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Spectra of a Compact Weighted Composition Operator on $CV_0(X, E)$

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Abstract

Let V be a system of weights on a completely regular Hausdorff space X and let E be a Hausdorff locally convex space. Denote by $CV_0(X, E)$ the weighted space of all continuous functions f from X into E such that v.f vanishes at infinity for all $v \in V$. Continuity and compactness of weighted composition operators on $CV_0(X, E)$ are characterized by Singh and Manhas. We study these operators and give imporved results for particular V, and also determine their spectra.

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1. INTRODUCTION:

The content of this note is in relation with the theory of weighted composition operators on spaces of continuous functions which are studied in

[1,2,3,4,5,8,12,14,15]. For details, we refer to the monograph [9] of Singh and Manhas.

The notation and terminology employed here agree with those of Singh and one of the author [13].

Throughout this note, X will atleast denote a completely regular Hausdorff space and E a Hausdorff locally convex topological vector space over $\mathbb{K}$, where the scalar field $\mathbb{K} \in \{\mathbb{C}, \mathbb{R}\}$. Let C(X,E) denote the space of all continuous functions from X into E and let cs(E) be the collection consisting of all continuous seminorms on E. For a system V of weights (see [13]) on X, the subspace $CV_0(X, E)$ of C(X,E) consisting of those f such that v.f vanishes at infinity (that is, the set $\{x \in X : v(x)p(f(x)) \geq \epsilon\}$ is compact for every $p \in cs(E)$ and $\epsilon > 0$) for each $v \in V$, is a locally convex space with the topology given by the family $\{||.||_{v,p} : v \in V$ and $p \in cs(E)\}$ of seminorms, where

$$||f||_{v,p} = sup\{v(x)p(f(x)) : x \in X\}, f \in CV_0(X, E).$$

When $E = \mathbb{K}$, we write $CV_0(X)$ in place of $CV_0(X, E)$. Further if E = (E,p) is a normed space, then the corresponding seminorm $||.||_{v,p}$ is denoted by $||.||_v$ for all $v \in V$. Also, it is known that when V is the system of all weightes on X which vanish at infinity, $CV_0(X, E)$ is the spacd $C_b(X, E)$ with the substrict topology β_0. For further details on weighted spaces of continuous functions, we refer to Ruess and Summers [7] as well as the reference listed therein.

The following result which is Proposition 2.2 of Singh Manhas and one of the author [12] will be used in the presentation of this note.

Proposition A: For a system V of weights on X and a locally convex topological vector space E, the following statements are equivalent:
(A.1) Each $v \in V$ vanish at infinity.
(A.2) For each $t \in E$, $1_t \in CV_0(X, E)$, where the function 1_t is defined by $1_t(x) = t$ for all $x \in X$.

The object $B_b(E)$ (respectively, $B_s(E)$) stands for the space B(E) of all operators from E to itself when it is equipped with the uniform (respectively, strong) operator topology, that is, the topology of uniform convergence on bounded (respectively, finite) subsets of E.

Let π be a B(E)-valued function on X and let T be a selfmap on X. Then a weighted composition operator (written WCO in short) on $CV_0(X, E)$ induced by the pair (π, T) is the operator, denoted by $W_{\pi,T}$, which has the following form:

$$W_{\pi,t}f(x) = \pi(x)f(T(x)) \text{ for each } f \in CV_0(X,E) \text{ and } x \in X.$$

2. CONTINUITY AND COMPACTNESS OF WCOs

The following result which is reported as Theorem 4.6 in the survey article [13] of Singh and one of the author has been proved by Singh and Manhas in [10].

Theorem B: Assume that X is also $k_{\mathbb{R}}$-space, let $\pi \in C(X, B_b(E))$ and T be a continuous selfmap on X. Then $W_{\pi,T}$ is a WCO on $CV_0(X,E)$ if and only if (i) for every $v \in V$ and $p \in cs(E)$, there exists $u \in V$ and $q \in cs(E)$ such that $v(x)p(\pi(x)t) \leq u(T(x))q(t)$ for all $x \in X$ and $t \in E$, and (ii) for every $v \in V, p \in cs(E), \epsilon > 0$ and compact subset K of X, the set $T^{-1}(K) \cap \{x \in X : v(x)p(\pi(x)t) \geq \epsilon\}$ is compact in X for all non-zero t in E.

