Stability and Hopf Bifurcation on a Neuron Network with Discrete and Distributed Delays

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Abstract

This paper is concerned with a two-neuron network model with both discrete and distributed delays. By regarding the discrete time delay as the bifurcation parameter, the stability of the equilibrium \((0, 0)\) and Hopf bifurcations are investigated. Finally, to verify our theoretical predictions, some numerical simulations are also included.

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1 Introduction

In recent years, a large number of neural networks models have been proposed and studied extensively since Hopfield constructed a simplified neural network. Based on the Hopfield neural network model, Marcus and Westervelt [3] argued that time delays always occur in the signal transmission and proposed a neural network model with delay. Afterward, a variety of artificial models has been established to describe neural networks with delays. In [4], Olien et al. discussed the dynamics of the following system with two discrete delays:

\[
\begin{aligned}
\dot{u}_1(t) &= -u_1(t) + a_{11}f(u_1(t - \tau_1)) + a_{12}f(u_2(t - \tau_2)), \\
\dot{u}_2(t) &= -u_2(t) + a_{21}f(u_1(t - \tau_1)) + a_{22}f(u_2(t - \tau_2)).
\end{aligned}
\]  

They investigated the stability and Hopf bifurcation of system (1) for several special case of parameters, and found that system (1) may undergo Hopf bifurcations at certain values of delays. The dynamics of systems similar to (1) have
been investigated extensively and many interesting results have been obtained (e.g. [6, 7, 10]).

However, as pointed out in [2], neural networks usually have a spatial extent due to presence of a multitude of parallel pathways with a variety of axon sizes and lengths (also see [5]), and hence there is a distribution of propagation delays over a period of time. Based on this idea, Zhou et al. [11] proposed and investigated a network is described by the following system:

\[
\begin{align*}
\dot{u}_1(t) &= -u_1(t) + a_{11}f(u_1(t-\tau)) + a_{12}f\left(\int_{t-\infty}^{t} F(t-s)u_2(s)ds\right), \\
\dot{u}_2(t) &= -u_2(t) + a_{21}f(u_1(t-\tau)) + a_{22}f\left(\int_{t-\infty}^{t} F(t-s)u_2(s)ds\right),
\end{align*}
\]  

(2)

where \( F(\cdot) \) is nonnegative bounded delay kernel defined on \([0, +\infty)\) which reflect the influence of the past states on the current dynamics.

Motivated by the above works, in the present paper, we consider an analogue of system (2) containing both discrete and diagonal distributed delays, which takes the following form:

\[
\begin{align*}
\dot{u}_1(t) &= -u_1(t) + a_{11}f\left(\int_{t-\infty}^{t} F(t-s)u_1(s)ds\right) + a_{21}f(u_1(t-\tau)) + a_{22}f\left(\int_{t-\infty}^{t} F(t-s)u_2(s)ds\right), \\
\dot{u}_2(t) &= -u_2(t) + a_{21}f(u_1(t-\tau)) + a_{22}f\left(\int_{t-\infty}^{t} F(t-s)u_2(s)ds\right),
\end{align*}
\]  

(3)

Generally speaking, neural network models that include the self-feedback terms with distributed time delays predict more complex dynamics than those without taking these factors into account. In this model, the presence of the distributed time delays must not affect the equilibrium values, so we normalize the kernel such that \( \int_{0}^{\infty} F(s)ds = 1 \). Following the ideas of Cushing et al. [1], the weak kernel \( F(s) = \alpha e^{-\alpha s}(\alpha > 0) \) and the strong kernel \( F(s) = \alpha^{2}se^{-\alpha s}(\alpha > 0) \) are frequently encountered in the literature. Due to the accompanying analytical convenience, in this paper, we consider system (3) with the weak kernel. Taking discrete delay \( \tau \) as the bifurcation parameter, we shall investigate the effect of the delay \( \tau \) on the dynamics of system (3).

This paper is organized as follows. In Section 2, we shall consider the stability of the zero equilibrium and the existence of local Hopf bifurcation. In order to verify our theoretical prediction, some numerical simulations are included in Section 3.

