Weak and Strong Convergence Theorem of Iterative Scheme for Generalized Equilibrium Problems and Fixed Point Problems of Asymptotically Strict Pseudo-Contraction Mappings

S. Sanhan\textsuperscript{a}, I. Inchan\textsuperscript{b,c} and W. Sanhan \textsuperscript{a,c}\textsuperscript{1}

\textsuperscript{a}Department of Mathematics, Faculty of Liberal Arts and Science, Kasetsart University Kamphaeng Saen Campus, Nakhonpathom 73140, Thailand

\textsuperscript{b}Department of Mathematics and Computer, Faculty of Science and Technology Uttaradit Rajabhat University, 53000, Thailand

\textsuperscript{c}Centre of Excellence in Mathematics, CHE, Si Ayutthaya Road, Bangkok 10400, Thailand

Abstract

In this paper, we introduce an iterative scheme for finding a common element of the set of fixed points of asymptotically k-strict pseudo-contractive mappings and the set of solution of the generalized equilibrium problems in a Hilbert space. Then, we prove weak and strong convergence theorems of the sequences generated by our proposed scheme. Our results extended and improve the results of Ceng, Al-Homidan, Ansari and Yao, [An iterative scheme for equilibrium problems and fixed point problems of strict pseudo-contraction mappings, J. Computational and Applied Mathematics, 223(2009) 967-974] and many other.

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\textsuperscript{1}Corresponding author;
Email addresses: winate_s@yahoo.com (W. Sanhan)
1 Introduction

Let $H$ be a real Hilbert space and let $C$ be a nonempty closed convex subset of $H$. A mapping $S$ of $C$ into itself is nonexpansive if $\|Sx - Sy\| \leq \|x - y\|, \forall x, y \in C$. The set of fixed points of $S$ is denoted by $F(S)$. Let $F$ be a bifunction of $C \times C$ into $\mathbb{R}$, where $\mathbb{R}$ is the real numbers. The equilibrium problem for $F : C \times C \to \mathbb{R}$ is to find $x \in C$ such that

$$F(x, y) \geq 0 \text{ for all } y \in C.$$  \hspace{1cm} (1)

The set of solutions of (1) is denoted by $EP(F)$. Numerous problems in physics, optimization, and economics reduce to find a solution of (1). In 1997, Combettes and Hirstoaga [4] introduced an iterative scheme of finding the best approximation to the initial data when $EP(F)$ is nonempty and proved a strong convergence theorem.

Let $A : C \to H$ be a mapping. The classical variational inequality, denoted by $VI(C, A)$, is to find $x^* \in C$ such that

$$\langle Ax^*, v - x^* \rangle \geq 0$$

for all $v \in C$. The variational inequality has been extensively studied in the literature. See, e.g. [8] and the references therein. Let $B : C \to H$ be a nonlinear mapping. Then, we consider the following equilibrium problem: Find $z \in C$ such that

$$F(z, y) + \langle Bz, y - z \rangle \geq 0, \forall y \in C$$  \hspace{1cm} (2)

The set of such $z \in C$ is denoted by $EP$, i.e.,

$$EP = \{z \in C : F(z, y) + \langle Bz, y - z \rangle \geq 0, \forall y \in C\}.$$ 

In the case of $B \equiv 0$, $EP$ is denoted by $EP(F)$. In the case of $F \equiv 0$, $EP$ is also denoted by $VI(C, A)$. A mapping $A$ of $C$ into $H$ is called $\alpha$-inverse-strongly monotone [2] if there exists a positive real number $\alpha$ such that

$$\langle Au - Av, u - v \rangle \geq \alpha \|Au - Av\|^2$$

for all $u, v \in C$.

We know that a Hilbert space $H$ satisfies Opial’s condition [12], that is, for any sequence $\{x_n\} \subset H$ with $x_n \rightharpoonup x$, the inequality

$$\liminf_{n \to \infty} \|x_n - x\| < \liminf_{n \to \infty} \|x_n - y\|$$

holds for every $y \in H$ with $y \neq x$. 

Recall that a mapping $T : C \rightarrow C$ is said to be a strict pseudo-contractive mapping \cite{2} if there exists a constant $0 \leq k < 1$ such that
\begin{equation}
\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(I - T)x - (I - T)y\|^2,
\end{equation}
for all $x, y \in C$. (If (1) holds, we also say that $T$ is a $k$-strict pseudo-contraction.)

It is known that if $T$ is $0$-strict pseudo-contractive mapping, $T$ is nonexpansive mapping.