In the next theorem we shall present an imporvement for Theorem B for the case when V consists of weights vanishing at infinity. Let us first note the following facts about a WCO $W_{\pi,T}$ on $CV_0(X,E)$.

If V consists of weights vanishing at infinity, then for each $t \in E$, the constant t-function 1_t belongs to $CV_0(X,E)$ (cf. Proposition A), and so $W_{\pi,T}1_t(x) = \pi(x)t$ for all $x \in X$. Further, if $\{x_\alpha\}$ is a net in X converging to some point $x_0 \in X$, then we have

$$p[\pi(x_\alpha)t - \pi(x_0)t] = p[W_{\pi,T}1_t(x_\alpha) - W_{\pi,T}1_t(x_0)] \to 0$$

for all $t \in E$ and $p \in cs(E)$. This implies that $\pi \in C(X, B_s(E))$. (Note that π is not necessarily continuous in the uniform operator toplogy. This fact is also noted by jamison and Rajagopalan [2, page 311] for the Banach space C(X,E)). This continuity of π shows that the set $N(\pi) = \{x \in X : \pi(x) \neq 0\}$ is open in X. Also, we shall see (in Theorem 1 below) that T is continuous on $N(\pi)$. But it is not necessarily continuous on $X \backslash N(\pi)$, because $W_{\pi,T}f$ is zero on $X \backslash N(\pi)$ for every $f \in CV_0(X,E)$ even if T is anyhow defined.

Theorem 1: Let $\pi : X \to B(E), T : X \to X$, and assuce that X is also a $k_{\mathbb{R}}$-space. Then the following conditins are sufficient for the pair (π, T) to induce a WCO on $CV_0(X,E)$:
(1.1) $\pi \in C(X, B_s(E))$;
(1.2) T is continuous on $N(\pi)$;

(1.3) for every $v \in V$ and $p \in cs(E)$, there exists $u \in V$ and $q \in cs(E)$ such that

$$v(x)p(\pi(x)t) \leq u(T(x))q(t) \text{ for all } x \in X \text{ and } t \in E; \qquad \text{and}$$

(1.4) for every $v \in V, p \in cs(E), \epsilon > 0$ and compact subset K of X, the set

$$T^{-1}(K) \cap \{x \in X : v(x)p(\pi(x)t) \geq \epsilon\} \text{ is compact for all nonzero t in E .}$$

Further more, the above conditions (1.1) - (1.4) are necessary if V consists of all weights on X which vanish at infinity.

Proof: In view of Theorem B and the above facts, it is only required to prove that the condition (1.2) is necessary. For this purpose, we suppose on the contrary that there exists an x_0 in $N(\pi)$ at which T is not continuous. Then there exists a net $\{x_a\}_{a\in A}$ in $N(\pi)$ such that $x_\alpha \to x_0$ but $T(x_\alpha)$ does not converge to $T(x_0)$. This implies that there exists an open neighbourhood G of $T(x_0$ and a subnet $\{x_\beta\}$ of $\{x_\alpha\}$ such that $T(x_\beta) \notin G$. By Nachbin Lemma [6,page 69], there exists $f \in CV_0(X)$ such that $0 \leq f \leq 1, f(T(x_0)) = 1$ and $f(X\backslash G) = \{0\}$. Let t be a nonzero vector in E such that $\pi(x_0)t \neq 0$ and choose $q \in cs(E)$ such that $q(\pi(x_0)t) = 1$. Define the function f_t by setting $f_t(x) = f(x)t$ for all $x \in X$. Then for every $p \in cs(E)$ and $\epsilon > 0$, the set $\{x \in X : v(x)p(f_t(x)) \geq \epsilon\}$ is compact because it is empty if p(t) = 0 and it is equal to $\{x \in X : v(x)|f(x)| \geq \epsilon/p(t)\}$ if $p(t) \neq 0$. Therefore, $f_t \in CV_0(X, E)$ with $f_t(T(x_0)) = t$ and $f_t(X\backslash G) = 0$. But for above $q \in cs(E)$, we have

$$q[W_{\pi,T}f_t(x_\beta) - W_{\pi,T}f_t(x_0)] = q[\pi(x_0)t] = 1,$$

a contradiction. Thus T must be continuous on $N(\pi)$.