## 2 Stability analysis and Hopf bifurcation

Let

\[
\begin{align*}
u_3(t) &= \int_{-\infty}^{t} \alpha e^{-\alpha(t-s)}u_1(s)ds, \\
u_4(t) &= \int_{-\infty}^{t} \alpha e^{-\alpha(t-s)}u_2(s)ds.
\end{align*}
\]
By the linear chain trick technique, system (3) can be transformed into the following system:

\[
\begin{cases}
\dot{u}_1(t) = -u_1(t) + a_{11}f(u_3(t)) + a_{12}f(u_2(t - \tau)), \\
\dot{u}_2(t) = -u_2(t) + a_{21}f(u_1(t - \tau)) + a_{22}f(u_4(t)), \\
\dot{u}_3(t) = \alpha u_1(t) - \alpha u_3(t), \\
\dot{u}_4(t) = \alpha u_2(t) - \alpha u_4(t).
\end{cases}
\] (4)

Suppose that \( f \in C^1 \), \( f(0) = 0 \). It is obvious that the origin \((0, 0, 0, 0)\) is an equilibrium of system (4). Linearizing system (4) about the origin yields the following linear system

\[
\begin{cases}
\dot{u}_1(t) = -u_1(t) + \alpha_1 u_3(t) + \alpha_{12} u_2(t - \tau), \\
\dot{u}_2(t) = -u_2(t) + \alpha_{21} u_1(t - \tau) + \alpha_{22} u_4(t), \\
\dot{u}_3(t) = \alpha u_1(t) - \alpha u_3(t), \\
\dot{u}_4(t) = \alpha u_2(t) - \alpha u_4(t),
\end{cases}
\] (5)

where \( \alpha_{ij} = a_{ij} f'(0) \), \( i, j = 1, 2 \). The associated characteristic equation of system (5) is

\[
\lambda^4 + a_3 \lambda^3 + a_2 \lambda^2 + a_1 \lambda + a_0 + (b_2 \lambda^2 + b_1 \lambda + b_0)e^{-2\lambda \tau} = 0,
\] (6)

where

\[
\begin{align*}
a_3 &= 2(1 + \alpha), \\
a_2 &= \alpha^2 + 4\alpha + 1 + \alpha(\alpha_{11} - \alpha_{22}), \\
a_1 &= \alpha(1 + \alpha)(2 + \alpha_{11} - \alpha_{22}), \\
a_0 &= \alpha^2(\alpha_{22} - 1)(\alpha_{11} - 1), \\
b_2 &= -\alpha_{12}\alpha_{21}, \quad b_1 = -2\alpha\alpha_{12}\alpha_{21}, \quad b_0 = -\alpha^2\alpha_{12}\alpha_{21}.
\end{align*}
\]

Note that when \( \tau = 0 \), (6) becomes

\[
\lambda^4 + a_3 \lambda^3 + (a_2 + b_2) \lambda^2 + (a_1 + b_1) \lambda + a_0 + b_0 = 0.
\] (7)

By the Routh-Hurwitz criterion, a set of the necessary and sufficient conditions for all roots of Eq. (7) to have negative real parts are given by

\[
D_1 = a_3 > 0,
\]

\[
D_2 = \det \begin{pmatrix} a_3 & a_1 + b_1 \\ 1 & a_2 + b_2 \end{pmatrix} = a_3(a_2 + b_2) - (a_1 + b_1) > 0,
\]

\[
D_3 = \det \begin{pmatrix} a_3 & a_1 + b_1 & 0 \\ 1 & a_2 + b_2 & a_0 + b_0 \\ 0 & a_3 & a_1 + b_1 \end{pmatrix} = a_3[(a_2 + b_2)(a_1 + b_1) - a_3(a_0 + b_0)]
\]

\[
- (a_1 + b_1)^2 > 0,
\]

\[
D_4 = \det \begin{pmatrix} a_3 & a_1 + b_1 & 0 & 0 \\ 1 & a_2 + b_2 & a_0 + b_0 & 0 \\ 0 & a_3 & a_1 + b_1 & 0 \\ 0 & 1 & a_2 + b_2 & a_0 + b_0 \end{pmatrix} = a_0 + b_0 > 0.
\]

A neuron network
Thus, one can immediately obtain the following result.