In this paper we will consider an iteration method of modified Mann for asymptotically $k$-strict pseudo-contractive mapping. We say that $T : C \rightarrow C$ is an asymptotically $k$-strict pseudo-contractive mapping if there exists a constant $0 \leq k < 1$ satisfying
\begin{equation}
\|T^nx - T^ny\|^2 \leq k_n\|x - y\|^2 + k\|(I - T^nx)x - (I - T^ny)y\|^2,
\end{equation}
for all $x, y \in C$ and for all $n \in \mathbb{N}$ where $k_n \geq 1$ for all $n$ such that $\lim_{n \to \infty} k_n = 1$. We see that the class of $k$-strict pseudo-contractive mappings is an asymptotically $k$-strict pseudo-contractive mapping if $k_n = 1$ for all $n \in \mathbb{N}$. Moreover, if $k = 0$, then $T$ is asymptotically nonexpansive mapping, i.e. there exists a sequence $k_n \geq 1$ for all $n$ such that $\lim_{n \to \infty} k_n = 1$ such that
\begin{equation}
\|T^n x - T^n y\| \leq k_n\|x - y\|,
\end{equation}
for all $x, y \in C$ and $n \geq 1$.

If $T$ is a nonexpansive self-mapping of $C$, then Mann’s algorithm generates, initializing with an arbitrary $x_1 \in C$, a sequence according to the recursive manner
\begin{equation}
x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n, \quad \forall n \geq 1,
\end{equation}
where $\{\alpha_n\}_{n=1}^{\infty}$ is a real sequence in the interval $(0, 1)$. Then the sequence $\{x_n\}$ converges weakly to a fixed point of $T$.

Very recently, Ceng, Homidan, Ansari and Yao \cite{3} introduced the sequence generated by an arbitrary element $x_1 \in H$ and follows by
\begin{equation}
\begin{cases}
F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n \rangle \geq 0, \forall y \in C \\
x_{n+1} = \alpha_n u_n + (1 - \alpha_n)S u_n, \quad \forall n \geq 1
\end{cases}
\end{equation}
where $S$ is $k$-strict pseudo-contractive mapping. Then under controls conditions they prove $\{x_n\}$ and $\{u_n\}$ weak and strong convergent to element in $F(S) \cap EP(F)$. 

\textbf{Weak and strong convergence theorem of iterative scheme 1979}
Motivated and inspired by the results of Ceng, Homidan, Ansari and Yao [3], in this paper we improve the iterative scheme (7) to a mapping $S$ is asymptotically $k$-strict pseudo-contractive mapping and for generalized equilibrium problems (2), then we prove the sequence converges weakly and strongly to common element of $F(S) \cap EP$.

2 Preliminary

Let $H$ be a real Hilbert space with norm $\| \cdot \|$ and inner product $\langle \cdot, \cdot \rangle$ and let $C$ be a closed convex subset of $H$. For every point $x \in H$, there exists a unique nearest point in $C$, denote by $P_Cx$, such that

$$\| x - P_Cx \| \leq \| x - y \|, \text{ for all } y \in C.$$ $P_C$ is called the metric projection of $H$ onto $C$. It is well known that $P_C$ is a nonexpansive mapping of $H$ onto $C$ and satisfied

$$\langle x - y, P_Cx - P_Cy \rangle \geq \| P_Cx - P_Cy \|^2$$

for every $x, y \in H$. Moreover, $P_Cx$ is characterized by the following properties:

$$\| x - y \|^2 \geq \| x - P_Cx \|^2 + \| y - P_Cx \|^2$$

for all $x \in H$, $y \in C$.

We collect some lemmas which will be used in the proof for the main result.

**Lemma 2.1.** [5] For a real Hilbert space $H$, the following identities hold:

(i) $\| x + y \|^2 = \| x \|^2 + \| y \|^2 + 2 \langle x, y \rangle$, $\forall x, y \in H$;

(ii) $\| x - y \|^2 = \| x \|^2 - \| y \|^2 - 2 \langle x, y \rangle$, $\forall x, y \in H$;

(iii) $\| tx + (1-t)y \|^2 = t\| x \|^2 + (1-t)\| y \|^2 - t(1-t)\| x - y \|^2$ for all $x, y \in H$ and $t \in [0, 1]$;

(iv) If $\{ x_n \}$ is a sequence in $H$ weakly convergent to $z$, then

$$\limsup_{n \to \infty} \| x_n - y \|^2 = \limsup_{n \to \infty} \| x_n - z \|^2 + \| z - y \|^2, \forall y \in H.$$

**Lemma 2.2.** [10] Let $T$ be an asymptotically $k$-strictly pseudo-contractive mapping defined on a bounded closed convex subset $C$ of a Hilbert space $H$. Assume that $\{ x_n \}$ is a sequence in $C$ with the properties
(i) \( x_n \rightharpoonup z \) and 
(ii) \( Tx_n - x_n \to 0 \).

Then \( (I - T)z = 0 \).

**Lemma 2.3.** \([5]\) Let \( C \) be a closed convex subset of a real Hilbert space \( H \). Given \( x \in H \) and \( z \in C \). Then \( z = P_C x \) if and only if there holds the inequality 
\[
\langle x - z, y - z \rangle \leq 0, \quad \forall y \in C.
\]

**Lemma 2.4.** \([11]\) Let \( \{r_n\}, \{s_n\} \) and \( \{t_n\} \) be a three nonnegative sequences satisfying the following condition:
\[
r_{n+1} \leq (1 + s_n)r_n + t_n, \quad \forall n \in \mathbb{N}.
\]

If \( \sum_{n=1}^{\infty} s_n < \infty \) and \( \sum_{n=1}^{\infty} t_n < \infty \), then the \( \lim_{n \to \infty} r_n \) exists.