Remark: It should be noted that in Theorem 1 above the $k_{\mathbb{R}}$-property on X is needed only to make the multiplication transformation M_π into (cf. Singh and Manhas [11]) and the assumption on V that weights vanish at infinity is needed only to make the condition (1.1) necessary. In case $\pi(x) = I$, the identity operator on X, for all $x \in X$, these assumptions are not needed (see, for example Theorem 4.2.28 of Singh and Manhas [9]).

As particular cases of Theorem 1, we obtain Theorem 2.3 of Singh and Summer [14], and a corrected version (cf. [11]) of Theorem 2.1 of Singh and Manhas [8].

Compact WCOs on (Banach) spaces of continuous functions have been the subject matter of several papers, see for example Kamowitz [3], Jamison and Rajagopalan [2], Chan [1], Takagi [15] and Lindstrom and Llavona [4]. See also [13] for details. In a recent work [5], Manhas and Singh have characterized compact WCOs on $CV_0(X, E)$ where V consists of weights vanishing at infinity. It contains the unweighted case due to Singh, Manhas and one of the author [12] as well as a result of [4].

In the following theorem, we give an improvement of [13, Theorem 5.18] which is proved by Manhas and Singh in [5].

Theorem 2: Let V be a system of weights on X satisfying (A.1) of Proposition A, and assume that X is a $k_{\mathbb{R}}$-space and E is a quasicomplete space. Let $\pi : X \to B(E)$ and $T : X \to X$ such that $W_{\pi,T}$ is a WCO on $CV_0(X, E)$. Then $W_{\pi,T}$ is compact if and only if the following conditions hold:
(2.1) $\pi \in C(X, B_b(E))$;
(2.2) each $\pi(x)$ is a compact operator on E;
(2.3) T is locally constant on $N(\pi)$.
Proof : In view of [5, Theorem 2.1] or [13, Theorem 5.18], it is only required to show that the condition (2.1) is necessary. For this, suppose that the map $\pi : X \to B_b(E)$ is not continuous at some point x_0 in X. Then there exists a net $\{x_\alpha\}$ in X such that $x_\alpha \to x_0$ but $\pi(x_\alpha)$ does not converge to $\pi(x_0)$ in $B_b(E)$. So we can find a seminorm $q \in cs(E)$, a nonzero vector t in E and a $\delta > 0$ such that

$$q[\pi(x_\alpha)t - \pi(x_0)t] > \delta.$$

By our assumption on weights, the constant t - function 1_t belongs to $CV_0(X, E)$ (cf. Proposition A). But

$$q[W_{\pi,T}1_t(x_\alpha) - W_{\pi,T}1_t(x_0)] = q[\pi(x_\alpha)t - \pi(x_0)t] > \delta,$$

which is a contradiction.

3. SPECTRA OF A COMPACT WCO:

The spectrum of a compact WCO on the Banach algebra C(X) has been studied by Kamowitz [3]. His result has been generalized to the vector-valued Banach space C(X,E) by Jamison and Rajagopalan in [2].

In the next theorem, we present it in the setting of $CV_0(X, E)$ where E is a Banach space with the norm $||.||$.

For a selfmap T on X, let T_n denote the nth iterate of T, that is, $T_0(x) = x$ and $T_n(x) = T(T_{n-1}(x))$ for all $x \in X$ and $n \geq 1$. A point $y \in X$ is called a fixed point of map T of order n if n is a positive integer, $T_n(y) = y$ and $T_k(y) \neq y$ for k = 1,2,..., n - 1.