**Lemma 2.1** Assume that (H) \( D_1 > 0, D_2 > 0, D_3 > 0, D_4 > 0 \). Then all the roots of Eq. (7) with \( \tau = 0 \) have always negative real parts.

Next, we shall investigate the distribution of roots of Eq. (6) with \( \tau > 0 \). Obviously, \( i\omega (\omega > 0) \) is a root of Eq. (6) if and only if \( \omega \) satisfies the following equation

\[
\omega^4 - ia_3\omega^3 - a_2\omega^2 + a_0 + (-b_2\omega^2 + ib_1\omega + b_0)(\cos 2\omega\tau - i\sin 2\omega\tau) = 0. 
\] (8)

Separating the real and imaginary parts of Eq. (8) gives the following equations

\[
\begin{align*}
\omega^4 - a_2\omega^2 + a_0 &= (b_2\omega^2 - b_0)\sin 2\omega\tau + b_1\omega\cos 2\omega\tau, \\
a_3\omega^3 - a_1\omega &= (b_2\omega^2 - b_0)\cos 2\omega\tau - b_1\omega\sin 2\omega\tau.
\end{align*} 
\] (9)

Adding up the squares of the corresponding sides of the above equations leads to

\[
(\omega^4 - a_2\omega^2 + a_0)^2 + (a_3\omega^3 - a_1\omega)^2 = (b_2\omega^2 - b_0)^2(b_1\omega)^2. 
\] (10)

It follows from (10) that

\[
\omega^8 + a\omega^6 + b\omega^4 + c\omega^2 + d = 0, 
\] (11)

where

\[
a = a_3^2 - 2a_2, \quad b = a_2^2 + 2a_0 - 2a_1a_3 - b_2^2, \\
c = a_1^2 - 2a_0a_2 + 2b_0b_1 - b_1^2, \quad d = a_0^2 - b_0^2.
\]

Let \( z = \omega^2 \), then (11) becomes

\[
h(z) \triangleq z^4 + az^3 + bz^2 + cz + d = 0. 
\] (12)

From (12), we have

\[
g(z) \triangleq \frac{dh(z)}{dz} = 4z^3 + 3az^2 + 2bz + c. 
\] (13)

Let \( y = z + \frac{a}{4} \), then the equation \( g(z) = 0 \) becomes \( y^3 + my + n = 0 \), where

\[
m = \frac{8b - 3a^2}{16}, \quad n = \frac{a^3 - 4ab + 8c}{32}.
\]
Define
\[ D = \frac{n^2}{4} + \frac{m^3}{27}, \quad \sigma = -1 + i\sqrt{3}, \quad z_i = y_i - \frac{a}{4}, \quad i = 1, 2, 3. \]
\[ y_1 = \frac{3}{2} - \frac{n}{2} + \sqrt{D} + \frac{3}{2} - \sqrt{D}, \quad y_2 = \frac{3}{2} - \frac{n}{2} + \sqrt{D\sigma} + \frac{3}{2} - \sqrt{D\sigma^2}, \]
\[ y_3 = \frac{3}{2} - \frac{n}{2} + \sqrt{D\sigma^2} + \frac{3}{2} - \sqrt{D\sigma}. \]

Assume that \( D > 0 \) then from the Cardano’s formulae for the third-degree algebra equation we know that the equation \( g(z) = 0 \) has only one real root \( z_i^* = z_1 \). If \( D = 0 \), then the equation \( g(z) = 0 \) have three real roots \( z_1, z_2 \) and \( z_3 \) (where \( z_2 = z_3 \)), and in this case we define \( z_2^* \) by \( \max\{z_1, z_2\} \). If \( D < 0 \), we know that the equation \( g(z) = 0 \) has three different real roots \( z_1, z_2 \) and \( z_3 \). In the last case, we define \( z_3^* \) by \( z_3^* = \max\{z_1, z_2, z_3\} \).

