Approximate Solutions for the Coupled Nonlinear Equations Using the Homotopy Analysis Method

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Abstract

In this article, the homotopy analysis method (HAM) is implemented to obtain the approximate solutions of the nonlinear evolution equations in mathematical physics. The results obtained by this method have a good agreement with one obtained by other methods. It illustrates the validity and the great potential of the homotopy analysis method in solving partial differential equations.

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1. Introduction

It is well known that nonlinear dynamical systems arise in various fields. A
wealth of methods have been developed to find these exact physically significant solutions of a partial differential equation though it is rather difficult. Some of the most important methods are Backlund transformation [1], the CK direct method [2,3], and the homogeneous balance method [4]. The homotopy analysis method (HAM) [5], is a powerful method to solve non-linear problems. Based on homotopy of topology, the validity of the HAM is independent of whether or not there exist small parameters in the considered equation. Therefore, the HAM can overcome the foregoing restrictions and limitations of perturbation techniques [6]. In recent years, this method has been successfully employed to solve many types of non-linear problems [7,8,9].

In this paper, the homotopy analysis method is used to solve the system of the approximate equations for long water waves [10,11], as follows:

\[
\begin{align*}
\left( \frac{1}{2} u_x - uu_x - v + u_{xx} \right) / 2 &= 0 \\
\left( \frac{1}{2} v_x - (uv)_x - v_{xx} \right) / 2 &= 0
\end{align*}
\]

with the solitary wave solutions [4]

\[
\begin{align*}
\phi (x,t) &= \alpha \tanh \left( (\alpha x + \beta t + \gamma) / 2 \right) / 2 + \alpha / 2 + c \\
\phi (x,t) &= \alpha^2 \sec h^2 \left( (\alpha x + \beta t + \gamma) / 2 \right) / 4
\end{align*}
\]

where \( \alpha, \gamma, c \) are arbitrary constants and \( \beta = c\alpha + \alpha^2 / 2 \).

2. Analysis of the method

For the convenience of the reader, we will first present a brief account of the HAM. Let us consider the following differential equation:

\[ N[u(x,t)] = 0 \]

where \( N \) is a nonlinear operator, \( u(x,t) \) is an unknown function, and \( x \) and \( t \) denote spatial and temporal independent variables. For simplicity, we ignore all boundary or initial conditions, which can be treated in the similar way. By means of generalizing the traditional homotopy method, we can construct the so-called zero-order deformation equation as:

\[
(1 - q) L[\phi (x,t; q) - u(x,t)] = q h H(x,t) N[\phi (x,t; q)]
\]

where \( q \in [0,1] \) is the embedding parameter, \( h \) is a non-zero auxiliary parameter,
$H(x,t) \neq 0$ is an auxiliary function, $L$ is an auxiliary linear operator, $u_0(x,t)$ is an initial guess of $u(x,t)$ and $\phi(x,t;q)$ is a unknown function. Obviously, when $q = 0$ and $q = 1$, it holds

$$\phi(x,t;0) = u_0(x,t), \quad \phi(x,t;1) = u(x,t)$$

respectively. Thus, as $q$ increases from 0 to 1, the solution $\phi(x,t;q)$ varies from the initial guess $u_0(x,t)$ to the solution $u(x,t)$. Expanding $\phi(x,t;q)$ in Taylor series with respect to $q$, we have

$$\phi(x,t;q) = u_0(x,t) + \sum_{m=1}^{\infty} u_m(x,t) q^m$$

where

$$u_m(x,t) = \frac{1}{m!} \frac{\partial^m \phi(x,t;q)}{\partial q^m} \bigg|_{q=0}$$

If the auxiliary linear operator, the initial guess, the auxiliary parameter $\mu$, and the auxiliary function are properly chosen, the series (5) converges at $q = 1$, then we have

$$u(x,t) = u_0 + \sum_{m=1}^{\infty} u_m(x,t)$$

which must be one of solutions of original nonlinear equations. According to the definition (6), the governing equation can be deduced from the zero-order deformation (4). Define the vector

$$\vec{u}_m(x,t) = \{u_0(x,t), u_1(x,t), \cdots, u_m(x,t)\}$$

Differentiating equation (4) $m$ times with respect to the embedding parameter $q$ and then setting $q = 0$ and finally dividing them by $m!$, we have the so-called $m$th-order deformation equation

$$L[u_m(x,t) - \chi_m u_{m-1}(x,t)] = hH(x,t) R_m \left( \vec{u}_{m-1} \right)$$
where
\[
R \left( u_{m-1} \right) = \frac{1}{(m-1)!} \frac{\partial^{m-1} N \left[ \phi(x,t;q) \right]}{\partial q^{m-1}} \bigg|_{q=0}
\]
(10)
and
\[
\gamma_{m} = \begin{cases} 0, & m \leq 1 \\ 1, & m > 1 \end{cases}
\]
(11)

In order to assess the advantages and accuracy of HAM, we consider the following application.