**Lemma 2.5.** \([10]\) Assume that \( C \) is a closed convex subset of a Hilbert space \( H \) and let \( T : C \to C \) be an asymptotically \( k \)-strictly pseudo-contraction. Then the following hold:

(i) For each \( n \geq 1 \), \( T^n \) satisfies the Lipschitz condition:
\[
\|T^n x - T^n y\| \leq L_n \|x - y\|
\]
for all \( x, y \in C \), where \( L_n = \frac{k + \sqrt{1 + \gamma_n (1 - k)}}{1 - k} \).

(ii) The demiclosedness principle holds for \( I - T \) in the sense that if \( \{x_n\} \) is a sequence in \( C \) such that \( x_n \rightharpoonup x \) and \( (I - T)x_n \to 0 \), then \( (I - T)x = 0 \).

(iii) The fixed point set \( F(T) \) is closed and convex so that the projection \( P_{F(T)} \) is well defined.

For solving the equilibrium problem for a bifunction \( F : C \times C \to \mathbb{R} \), let us assume that \( F \) satisfies the following condition:

(A1) \( F(x, x) = 0 \) for all \( x \in C \);

(A2) \( F \) is monotone, i.e., \( F(x, y) + F(y, x) \leq 0 \) for all \( x, y \in C \);

(A3) for each \( x, y \in C \),
\[
\lim_{t \to 0} F(tz + (1 - t)x, y) \leq F(x, y);
\]

(A4) for each \( x \in C \), \( y \mapsto F(x, y) \) is convex and lower semicontinuous.

The following lemma appears implicitly in \([1]\).
Lemma 2.6. \cite{1, 6} Let $C$ be a nonempty closed convex subset of $H$ and let $F$ be a bifunction of $C \times C$ into $\mathbb{R}$ satisfying (A1)-(A4). Let $r > 0$ and $x \in H$. Then, there exists $z \in C$ such that
\[
F(z, y) + \frac{1}{r} \langle y - z, z - x \rangle \geq 0 \quad \text{for all } y \in C.
\]

The following lemma was also given in \cite{1}.

Lemma 2.7. \cite{1, 6, 7} Assume that $F : C \times C \to \mathbb{R}$ satisfies (A1)-(A4), and let $r > 0$ and $x \in H$. Then, there exists unique $z \in C$ such that
\[
F(z, y) + \frac{1}{r} \langle y - z, z - x \rangle \geq 0 \quad \text{for all } y \in C.
\]

Further, if $T_r x = \{ z \in C : F(z, y) + \frac{1}{r} \langle y - z, z - x \rangle \geq 0, \forall y \in C \}$, then the following hold:

1. $T_r$ is single-valued;
2. $T_r$ is firmly nonexpansive, i.e., $\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y \rangle$, for any $x, y \in H$;
3. $F(T_r) = EP(F)$;
4. $EP(F)$ is closed and convex;

3 Weak convergence theorems

In this section, we prove a strong convergence theorem of the hybrid parallel method for a family of finitely strictly pseudo-contractive mappings in a real Hilbert space.

Theorem 3.1. Let $C$ be a nonempty closed convex subset of a real Hilbert space $H$. Let $F : C \times C \to \mathbb{R}$ be a bifunction satisfying (A1)-(A4). Let $B$ be an $\beta$-inverse strongly monotone mapping of $C$ into $H$ and let $S : C \to C$ be an asymptotically $k$-strictly pseudo-contractive self mapping for some $0 \leq k < 1$ such that $\sum_{n=1}^{\infty} (k_n - 1) < \infty$ and $F(S) \cap EP \neq \emptyset$. Let $\{x_n\}$ and $\{u_n\}$ be sequences generated initially by an arbitrary element $x_1 \in H$ and then by
\[
\begin{align*}
F(u_n, y) + \langle Bx_n, y - u_n \rangle + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0, \forall y \in C \\
x_{n+1} = \alpha_n u_n + (1 - \alpha_n) S^n u_n, \quad \forall n \geq 1
\end{align*}
\]
where \( \{\alpha_n\} \) and \( \{r_n\} \) satisfy the following conditions:

(i) \( \{\alpha_n\} \subset [\alpha, \gamma] \) for some \( \alpha, \gamma \in (k, 1) \);

(ii) \( \{r_n\} \subset [0, 2\beta] \) and \( \lim \inf r_n > 0 \).

Then, the sequence \( \{x_n\} \) and \( \{u_n\} \) converges weakly to an element of \( F(S) \cap EP \).