According to Lemma 2.1 in Yan and Li [9], we have the following

**Lemma 2.2 (i)** If \( d < 0 \), then (12) has at least one positive root.

(ii) If \( d \geq 0 \), then (12) has no positive root if one of the following conditions holds:

(a) \( D > 0 \) and \( z_1^* > 0 \); (b) \( D = 0 \) and \( z_2^* < 0 \); (c) \( D < 0 \) and \( z_3^* < 0 \).

(iii) If \( d \geq 0 \), then (12) has at least a positive root if one of the following conditions holds:

(a) \( D > 0 \), \( z_1^* > 0 \) and \( h(z_1^*) < 0 \); (b) \( D = 0 \), \( z_2^* > 0 \) and \( h(z_2^*) < 0 \); (c) \( D < 0 \), \( z_3^* > 0 \) and \( h(z_3^*) < 0 \).

Without loss of generality, suppose that equation (12) have four positive real roots, denoted by \( z_1, z_2, z_3, z_4 \), respectively. Then (11) should also have four positive real roots
\[ \omega_1 = \sqrt{z_1}, \quad \omega_2 = \sqrt{z_2}, \quad \omega_3 = \sqrt{z_3}, \quad \omega_4 = \sqrt{z_4}. \]

From the first equation of (9), we have
\[
\begin{align*}
\sin 2\omega_k \tau &= \frac{(\omega^4 - a_1\omega_k)(b_2\omega_k^2 - b_0) - b_1\omega_k(\omega^4 - a_2\omega_k^2 + a_0)}{(b_2\omega_k^2 - b_0)^2 + b_1^2\omega_k^2} \triangleq \frac{A_1}{(b_2\omega_k^2 - b_0)^2 + b_1^2\omega_k^2} \tag{14}
\cos 2\omega_k \tau &= \frac{(\omega^4 - a_2\omega_k^2 + a_0)(b_2\omega_k^2 - b_0) + b_1\omega_k(\omega^4 - a_1\omega_k)}{(b_2\omega_k^2 - b_0)^2 + b_1^2\omega_k^2} \triangleq \frac{A_2}{(b_2\omega_k^2 - b_0)^2 + b_1^2\omega_k^2}.
\end{align*}
\]
Define
\[ \tau_k^j = \frac{1}{2\omega_k} \left[ \arctan \frac{A_1}{A_2} + j\pi \right], \quad (15) \]
where \( k = 1, 2, 3, 4, \) \( j = 0, 1, 2, \ldots \). Then \((\tau_k^j, \omega_k)\) are solutions of equation (8) and \( \lambda = \pm i\omega_k \) are a pair of purely imaginary roots of (6) with \( \tau = \tau_k^j \). Define

\[
\tau_0 = \tau_{k_0}^0 = \min_{1 \leq k \leq 4} \{\tau_k^0\}, \quad \omega_0 = \omega_{k_0},
\]

where \( k_0 \in \{1, 2, 3, 4\} \). Then \( \tau_0 \) is the first value of \( \tau \) such that (6) have purely imaginary roots.

Let \( \lambda(\tau) = \alpha(\tau) \pm i\omega(\tau) \) be the root of Eq.(6) near \( \tau = \tau_k^j \) satisfying \( \alpha(\tau_k^j) = 0, \omega(\tau_k^j) = \omega_0(j = 0, 1, 2, \ldots) \).

**Lemma 2.3** [8]. Suppose \( h'(z_k) \neq 0 \), where \( h(z) \) is defined by (12), then the following transversality conditions are satisfied:

\[
\frac{d\text{Re}\lambda(\tau)}{d\tau} \bigg|_{\tau = \tau_k^j} \neq 0.
\]

Moreover, the sign of \( \frac{d\text{Re}\lambda(\tau)}{d\tau} \bigg|_{\tau = \tau_k^j} \) is consistent with that of \( h'(z_k) \).

