x)}{u_1}, \frac{F_2(x)}{u_2}, ..., \frac{F_m(x)}{u_m}\}$$

$$\leq \inf_{x \in M} \max \{\lambda^*, \frac{F_1(x^*)}{u_1}, \frac{F_2(x^*)}{u_2}, ..., \frac{F_m(x^*)}{u_m}\}$$

$$\leq \max \{\lambda^*, \frac{F_1(x^*)}{a_1\hat{y}_1}\lambda^*, \frac{F_2(x^*)}{a_2\hat{y}_2}\lambda^*, ..., \frac{F_m(x^*)}{a_m\hat{y}_m}\lambda^*\}$$

$$\leq \lambda^* < \lambda(x)$$

So $\lambda(x) \geq \phi(u), \forall u \in E, \forall x \in M$ and hence $\lambda(x^*) \leq \phi(u^*)$.

3 Conclusion

Multi-objective programming problem has drawn the attention of many researchers since last few decades. Most of the methods to solve Multi objective programming problem are valid under convexity assumption. In this paper a particular type of nonlinear Lagrange dual function for a Multi-objective programming problem is formulated and proved that this dual can be a strong dual without any assumption of convexity and generalized convexity conditions. Theoretically this dual makes sense but it is not so easy to solve such types of dual numerically. Using the present development there is scope to construct alternate version of nonlinear function of dual variables like least square, quadratic type etc. to make nonlinear Lagrange dual more useful.
References


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