**Proof.** We show that \( \lim_{n \to \infty} \|x_n - p\| \) exists for each \( p \in F(S) \cap EP \). Let \( p \in F(S) \cap EP \). From the definition of \( T_r \), we have \( u_n = T_{r_n}(x_n - r_nBx_n) \). Since \( 0 \leq r_n \leq 2\beta \), we have

\[
\|u_n - p\|^2 = \|T_{r_n}(x_n - r_nBx_n) - T_{r_n}(p - r_nBp)\|^2 \\
\leq \|(x_n - r_nBx_n) - (p - r_nBp)\|^2 \\
\leq \|(x_n - p) - r_n(Bx_n - Bp)\|^2 \\
\leq \|x_n - p\|^2 - 2r_n\langle x_n - p, Bx_n - Bp \rangle + r_n^2\|Bp - Bx_n\|^2 \\
\leq \|x_n - p\|^2 - 2r_n\beta\|Bx_n - Bp\|^2 + r_n^2\|Bp - Bx_n\|^2 \\
\leq \|x_n - p\|^2. \tag{13}
\]

Since \( S \) is asymptotically \( k \)-strictly pseudo-contractive, we have \( \{\alpha_n\} \in [\alpha, \beta], \alpha, \beta \in (k, 1) \) and \( k \leq \alpha_n \), we have

\[
\|x_{n+1} - p\|^2 = \|\alpha_n u_n + (1 - \alpha_n)S^\alpha u_n - p\|^2 \\
= \|\alpha_n (u_n - p) + (1 - \alpha_n)(S^\alpha u_n - p)\|^2 \\
= \alpha_n\|u_n - p\|^2 + (1 - \alpha_n)\|S^\alpha u_n - p\|^2 - \alpha_n(1 - \alpha_n)\|u_n - S^\alpha u_n\|^2 \\
\leq \alpha_n\|u_n - p\|^2 + (1 - \alpha_n)\left[k_n\|u_n - p\|^2 + k\|u_n - S^\alpha u_n\|^2\right] \\
- \alpha_n(1 - \alpha_n)\|u_n - S^\alpha u_n\|^2 \\
\leq k_n\|u_n - p\|^2 - (1 - \alpha_n)(\alpha_n - k)\|u_n - S^\alpha u_n\|^2 \tag{14} \\
\leq k_n\|x_n - p\|^2 - (1 - \alpha_n)(\alpha_n - k)\|u_n - S^\alpha u_n\|^2 \tag{15} \\
\leq (1 + (k_n - 1))\|x_n - p\|^2. \tag{16}
\]

It follows from Lemma 2.4 that \( \lim_{n \to \infty} \|x_n - p\| \) exists and hence \( \{x_n\} \) is bounded and we also obtain that \( \{u_n\} \) is bounded. Also, from (14) and \( \lim_{n \to \infty} k_n = 1 \) it follows that

\[
(1 - \gamma)(\alpha - k)\|u_n - S^\alpha u_n\|^2 \leq (1 - \alpha_n)(\alpha_n - k)\|u_n - S^\alpha u_n\|^2 \\
\leq k_n\|x_n - p\|^2 - \|x_{n+1} - p\|^2. \tag{17}
\]

This implies that

\[
\lim_{n \to \infty} \|u_n - S^\alpha u_n\| = 0. \tag{18}
\]
From (14) and (12), we have
\[
\|x_{n+1} - p\|^2 \leq k_n \|u_n - p\|^2 - (1 - \alpha_n)(\alpha_n - k)\|u_n - S^n u_n\|^2
\]
\[
\leq k_n \left[ \|x_n - p\|^2 - r_n(2\beta - r_n)\|Bx_n - Bp\|^2 \right].
\]
It follows that
\[
r_n(2\beta - r_n)\|Bx_n - Bp\|^2 \leq \|x_n - p\|^2 - \frac{1}{k_n} \|x_{n+1} - p\|^2.
\]
So, from the existence of \( \lim_{n \to \infty} \|x_n - p\| \), \( \lim_{n \to \infty} k_n = 1 \) and \( \lim \inf r_n > 0 \), we have
\[
\lim_{n \to \infty} \|Bx_n - Bp\| = 0. \tag{19}
\]
From \( T_n \) is firmly nonexpansive and by using Lemma 2.7, we have
\[
\|u_n - p\|^2 = \|T_n(x_n - r_n Bx_n) - T_n(p - r_n Bp)\|^2
\]
\[
\leq \langle (x_n - r_n Bx_n) - (p - r_n Bp), u_n - p \rangle
\]
\[
= \frac{1}{2}(\|x_n - r_n Bx_n\|^2 + \|u_n - p\|^2 - \|x_n - r_n Bx_n\|^2 - \|u_n - p\|^2)
\]
\[
\leq \frac{1}{2}(\|x_n - p\|^2 + \|u_n - p\|^2 - \|x_n - u_n\|^2 - \|r_n(Bx_n - Bp)\|^2)
\]
\[
= \frac{1}{2}(\|x_n - p\|^2 + \|u_n - p\|^2 - \|x_n - u_n\|^2 + 2r_n\langle x_n - u_n, Bx_n - Bp \rangle
\]
\[
- r_n^2\|Bx_n - Bp\|^2).
\]
Thus, we obtain
\[
\|u_n - p\|^2 \leq \|x_n - p\|^2 - \|x_n - u_n\|^2 + 2r_n\langle x_n - u_n, Bx_n - Bp \rangle - r_n^2\|Bx_n - Bp\|^2. \tag{20}
\]
From (14) and (20), we have
\[
\|x_{n+1} - p\|^2 \leq k_n \left[ \|x_n - p\|^2 - \|x_n - u_n\|^2 + 2r_n\langle x_n - u_n, Bx_n - Bp \rangle - r_n^2\|Bx_n - Bp\|^2 \right]
\]
\[- (1 - \alpha_n)(\alpha_n - k)\|u_n - S^n u_n\|^2.
\]
It follows that
\[
\|x_n - u_n\|^2 \leq \|x_n - p\|^2 - \frac{1}{k_n} \|x_{n+1} - p\|^2 + 2r_n\langle x_n - u_n, Bx_n - Bp \rangle
\]
\[- r_n^2\|Bx_n - Bp\|^2 - \frac{1}{k_n}(1 - \alpha_n)(\alpha_n - k)\|u_n - S^n u_n\|^2.
\]
By using \( \lim_{n \to \infty} \|x_n - p\| \) exists, (18), (19) and boundedness of \( \{x_n\} \) and \( \{u_n\} \), we have