**3. Application**

we shall deal with the system of the approximate equations for long water waves (1). For simplicity, \( \alpha = 2, \gamma = 0, c = 0 \) are used in the analyses. For application of homotopy analysis method, we choose the initial conditions
\[
\begin{cases}
u_{0} = \nu(x,0) = \sec h^{-1}(x) \\
u_{0} = \nu(x,0) = \sec h^{-1}(x)
\end{cases}
\]
(12)
and the auxiliary linear operators
\[
L_{x} \left[ \phi(x,t;q) \right] = \frac{\partial \phi(x,t;q)}{\partial t}, \quad L_{x} \left[ \varphi(x,t;q) \right] = \frac{\partial \varphi(x,t;q)}{\partial t}
\]
(13)
with the property \( L_{x} \left[ C_{i} \right] = 0 \), \( L_{x} \left[ C_{i} \right] = 0 \) where \( C_{i} \) and \( C_{i} \) are constant coefficients, \( \phi \) and \( \varphi \) are real functions. Furthermore, we define the nonlinear operators
\[
N_{x} \left[ \phi(x,t;q), \varphi(x,t;q) \right] = \frac{\partial \phi(x,t;q)}{\partial t} - \phi(x,t;q) \frac{\partial \phi(x,t;q)}{\partial x} - \varphi(x,t;q) + \frac{1}{2} \frac{\partial^{2} \phi(x,t;q)}{\partial x^{2}}
\]
(14),
\[
N_{x} \left[ \phi(x,t;q), \varphi(x,t;q) \right] = \frac{\partial \varphi(x,t;q)}{\partial t} - \phi(x,t;q) \frac{\partial \varphi(x,t;q)}{\partial x} - \varphi(x,t;q) + \frac{1}{2} \frac{\partial^{2} \varphi(x,t;q)}{\partial x^{2}}
\]
(15)
where \( q \in [0,1] \), \( \phi(x,t;q) \) and \( \varphi(x,t;q) \) are real functions of \( x, t \) and \( q \). Let
Approximate solutions

$h_u, h_v$ denote the nonzero auxiliary parameters. Using the above definition, with assumption $H_u(x, t) = 1, H_v(x, t) = 1$, we construct the zero-order deformation equations as follows,

\[
(1-q) L_u \left[ \phi(x, t; q) - u_0(x, t) \right] = q h_u N_u (\phi(x, t; q), \varphi(x, t; q)),
\]

\[
(1-q) L_v \left[ \varphi(x, t; q) - v_0(x, t) \right] = q h_v N_v (\phi(x, t; q), \varphi(x, t; q))
\]

obviously, when $q = 0$ and $q = 1$, it is clear that

\[
\phi(x, t; 0) = u_0(x, t), \varphi(x, t; 0) = v_0(x, t), \phi(x, t; 1) = u(x, t), \varphi(x, t; 1) = v(x, t)
\]

Both of $h_u$ and $h_v$ are properly chosen so that the terms

\[
u_u(x, t) = \frac{1}{n!} \frac{\partial^n \phi(x, t; q)}{\partial q^n} \bigg|_{q=0} \text{ and } \nu_v(x, t) = \frac{1}{n!} \frac{\partial^n \varphi(x, t; q)}{\partial q^n} \bigg|_{q=0}
\]

exist for $n \geq 1$ and the power series of $q$ in the following forms

\[
\phi(x, t; q) = u_0(x, t) + \sum_{n=1}^{\infty} \nu_u(x, t) q^n, \quad \varphi(x, t; q) = v_0(x, t) + \sum_{n=1}^{\infty} \nu_v(x, t) q^n
\]

are convergent at $q = 1$. So using (19), we obtain

\[
u_u(x, t) = u_0(x, t) + \sum_{n=1}^{\infty} u_n(x, t), \quad \nu_v(x, t) = v_0(x, t) + \sum_{n=1}^{\infty} v_n(x, t)
\]

According to the fundamental theorem of HAM, we have the $n$th-order deformation equation

\[
L_u \left[ u_n(x, t) - \nabla_u u_{n-1}(x, t) \right] = h_u R^u_n \left[ u_n, v_{n-1} \right],
\]

\[
L_v \left[ v_n(x, t) - \nabla_v v_{n-1}(x, t) \right] = h_v R^v_n \left[ u_{n-1}, v_{n-1} \right],
\]

where

\[
R^u_n \left[ u_{n-1}, v_{n-1} \right] = \frac{\partial u_{n-1}(x, t)}{\partial t} - \sum_{i=0}^{n-1} u_i(x, t) \frac{\partial u_{n-1-i}(x, t)}{\partial x} - \frac{\partial v_{n-1}(x, t)}{\partial x} + \frac{1}{2} \frac{\partial^2 u_{n-1}(x, t)}{\partial x^2}
\]

\[
R^v_n \left[ u_{n-1}, v_{n-1} \right] = \frac{\partial v_{n-1}(x, t)}{\partial t} - \sum_{i=0}^{n-1} u_i(x, t) \frac{\partial v_{n-1-i}(x, t)}{\partial x} - \sum_{i=0}^{n-1} v_i(x, t) \frac{\partial u_{n-1-i}(x, t)}{\partial x} - \frac{1}{2} \frac{\partial^2 v_{n-1}(x, t)}{\partial x^2}
\]
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and $\chi_n$ is defined by (11).