Theorem 3 : Under the hypothesis of Theorem 2, let $W_{\pi,T}$ be a compact weighted composition operator on $CV_0(X,E)$, where E is a Banach space, and $\pi(X)$ is a bounded subset of B(E). Let

$$H = \{\alpha : \alpha^n = \pi(y)\pi(T(y))...\pi(T_{n-1}(y))$$

for some fixed point y of T or order n $\}$.
Then $\sigma(W_{\pi,T})\backslash\{0\} = H$.

We prove the theorem by adoping the method due to Kamowitz [3].

Proposition 4 : Let V be the system of all weights vanishing at infinity and let $W_{\pi,T}$ be a WCO on $CV_0(X,E)$. Suppose n is a positive integer and y is a fixed point of T of order n. Then $\alpha \in H$ implies that $\alpha \in \sigma(W_{\pi,T})$.
Proof : Suppose α satisfies $\alpha^n = \pi(y)\pi(T(y))...\pi(T_{n-1}(y))$. If $\alpha = 0$, we have $\pi(T_k(y)) = 0$ for some positive integer $k \leq n-1$. So $W_{\pi,T}$ is not onto because, for every non-zero t in E, $l_t \in CV_0(X,E)$ but there is no $f \in CV_0(X,E)$ with $W_{\pi,T}1_t = f$. Hence $\alpha = 0 \in \sigma(W_{\pi,T})$.

If $\alpha \neq 0$, then we have $\pi(T_k(y)) \neq 0$ for all k. If n = 1, we have $\alpha = \pi(y)$ and T(y) = y. According to Nachbin Lemma [6, page 69], we choose a function $g \in CV_0(X)$ such that g(y) = 1. For $0 \neq t \in E$, consider the function $g_t \in CV_0(X,E)$ which is defined as $g_t(x) = g(x)t$ for all $x \in X$. But there exists no $f \in CV_0(X,E)$ with $(\alpha - W_{\pi,T})f = g_t$. Hence $\alpha \in \sigma(W_{\pi,T})$.
For $n \geq 2$, let $h = g_t \in CV_0(X,E)$, where $0 \neq t \in E$, such that

$$(*) \qquad \alpha^{n-1}h(y) + \sum_{k=1}^{n-1} \alpha^{n-k-1}\pi(y)...\pi(T_{k-1}(y))h(T_k(y)) = t \neq 0.$$

Such a function h can be choosen according to Nachbin Lemma and the fact that all the coefficients of $g(T_j(y))$ in (*) are nonzero.
But if $f \in CV_0(X,E)$ such that $(\alpha - W_{\pi,T})f = h$, then by induction

$$(**)\alpha^n f(x) - \pi(x)...\pi(T_{n-1}(x))f(T_n(x)) = \alpha^{n-1}h(x) + \sum_{k=1}^{n-1} \alpha^{n-k-1}\pi(y)...\pi(T_{k-1}(x))h(T_k(x)).$$

for all $x \in X$. Since $T_n(y) = y$, evaluating (**) at y gives

$$\begin{aligned}[\alpha^n - \pi(y)...\pi(T_{n-1}(y))]f(y) &= \alpha^{n-1}h(y) + \sum_{k=1}^{n-1} \alpha^{n-k-1}\pi(y)...\pi(T_{k-1}(y))h(T_k(y)) \\ &= t \neq 0,\end{aligned}$$

a contradiction since $\alpha^n = \pi(y)...\pi(T_{n-1}(y))$. Therefore $\alpha \in \sigma(W_{\pi,T})$.

Remark : If $W_{\pi,T}$ is a compact WCO on $CV_0(X,E)$, Then each nonzero α in $\sigma(W_{\pi,T})$ in the preceding proposition is an eigen value.

In the next Proposition we will show that if $W_{\pi,T}$ is compact, then $\sigma(W_{\pi,T})-\{0\}$ consists only of the nonzero $\alpha's$ in Proposition 4.

