Capture Zone in the Herding Pursuit Evasion Games

A. Delavar Khalafi
Faculty of Mathematics
Yazd University, Yazd, Iran
delavarkh@mail.ipm.ir

M. Ranjbar Toroghi
Faculty of Mathematics
Yazd University, Yazd, Iran
mranjbar@stu.yazduni.ac.ir

Abstract
A herding pursuit evasion problem is studied where the agent pursuer is considered the control action for moving the agent evader to a fixed location using the dynamics of their interaction, such that a norm characterizing distance traveled by both the pursuer and the evader is minimized. The problem is solved using calculus of variations. Simulation results for the system are given for different initial conditions. Finally, the concept of capture zone is introduced and is shown that with limited $\varepsilon$-neighbourhood, the range of final angle is small and final time $t_f$ increases.

Mathematics Subject Classification: 91A15, 91A23, 91A05

Keywords: Pursuit evasion, Herding problem, Calculus of variations, Optimal control law, Capture zone

1 Introduction

Contests of pursuit and evasion are among the most widespread, challenging, and important optimization problems that confront mobile agents, and represent some of the most important potential applications for robots and other artificial autonomous agents. In a typical contest of this sort, a predator chases a prey animal around until the prey is captured. Pursuit evasion problems have been studied and solved using various optimization techniques such as dynamic programming [4-5], calculus of variations and
optimal control [1,3]. Also Many of the pursuit evasion problems are set in the differential games theory where the modeling of the system is done using differential or difference equations. In this models, pursuer’s aim is to hunt, or intercept the evader and the typical termination state is capture [6,13].

Unlike these models, there are pursuit evasion problems that uses a different view and called the herding pursuit evasion games. The termination state for the herding problem needs that the evader to enter the pen. In this model, the aim of the pursuer is to drive the evader to a certain location in the x-y plane [12,14]. Another type of herding problem is the study of shep-herding behaviors in which one group (the shepherds) try to control the motion of another group (the flock) [9-10].

The present paper is organized as follows. In section 2 we introduced the model of the problem. The solution for the optimal trajectory of the pursuer is derived and it is illustrated how it satisfies the necessary and sufficient conditions for a minimizing curve. Simulation results for the system are given in section 3 for different initial conditions. The concept of capture zone is introduced and finally an example is presented.

2 System Model and finding optimal trajectory

Figure (1) gives a summary representation of our model. As shown in the figure, the initial position of the pursuer is \((x_{p0}, y_{p0})\) and that of the evader is \((x_{e0}, y_{e0})\). Beginning at these initial positions, the pursuer is supposed to drive the evader to the \((0, 0)\) position in the \(x - y\) grid through the shortest path.

Figure 1: System Model
The associated dynamics of the problem are given below [12]:

\[\dot{y}_e = \dot{x}_e \tan \theta \quad (1)\]
\[\dot{x}_e^2 + \dot{y}_e^2 = 1 \quad (2)\]
\[\begin{align*}
  x_p &= x_e + r_0 \cos \theta \\
  y_p &= y_e + r_0 \sin \theta 
\end{align*} \quad (3)\]
\[\begin{align*}
  \int_0^{t_f} \dot{x}_e dt &= -x_{e_0} \\
  \int_0^{t_f} \dot{y}_e dt &= -y_{e_0} \quad (4)
\end{align*}\]

Based on the above dynamic equations of the system, the evader moves away from the pursuer according to equations (2) and (4), that show that the direction of the motion of the evader is in the straight line joining the pursuer and the evader. We have chosen the normalized velocity of the evader to be 1 unit. Notice that the distance between the two agents is always the same and equals \(r_0\). The goal of the pursuer is to drive the evader form the given initial position to the final one, following the above constraints, such that a norm characterizing distance traveled by both the pursuer and the evader is minimized. The previous statement can be represented by the following objective function to be minimized:

\[J = \min \int_0^{t_f} (\dot{x}_e^2 + \dot{y}_e^2 + \dot{x}_p^2 + \dot{y}_p^2) dt \quad (5)\]