Now, the solution of the $n$th-order deformation equation (22) for $n \geq 1$ becomes

$$u_n(x,t) = \chi_n u_{n-1}(x,t) + h^L \left[ R_n^0 \left( \frac{u_{n-1}}{u_{n-1}}, \frac{v_{n-1}}{v_{n-1}} \right) \right],$$

$$v_n(x,t) = \chi_n v_{n-1}(x,t) + h^L \left[ R_n^0 \left( \frac{u_{n-1}}{u_{n-1}}, \frac{v_{n-1}}{v_{n-1}} \right) \right].$$

(25)

Note that the solutions series (21) contain two auxiliary $h_u$ and $h_v$. For simplicity, let $h_u = h_v = h$, then the approximations of $u(x,t)$ and $v(x,t)$ are only dependent on $h$. We write the differential equations need to calculate $u_1, u_2, u_3, \cdots, u_n$ and $v_1, v_2, v_3, \cdots, v_n$ as follows

$$u_i = \frac{1}{2} \frac{\partial^2 u_i}{\partial x^2} - hu_i \frac{\partial u_i}{\partial t} + h^2 \frac{\partial u_i}{\partial x} - h \frac{\partial v_i}{\partial x},$$

$$v_i = \frac{1}{2} \frac{\partial^2 v_i}{\partial x^2} + hu_i \frac{\partial v_i}{\partial t} - h \frac{\partial u_i}{\partial x} - hu_i \frac{\partial u_i}{\partial x},$$

$$u_{i+1} = (h+1) \frac{\partial u_i}{\partial t} - h \frac{\partial v_i}{\partial x} + \frac{1}{2} \frac{\partial^2 u_i}{\partial x^2} - h \left( u_i \frac{\partial u_i}{\partial x} + u_i \frac{\partial v_i}{\partial x} \right),$$

$$v_{i+1} = (h+1) \frac{\partial v_i}{\partial t} + \frac{1}{2} \frac{\partial^2 v_i}{\partial x^2} - h \left( v_i \frac{\partial u_i}{\partial x} + v_i \frac{\partial v_i}{\partial x} \right),$$

(26)

The $u_i$ and $v_i, (i = 1, 2, 3, \cdots)$ components have been obtained using the maple package.

It is important to ensure that the series solutions are convergent. We investigate the influence of the auxiliary parameter $h$ on the convergence of the
series by plotting the so-called $h$-curves. The $h$ curves of $u, u_{tt}, u_{xx}, v, v_{tt}$, and $v_{xx}$ at the point $(x,t) = (13.5, 5.5)$ are depicted in Fig. 1 and Fig. 2. The valid region of $h$ is a horizontal line segment. It is observed the valid region for $h$ is $-2.5 < h < 2.5$ as shown in Fig. 1 and Fig. 2.

![Fig. 1](image1.png)  ![Fig. 2](image2.png)

**Fig. 1.** $h$-curves of $u$ and its derivatives with different $h$ in the case of $(x,t) = (13.5, 5.5)$ and 8th-order approximation.

**Fig. 2.** $h$-curves of $v$ and its derivatives with different $h$ in the case of $(x,t) = (13.5, 5.5)$ and 8th-order approximation.

In order to verify the validity of the HAM solution, the three-dimensional plots of the absolute error between the exact solutions and the solutions series obtained by HAM for $h = -1.8$ are shown in Fig. 3 and Fig. 4. As shown in these figures, the behavior of the approximate solutions obtained by homotopy analysis method agree well with one obtained by the exact solutions (2).

![Fig. 3](image3.png)  ![Fig. 4](image4.png)

**Fig. 3** Absolute error for the 8th-order approximation by HAM for $u(x,t)$ and $h = -1.8$.

**Fig. 4** Absolute error for the 8th-order approximation by HAM for $u(x,t)$ and $h = -1.8$. 
4. Conclusion

In this paper, the homotopy analysis method (HAM) is used to obtain the approximate solutions of the system of the approximate equations for long water waves. The results obtained by this method have a good agreement with one obtained by other methods. The advantages of the HAM are illustrated. It is easy to see that the HAM is a very powerful and efficient technique in finding analytical solutions of wide classes of nonlinear partial differential equations.

References


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