\[
\lim_{n \to \infty} \|x_n - u_n\| = 0. \tag{21}
\]

Next, we show that \( \|x_{n+1} - x_n\| \to 0 \). In fact, from (21), we note that

\[
\|x_{n+1} - x_n\| \leq \|x_{n+1} - u_n\| + \|x_n - u_n\| = \|\alpha_n u_n + (1 - \alpha_n)S^n u_n - u_n\| + \|x_n - u_n\|
= (1 - \alpha_n)\|u_n - S^n u_n\| + \|x_n - u_n\|.
\]

From (18) and (21), we obtain

\[
\lim_{n \to \infty} \|x_{n+1} - x_n\| = 0. \tag{22}
\]

Next, we show that \( \lim_{n \to \infty} \|Su_n - u_n\| = 0 \). From Lipschitz condition on \( S \), we have

\[
\|Su_n - u_n\| \leq \|Su_n - S^{n+1} u_n\| + \|S^{n+1} u_n - S^{n+1} u_{n+1}\| + \|S^{n+1} u_{n+1} - u_{n+1}\|
+ \|u_{n+1} - u_n\|
\leq L_1 \|u_n - S^n u_n\| + L_{n+1} \|u_n - u_{n+1}\| + \|S^{n+1} u_{n+1} - u_{n+1}\|
\leq L_1 \|u_n - S^n u_n\| + (1 + L_{n+1}) \|u_{n+1} - u_n\| + \|S^{n+1} u_{n+1} - u_{n+1}\|
\leq L_1 \|u_n - S^n u_n\| + (1 + L_{n+1}) \|u_{n+1} - x_{n+1}\| + \|x_{n+1} - u_n\|
+ \|S^{n+1} u_{n+1} - u_{n+1}\|
\leq L_1 \|u_n - S^n u_n\| + (1 + L_{n+1}) \|u_{n+1} - x_{n+1}\|
+(1 - \alpha_n)\|u_n - S^n u_n\| + \|S^{n+1} u_{n+1} - u_{n+1}\|.
\]

From (18) and (21), we have

\[
\lim_{n \to \infty} \|Su_n - u_n\| = 0. \tag{23}
\]

Since \( \{u_{n_i}\} \) is bounded, there exists a subsequence \( \{u_{n_{i_j}}\} \) of \( \{u_{n_i}\} \) such that \( u_{n_{i_j}} \to w \). Without loss of generality, we can assume that \( u_{n_i} \to w \). Since \( C \) is closed and convex, \( w \in C \). Next, we show that \( w \in F(S) \cap EP \). It follows by (38) and (A2) that

\[
\langle Bx_n, y - u_n \rangle + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq F(y, u_n)
\]

and hence

\[
\langle Bx_n, y - u_{n_i} \rangle + \langle y - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}} \rangle \geq F(y, u_{n_i}) \tag{24}
\]
Put $y_t = ty + (1 - t)w$ for all $t \in (0, 1]$ and $y \in C$. Since $y \in C$ and $w \in C$, we have $y_t \in C$. So, from (21), we have

\[ \langle y_t - u_n, By_t \rangle - \langle y_t - u_n, B y_t \rangle = 0 \geq -\langle y_t - u_n, B x_n \rangle - \langle y - u_n, \frac{u_n - x_n}{r_n} \rangle + F(y_t, u_n) \]

and hence

\[ \langle y_t - u_n, By_t \rangle \geq \langle y_t - u_n, B y_t \rangle - \langle y_t - u_n, B x_n \rangle - \langle y - u_n, \frac{u_n - x_n}{r_n} \rangle + F(y_t, u_n) \]

\[ = \langle y_t - u_n, B y_t - B u_n \rangle + \langle y_t - u_n, B u_n - B x_n \rangle - \langle y - u_n, \frac{u_n - x_n}{r_n} \rangle + F(y_t, u_n). \]

Since $\|u_n - x_n\| \to 0$, it follows that $\|B u_n - B x_n\| \to 0$. Further, from monotonicity of $B$, we get

\[ \langle y_t - u_n, By_t - B u_n \rangle \geq 0. \]