Since the multiplicity of roots with positive real parts of Eq.(6) can change only if a root appears on or crosses the imaginary axis as time delay \( \tau \) varies, by Lemma 2.3, we have the following result.

**Lemma 2.4** Suppose that equation (12) has at least one positive root. If \( \tau \in (\tau_k^j, \tau_k^{j+1}) \), then Eq. (6) has at least \( 2(j + 1)(j = 0, 1, 2, \ldots) \) roots with positive real part.

By Lemmas 2.1-2.4, we have the following result regarding on stability and bifurcation of system (1).

**Theorem 2.5** Suppose that (H) holds, we have the following:

(i) All roots of (6) have negative real parts and the zero solution of system (4) is absolutely stable if \( d \geq 0 \) and one of the following conditions is satisfied:

(a) \( D > 0 \) and \( z_1^* \leq 0 \); (b) \( D = 0 \) and \( z_2^* \leq 0 \); (c) \( D < 0 \) and \( z_3^* \leq 0 \).

(ii) All roots of (6) have negative real parts and the zero solution of system (4) is asymptotically stable for \( \tau \in [0, \tau_0) \) if \( d < 0 \) or \( d \geq 0 \) and one of the following conditions is satisfied:

(a) \( D > 0, z_1^* > 0 \) and \( h(z_1^*) < 0 \); (b) \( D = 0, z_2^* > 0 \) and \( h(z_2^*) < 0 \); (c) \( D < 0, z_3^* \leq 0 \) and \( h(z_3^*) < 0 \).

(iii) if the conditions as stated in (ii) are satisfied, and \( h'(z_k) \neq 0 \), then system (4) undergoes Hopf bifurcations at \( \tau = \tau_k^j(k = 1, 2, 3, 4, j = 0, 1, 2, \ldots) \).
3 A numerical example

In this section, we give some numerical simulations to illustrate our results. As an example, we consider system (3) with \( f(\cdot) = \tanh(\cdot) \), \( a_{11} = -0.5 \), \( a_{12} = -1.8 \), \( a_{21} = 1.3 \), \( a_{22} = 1.7 \), and kernel \( F(s) = e^{-s} \), i.e., \( \alpha = 1 \), then (3) becomes the following system

\[
\begin{align*}
\dot{u}_1(t) &= -u_1(t) - 0.5 \tanh \left[ \int_{-\infty}^{t} e^{-(t-s)} u_1(s) ds \right] - 1.8 \tanh(u_2(t - \tau)), \\
\dot{u}_2(t) &= -u_2(t) + 1.3 \tanh(u_1(t - \tau)) + 1.7 \tanh \left[ \int_{-\infty}^{t} e^{-(t-s)} u_2(s) ds \right].
\end{align*}
\]

(16)

Fig. 1. The phase graph of (16) with \( \tau = 0.55 \) and initial data \( u_1(t) = u_2(t) = 0.2 \), \( t \in [-0.55, 0] \) in the \( u_1 - u_2 \) plane.

Fig. 2. The phase graph of system (16) with \( \tau = 0.6 \).

It is easy to verify that hypothesis (H) holds, and (12) becomes

\[ h(z) = z^4 + 8.4z^3 + 10.0644z^2 - 35.6648z - 4.3731 = 0. \]

In this case, \( h(z) = 0 \) has only one positive real root \( z_0 = 1.5237 \), and then

\[ \omega_0 = 1.2344, \quad \tau_0 = 0.574, \quad h'(z_0) = 67.6614 \neq 0. \]
Thus from Theorem 2.5 we know that the zero solution of system (16) is asymptotically stable when $0 < \tau < \tau_0 = 0.574$, (see Fig.1). The system (16) also undergoes a Hopf bifurcation at the origin $(0,0)$ when $\tau$ crosses through increasingly the critical value $\tau_0 = 0.574$ (see Fig.2).

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**References**


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