### 2.1 Solving for Optimal Trajectory

Examining figure (1), with the given system equations, shows that the main control variable that the pursuer can use to achieve the objective is its relative position angle \(\theta\). This, in turn can be directly controlled via its rate of change. Therefore, the first step in obtaining the optimal trajectory is to express the objective function as well as the system constraints as functions of \(\theta\). By substituting (1) in (2) and simplifying, we get

\[\dot{x}_e^2 = \cos^2 \theta, \quad \dot{y}_e^2 = \sin^2 \theta \quad (6)\]

By differentiating equation (3), using equations (6) and then substituting in equation (5) of the objective function, the integrand, \(L\), becomes

\[L = (\dot{x}_e^2 + \dot{y}_e^2 + \dot{x}_p^2 + \dot{y}_p^2) = 1 + \left(\dot{x}_e - r_0 \dot{\theta} \sin \theta\right)^2 + \left(\dot{y}_e + r_0 \dot{\theta} \cos \theta\right)^2\]
\[= 2 + r_0^2 \dot{\theta}^2 \quad (7)\]

Therefore, the original model of the problem can be transformed into the following equivalent one.

\[J = \min \int_0^{t_f} \left(2 + r_0^2 \dot{\theta}^2\right) dt \quad (8)\]
subjected to the following constraints:
\[ \theta(0) = \theta_0 = \tan^{-1} \left( \frac{y_{p_0} - y_{e_0}}{x_{p_0} - x_{e_0}} \right), \quad \theta(t_f) = \theta_f \]

where \( \theta_f \) is free. The vertical and horizontal components of the evader’s velocity have to satisfy
\[
\begin{align*}
\int_0^{t_f} \cos \theta \, dt &= x_{e_0} \\
\int_0^{t_f} \sin \theta \, dt &= y_{e_0}
\end{align*}
\]
Clearly, the Lagrangian of equation (8) satisfies the condition \( L_{\theta \dot{\theta}} = 2 \lambda_0^2 \geq 0 \). Therefore, the solution of the Euler’s differential equation provides a solution of (8). Combining the constraints given by (9), the problem model becomes an isoperimetric model whose Lagrangian is given by;
\[
L = 2 + \left( r_0 \dot{\theta} \right)^2 + \lambda_1 \sin \theta + \lambda_2 \cos \theta
\]
Assuming that \( \theta_* \) is the optimizer of the equation (10), it has to satisfy the Euler-Lagrange differential equation
\[
2r_0^2 \ddot{\theta}_* = -\lambda_2 \sin \theta_* + \lambda_1 \cos \theta_*
\]
Where, \( \theta_* \) is the optimal angle of the pursuer with respect to the evader position at any time \( t \). Since the final angle of arrival, \( \theta_f \), is free, the transversality condition has to be satisfied at the final time \( t_f \); i.e. \( L_{\dot{\theta}} = 0 \) at \( t_f \).
Since, the control variable \( \dot{\theta} \) is completely state dependent and has no explicit dependence on time, we can use
\[
v = \dot{\theta}, \quad \ddot{\theta} = v \frac{dv}{d\theta}
\]
From equations (11) and (12), we get;
\[
\frac{\lambda_2}{\lambda_1} = \frac{\cos \theta_f}{\sin \theta_f}
\]
Substituting equation (13) into equation (11) gives;
\[
\ddot{\theta}_* = \frac{1}{2r_0^2} \sin (\theta_* - \theta_f)
\]
By equation (12) and (14), we get
\[
\dot{\theta}_*^2 = \frac{2}{r_0^2} \sin^2 \left( \frac{\theta_* - \theta_f}{2} \right)
\]
Solution of the differential equation (15) gives the family of all extremal curves. Hence, checking for Legendre necessary condition, we find that \( L_{\dot{\theta}} = 2r_0^2 > 0 \). This means that the solution obtained by solving Euler’s differential equation provides a weak minimum of the objective function \( J \).