Proposition 5 : Undr the hypothesis of Theorem 2, suppose $W_{\pi,T}$ is a compact WCO on $CV_0(X,E)$, where $\pi(X)$ is a bounded subset of $B(E)$. If $\alpha \neq 0$ and $\alpha \notin H$, then α is not an eigen value of $W_{\pi,T}$.

Proof : Let $\alpha \neq 0$, and for each positive integer n, let $\alpha^n \neq \pi(y)...\pi(T_{n-1}(y))$, where y is a fixed point of T of order n. Suppose $W_{\pi,T}f = \alpha f$. We show that f = 0. First we show that f(y) = 0.
Since $W_{\pi,T}f = \alpha f$, we have

$$\begin{aligned} \alpha^n f(x) &= \alpha^{n-1} W_{\pi,T} f(x) = \alpha^{n-1}\pi(x) f(T(x)) \\ &= \alpha^{n-2}\pi(x)\pi(T(x)) f(T_2(x)) = ... \\ &= \pi(x)\pi(T(x))...\pi(T_{n-1}(x)) f(T_n(x)) \end{aligned}$$

for all $x \in X$. In particular, for x = y, this equation gives

$$\alpha^n f(y) = \pi(y)...\pi(T_{n-1}(y)) f(y)$$

because $T_n(y) = y$. Since $\alpha^n \neq \pi(y)...\pi(T_{n-1}(y))$, it follows that f(y) = 0.

Next let $z \in X$ and set $Z = \{z, T(z), T_2(z), ...\}$. If Z is finite, then for some positive integers m and n with $m < n$, say we have $T_n(z) = T_m(z)$. This implies that $T_{n-m}(T_m(z)) = T_m(z)$, that is, $T_m(z)$ is a fixed point of T_{n-m}. As seen above, we have $f(T_m(z)) = 0$. Now

$$\pi(z)...\pi(T_{m-1}(z)) f(T_m(z)) = \alpha^m f(z)$$

implies f(z) = 0 since $\alpha \neq 0$.

We now consider the remaining case that Z is infinite. Let us assume that $\{k : ||\pi(T_k(z))|| \geq \epsilon\}$ is infinite for some $\epsilon > 0$. Then there exists a sequence $\{T_k(z)\}$ satisfying $||\pi(T_k(z))|| \geq \epsilon$ and a limit point z' of $\{T_k(z)\}$ with $||\pi(z')|| \geq \epsilon$ (since $\pi \in C(X, B_b(E))$). This implies that for a given neighbourhood U of z', there existrs a K such that $T_k(z) \in U$ for all $k \geq K$. Since T is constant on U, we have $T(T_k(z)) = T(z')$ for all $k \geq K$, contradicting the assumption that Z is infinite. Hence for any $\epsilon > 0$, there exists a K such that $||\pi(T_k(z))|| < \epsilon$ for all $k \geq K$. Now each $k \geq K$, we have

$$\begin{aligned} ||\alpha^k f(z)|| &= ||\pi(z)...\pi(T_{K-1}(z))\pi(T_K(z))...\pi(T_{k-1}(z)) f(T_k(z))|| \\ &< ||pi(z)||...||(T_{K-1}(z))||.\epsilon^{k-K}||f(T_k(z))||. \\ &\geq \beta^{K-1}.\epsilon^{k-K}\frac{1}{v(T_k(z))}.||f||_v, \end{aligned}$$

where $v(T_k(z)) > 0$ and $\beta = max.\{||\pi(T_j(z))|| : 1 \leq j \leq K-1\}$. So

$$||f(z)|| < \frac{\beta^{K-1}.||f||_v}{\epsilon^K v(T_k(z))}.(\frac{\epsilon}{|\alpha|})^k.$$

Take $\epsilon = \frac{|\alpha|}{2}$ and let $k \to \infty$, this inequality yields f(z) = 0.

The proof of Theorem 3 follows from Proposition 4 and 5 since every nonzero element in the spectum of a compact operator is an eigen value.

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