So, from (A4), we have

\[ \langle y_t - w, By_t \rangle \geq F(y_t, w), \tag{25} \]

as $i \to \infty$. From (A1), (A4) and (21), we have

\[ 0 = F(y_t, y_t) \leq tF(y_t, y) + (1 - t)F(y_t, w) \leq tF(y_t, y) + (1 - t)\langle y_t - w, By_t \rangle \]

\[ \leq tF(y_t, y) + (1 - t)\langle y - w, By_t \rangle \]

and hence $0 \leq F(y_t, y) + (1 - t)\langle y - w, B y_t \rangle$. Letting $t \to 0$, we have for each $y \in C$, $0 \leq F(w, y) + \langle y - w, Bw \rangle$. This implies that $w \in EP$. Next, we show that $w \in F(S)$. Since $S$ is asymptotically $k$-strict contraction mapping, by Lemma 2.5 (ii), we know that the mapping $I - S$ is demiclosed at zero. Note that $\|u_n - S u_n\| \to 0$ and $u_{n_j} \to w$. Thus, $w \in F(S)$. Consequently, we deduce that $w \in F(S) \cap EP$. Since $w$ was an arbitrary element, we conclude that $\omega_w(x_n) \subset F(S) \cap EP$. We claim that $\{x_n\}$ and $\{u_n\}$ converge weakly to an element of $F(S) \cap EP$. We take $w_1, w_2 \in \omega_w(x_n)$ arbitrarily and let $\{x_{k_i}\}$ and $\{x_{m_j}\}$ be subsequences of $\{x_n\}$ such that $x_{k_i} \to w_1$ and $x_{m_j} \to w_2$, respectively. Since $\lim_{n\to\infty} \|x_n - p\|$ exists for each $p \in F(S) \cap EP$ and since
$w_1, w_2 \in F(S) \cap EP$, by Lemma 2.1 (iv), we obtain

\[
\lim_{n \to \infty} \|x_n - w_1\|^2 = \lim_{j \to \infty} \|x_{m_j} - w_1\|^2 = \lim_{j \to \infty} \|x_{m_j} - w_2\|^2 + \|w_2 - w_1\|^2 = \lim_{i \to \infty} \|x_k - w_2\|^2 + \|w_2 - w_1\|^2 = \lim_{i \to \infty} \|x_k - w_1\|^2 + 2\|w_2 - w_1\|^2 = \lim_{n \to \infty} \|x_n - w_1\|^2 + \|w_2 - w_1\|^2.
\]

Hence $w_1 = w_2$. This shows that $\omega_w(x_n)$ is a single-point set. This completes the proof. \( \Box \)

Consequence of the Theorem 3.1, we can obtain the corollaries.

**Corollary 3.2.** [3] Let $C$ be a nonempty closed convex subset of a real Hilbert space $H$. Let $F : C \times C \to \mathbb{R}$ be a bifunction satisfying (A1)-(A4). Let $S : C \to C$ be a $k$-strictly pseudo-contractive self mapping for some $0 \leq k < 1$ such that $F(S) \cap EP(F) \neq \emptyset$. Let $\{x_n\}$ and $\{u_n\}$ be sequences generated initially by an arbitrary element $x_1 \in H$ and then by

\[
F(u_n, y) + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0, \forall y \in C
\]

\[x_{n+1} = \alpha_n u_n + (1 - \alpha_n) Su_n, \quad \forall n \geq 1 \tag{26}\]

where $\{\alpha_n\}$ and $\{r_n\}$ satisfy the following conditions:

(i) $\{\alpha_n\} \subset [\alpha, \beta]$ for some $\alpha, \beta \in (k, 1)$;

(ii) $\{r_n\} \subset (0, \infty)$ and $\lim \inf r_n > 0$.

Then, the sequence $\{x_n\}$ and $\{u_n\}$ converge weakly to an element of $F(S) \cap EP(F)$.

**Proof.** Put $k_n = 1$ for all $n \in \mathbb{N}$ and $B \equiv 0$ in Theorem 3.1, we have $\{x_n\}$ generated by (26) converges weakly an element of $F(S) \cap EP(F)$. \( \Box \)

**Corollary 3.3.** Let $C$ be a nonempty closed convex subset of a real Hilbert space $H$. Let $F : C \times C \to \mathbb{R}$ be a bifunction satisfying (A1)-(A4). Let $B$ be an $\beta$-inverse strongly monotone mapping of $C$ into $H$ and let $S : C \to C$ be an asymptotically nonexpansive self mapping such that $F(S) \cap EP \neq \emptyset$. Let $\{x_n\}$ and $\{u_n\}$ be sequences generated initially by an arbitrary element $x_1 \in H$ and then by

\[
\begin{cases}
F(u_n, y) + \langle B x_n, y - u_n \rangle + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0, \forall y \in C \\
x_{n+1} = \alpha_n u_n + (1 - \alpha_n) S^n u_n, \quad \forall n \geq 1 
\end{cases} \tag{27}
\]
where \( \{\alpha_n\} \) and \( \{r_n\} \) satisfy the following conditions:

(i) \( \{\alpha_n\} \subset [\alpha, \beta] \) for some \( \alpha, \beta \in (0, 1) \);

(ii) \( \{r_n\} \subset (0, \infty) \) and \( \lim \inf r_n > 0 \).