According to the Weiestrass condition, in order for that extremal \( \theta_* \) to give a strong minimum of the objective function \( J \), it is sufficient that \( \theta_* \) is a member of a field of extremals which is satisfied from equation (15). To satisfy the Weiestrass condition we notice that

\[
E(t, \theta, \dot{\theta}, q) = L(t, \theta, q) - L(t, \theta, \dot{\theta}) - (q - \dot{\theta})L_\theta(t, \theta, \dot{\theta})
\]

\[
= (r_0\dot{\theta} - r_0q)^2 \geq 0 \quad (16)
\]

Therefore, \( \theta_* \) provides a strong minimum for our objective function \( J \). Based on this analysis we obtain the nonlinear feedback control law for the pursuit-evasion problem as:

\[
\dot{\theta} = -\frac{\sqrt{2}}{r_0} \sin\left(\frac{\theta - \theta_f}{2}\right) \quad (17)
\]

We can show that the optimal control law (17) is a stabilizing control law.

**Theorem 2.1.** the optimal feedback control law (17) is a stabilizing control law and due to it, pursuer drives evader to origin.

**proof.** consider a candidate Lyapunov function as

\[
V = e^2 \quad (18)
\]

where

\[
e = \left(\frac{\theta - \theta_f}{2}\right)
\]

Differentiating (18) with respect to time along (17) we get;

\[
\dot{V} = 2e\dot{e} = -\frac{\sqrt{2}}{r_0} e \sin(e) \leq 0 \quad (19)
\]

For \( 0 \leq e \leq \frac{\pi}{2} \) the largest invariant subset of the set \( E = \{ e : \dot{V} = 0 \} \) is given by \( e = 0 \). Therefore, the LaSalle’s theorem implies that the pursuer drives the evader to the origin [7].

3 Simulation Results and concept of capture zone

Solving process for \( \theta_* \) involves several difficulties such as the non-linear nature of the differential equation given in (17), the unknown final angle, \( \theta_f \), and the
final time $t_f$. To overcome such problems, the optimal value $\theta*$ in (17) must be obtained by numerical methods along with iterative guessed values of $\theta_f$ and $t_f$. Due the discretization process of both time and $\theta_f$, we put a threshold ($\varepsilon$-neighbourhood) around the origin, such that once the evader is within this $\varepsilon$-neighbourhood, the simulation will be finished. In this paper, we consider $\varepsilon$-neighbourhood as a square in nonnegative region of plane.

In figure (2), optimal trajectory of evader with initial point $(x_0, y_0) = (3.5, 3.5)$ and $(x_e, y_e) = (3, 3.5)$ is presented.

![Figure 2: Optimal trajectory of evader](image)

The forgoing optimal trajectory is related to the case that evader reaches to point $(0, 0)$ for the first time. In this case, we have $t_f = 4.75s$ and $\theta_f = 58^\circ$. If pursuer wants to directed evader to point $(0, 0.5)$, simulation resulting will be final time $t_f = 4.4s$ and final angle $\theta_f = 53.8^\circ$. When the evader be directed to point $(0.5, 0)$, will be achieved $t_f = 4.5s$ and $\theta_f = 64.5^\circ$. For case that evader reached to $(0.5, 0.5)$, angle and final time is obtained $60.7^\circ$ and 4.05s, respectively.

Now, consider the initial point of pursuer and evader as $(x_0, y_0) = (7.5, 8)$ and $(x_e, y_e) = (7.5, 7)$. Then the similar results presented in table (1).

<table>
<thead>
<tr>
<th>Position</th>
<th>$(0, 0)$</th>
<th>$(0, 0.5)$</th>
<th>$(0.5, 0)$</th>
<th>$(0.5, 0.5)$</th>
<th>$(0.25, 0.5)$</th>
<th>$(0.5, 0.25)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$t_f$</td>
<td>10.5</td>
<td>10.2</td>
<td>10.15</td>
<td>9.8</td>
<td>10</td>
<td>9.95</td>
</tr>
<tr>
<td>$\theta_f$</td>
<td>35.8</td>
<td>33.2</td>
<td>37.8</td>
<td>35.1</td>
<td>34.1</td>
<td>36.4</td>
</tr>
</tbody>
</table>

Table 1: results simulation for $(x_0, y_0) = (7.5, 8)$ and $(x_e, y_e) = (7.5, 7)$.