Then, the sequence \( \{x_n\} \) and \( \{u_n\} \) converges weakly to an element of \( F(S) \cap EP \).

**Proof.** We know that asymptotically nonexpansive mapping is an asymptotically \( 0 \)-strictly pseudo-contractive mapping. \( \square \)

**Theorem 3.4.** Let \( C \) be a nonempty closed convex subset of a real Hilbert space \( H \). Let \( F : C \times C \to \mathbb{R} \) be a bifunction satisfying (A1)-(A4). Let \( B \) be an \( \beta \)-inverse strongly monotone mapping of \( C \) into \( H \) and let \( S : C \to C \) be an asymptotically \( k \)-strictly pseudo-contractive self mapping for some \( 0 \leq k < 1 \) such that \( \sum_n (k^n - 1) < \infty \) and \( F(S) \cap EP \neq \emptyset \). Let \( \{x_n\} \) and \( \{u_n\} \) be sequences generated initially by an arbitrary element \( x_1 \in H \) and then by

\[
\begin{align*}
F(u_n, y) + \langle Bx_n, y - u_n \rangle + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle & \geq 0, \quad \forall y \in C \\
x_{n+1} = \alpha_n u_n + (1 - \alpha_n)S^\alpha u_n, \quad \forall n \geq 1
\end{align*}
\]

where \( \{\alpha_n\} \) and \( \{r_n\} \) satisfy the following conditions:

(i) \( \{\alpha_n\} \subset [\alpha, \gamma] \) for some \( \alpha, \gamma \in (k, 1) \);

(ii) \( \{r_n\} \subset (0, \infty) \) and \( \lim \inf r_n > 0 \).

Then, the sequence \( \{x_n\} \) and \( \{u_n\} \) converges strongly to an element of \( F(S) \cap EP \) if and only if \( \lim \inf \frac{n}{\infty} d(x_n, F(S) \cap EP) = 0 \), where \( d(x_n, F(S) \cap EP) \) denote the metric distance from the point \( x_n \) to \( F(S) \cap EP \).

**Proof.** From the proof in Theorem 3.1, we know that \( \lim_{n \to \infty} \|x_n - p\| \) exists each \( p \in F(S) \cap EP \) and \( \lim_{n \to \infty} \|x_n - u_n\| = 0 \). Hence \( \{x_n\} \) is bounded. The necessity is apparent. We show the sufficiency. Suppose that

\[
\lim_{n \to \infty} d(x_n, F(S) \cap EP) = 0
\]

From (16) we have

\[
\|x_{n+1} - p\| \leq \|x_n - p\|.
\]

Taking the infimum over \( p \in F(S) \cap EP \), we have

\[
d(x_{n+1}, F(S) \cap EP) \leq d(x_n, F(S) \cap EP)
\]
and hence $\lim_{n \to \infty} d(x_n, F(S) \cap EP)$ exists. Thus, we have

$$\lim_{n \to \infty} d(x_n, F(S) \cap EP) = \lim_{n \to \infty} \inf d(x_n, F(S) \cap EP) = 0. \quad (31)$$

Now, it follows from (29), that for all $p \in F(S) \cdot EP$

$$\|x_{n+m} - p\| \leq \|x_{n+m} - p\| + \|x_n - p\| \leq 2\|x_n - p\|. \quad (32)$$

Taking the infimum over all $p \in F(S) \cap EP$, from (32) we obtain

$$\|x_{n+m} - p\| \leq 2d(x_n, F(S) \cap EP). \quad (33)$$

Thus $\{x_n\}$ is a cauchy sequence. Suppose $x_n \rightharpoonup w \in H$. Then

$$d(w, F(S) \cap EP) = \lim_{n \to \infty} d(x_n, F(S) \cap EP) = 0.$$  

Since $F(S) \cap EP$ is closed and convex, $w \in F(S) \cap EP$. Since $\lim_{n \to \infty} \|x_n - u_n\| = 0$, then we have both sequence $\{x_n\}$ and $\{u_n\}$ are converge strongly to an element $w$ of $F(S) \cap EP$. \hfill \Box

Consequence of the Theorem 3.1, we can obtain the corollaries.

**Corollary 3.5.** [3] Let $C$ be a nonempty closed convex subset of a real Hilbert space $H$. Let $F : C \times C \to \mathbb{R}$ be a bifunction satisfying (A1)-(A4). Let $S : C \to C$ be a $k$-strictly pseudo-contractive self mapping for some $0 \leq k < 1$ such that $F(S) \cap EP(F) \neq \emptyset$. Let $\{x_n\}$ and $\{u_n\}$ be sequences generated initially by an arbitrary element $x_1 \in H$ and then by

$$\begin{cases} 
F(u_n, y) + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0, \forall y \in C \\
x_{n+1} = \alpha_n u_n + (1 - \alpha_n) Su_n, \quad \forall n \geq 1 
\end{cases} \quad (34)$$

where $\{\alpha_n\}$ and $\{r_n\}$ satisfy the following conditions:

(i) $\{\alpha_n\} \subset [\alpha, \beta]$ for some $\alpha, \beta \in (k, 1)$;

(ii) $\{r_n\} \subset (0, \infty)$ and $\lim inf r_n > 0$.

Then, the sequence $\{x_n\}$ and $\{u_n\}$ converge strongly to an element of $F(S) \cap EP(F)$ if and only if

$$\lim_{n \to \infty} \inf d(x_n, F(S) \cap EP(F)) = 0. \quad \Box$$

## 4 Applications

Using the Theorem 3.1 we can applied the following Theorems.
Theorem 4.1. Let $C$ be a nonempty closed convex subset of $H$ $S : C \to C$ be a $k$-strictly pseudo-contraction mapping for some $0 \leq k < 1$ such that $F(S) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by an arbitrary $x_1 \in H$ and then by

$$x_{n+1} = \alpha_n P_Cx_n + (1 - \alpha_n)SPCx_n, \quad \forall n \geq 1,$$

where $\{\alpha_n\} \subset [\alpha, \beta]$ for some $\alpha, \beta \in (k, 1)$. Then, $\{x_n\}$ converges weakly to an element of $F(S)$.

Proof. Put $F(x, y) = 0$ for all $x, y \in C$, $B \equiv 0$ and $r_n = 1$ for all $n \in \mathbb{N}$ in Theorem 3.1. Then by Lemma 2.3, we have $u_n = P_Cx_n$. So, from Theorem 3.1, the sequence $\{x_n\}$ generated by (35) converges weakly to an element of $F(S)$.

Theorem 4.2. Let $C$ be a nonempty closed convex subset of a real Hilbert space $H$. Let $F : C \times C \to \mathbb{R}$ be a bifunction satisfying (A1)-(A4). Let $S : C \to C$ be a nonexpansive self mapping such that $F(S) \cap EP(F) \neq \emptyset$. Let $\{x_n\}$ and $\{u_n\}$ be sequences generated initially by an arbitrary element $x_1 \in H$ and then by

$$x_{n+1} = \alpha_n u_n + (1 - \alpha_n)Su_n, \quad \forall n \geq 1 \quad (36)$$

where $\{\alpha_n\}$ and $\{r_n\}$ satisfy the following conditions:

(i) $\{\alpha_n\} \subset [\alpha, \beta]$ for some $\alpha, \beta \in (0, 1)$;

(ii) $\{r_n\} \subset (0, \infty)$ and $\lim \inf r_n > 0$. Then, $\{x_n\}$ and $\{u_n\}$ converge weakly to an element of $F(S) \cap EP(F)$.

Proof. If $k_n = 1$ for all $n \in \mathbb{N}$ and $k = 0$ of a mapping $S$ in (4), we have $S$ is nonexpansive mapping. Then the conclusion follows immediately from Theorem 3.1. 

Using the Theorem 3.4 we can applied the following Theorems.

Theorem 4.3. Let $C$ be a nonempty closed convex subset of $H$ $S : C \to C$ be a $k$-strictly pseudo-contraction mapping for some $0 \leq k < 1$ such that $F(S) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by an arbitrary $x_1 \in H$ and then by

$$x_{n+1} = \alpha_n P_Cx_n + (1 - \alpha_n)SPCx_n, \quad \forall n \geq 1,$$

where $\{\alpha_n\} \subset [\alpha, \beta]$ for some $\alpha, \beta \in (k, 1)$. Then, $\{x_n\}$ converges strongly to an element of $F(S)$ if and only if $\lim \inf \limits_{n \to \infty} d(x_n, F(S)) = 0$. 

**Theorem 4.4.** Let $C$ be a nonempty closed convex subset of a real Hilbert space $H$. Let $F : C \times C \to \mathbb{R}$ be a bifunction satisfying (A1)-(A4). Let $S : C \to C$ be a nonexpansive self mapping such that $F(S) \cap EP(F) \neq \emptyset$. Let \{${x_n}$\} and \{${u_n}$\} be sequences generated initially by an arbitrary element $x_1 \in H$ and then by

$$
\begin{aligned}
F(u_n, y) + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle &\geq 0, \forall y \in C \\
x_{n+1} = \alpha_n u_n + (1 - \alpha_n) Su_n, \quad \forall n \geq 1 
\end{aligned}
$$

(38)

where \{${\alpha_n}$\} and \{${r_n}$\} satisfy the following conditions :

(i) \{${\alpha_n}$\} $\subset [\alpha, \beta]$ for some $\alpha, \beta \in (0, 1)$;

(ii) \{${r_n}$\} $\subset (0, \infty)$ and $\lim \inf r_n > 0$.

Then, the sequence \{${x_n}$\} and \{${u_n}$\} converges strongly to an element of $F(S) \cap EP(F)$ if and only if

$$
\lim \inf_{n \to \infty} d(x_n, F(S) \cap EP(F)) = 0.
$$

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