For initial point above, simulation is repeated and shows that there are certain limits on $\theta_f$. In figure (3) trajectory of evader with $\theta_f = 75^\circ$ is shown.

We see that the pursuer could not steer evader to origin. Considering different $\theta_f$, the evader will tend to different location in grid that results are summarized in the table (2).
Figure 3: trajectory of evader with $\theta_f = 75^\circ$

<table>
<thead>
<tr>
<th>$\theta_f$</th>
<th>0</th>
<th>15</th>
<th>30</th>
<th>45</th>
<th>60</th>
<th>75</th>
<th>90</th>
</tr>
</thead>
<tbody>
<tr>
<td>$t_f$</td>
<td>8.25</td>
<td>8.8</td>
<td>9.85</td>
<td>9.05</td>
<td>7.8</td>
<td>7.2</td>
<td>7.05</td>
</tr>
<tr>
<td>Position</td>
<td>(0,5)</td>
<td>(0,3.2)</td>
<td>(0,1.05)</td>
<td>(2.03,0)</td>
<td>(4.3,0)</td>
<td>(6.0)</td>
<td>(7.5,0)</td>
</tr>
</tbody>
</table>

Table 2: enter the evader to different location with different $\theta_f$

Inspection of the simulation results shows that not only a clear symmetry of $\theta_f$ about the line connecting the initial position of the evader to the origin, but also there is a relationship between the $\theta_f$, $\theta_0$ and the slope of the symmetry line (angle $\alpha_0$). Trying to estimate this dependence of $\theta_f$, on $\theta_0$ and $\alpha_0$, we plot $(\theta_f - \theta_0)$ versus $(\alpha_0 - \theta_0)$ and come up with the results shown in figure (4) where the plots are drawn for different values of $\alpha_0$.

Figure 4: Dependence of $(\theta_f - \theta_0)$ on $(\alpha_0 - \theta_0)$

Based on the plots, we consider a linear approximation of the dependence of $(\theta_f - \theta_0)$ on $(\alpha_0 - \theta_0)$ given as:

$$\theta_f - \theta_0 = k(\alpha_0 - \theta_0) + \beta$$  \hspace{1cm} (20)
where, \( \alpha_0 = \tan^{-1}\left(\frac{y_0}{x_0}\right) \), \( \theta_0 = \tan^{-1}\left(\frac{y_{p0} - y_0}{x_{p0} - x_0}\right) \) and by plotting we have:

\[
k = \begin{cases}
0.55 & -10 \leq \alpha_0 - \theta_0 \leq 10 \\
1.35 & -139 \leq \alpha_0 - \theta_0 \leq -10 \text{ or } 10 \leq \alpha_0 - \theta_0 \leq 139 \\
-38.9 & -148 \leq \alpha_0 - \theta_0 \leq -139 \text{ or } 139 \leq \alpha_0 - \theta_0 \leq 148 \\
1 & -180 \leq \alpha_0 - \theta_0 \leq -148 \text{ or } 148 \leq \alpha_0 - \theta_0 \leq 180
\end{cases}
\]

and

\[
\beta = \begin{cases}
318 & -180 \leq \alpha_0 - \theta_0 \leq -148 \\
-5587 & -148 \leq \alpha_0 - \theta_0 \leq -139 \\
8 & -139 \leq \alpha_0 - \theta_0 \leq -10 \\
0 & -10 \leq \alpha_0 - \theta_0 \leq 10 \\
-8 & 10 \leq \alpha_0 - \theta_0 \leq 139 \\
5587 & 139 \leq \alpha_0 - \theta_0 \leq 148 \\
-318 & 148 \leq \alpha_0 - \theta_0 \leq 180
\end{cases}
\]

Since the value of \( \theta_f \) is the same for all the intermediate values of \( \theta_0 \) and \( \alpha_0 \) on the system integral curves, we replace (20) by

\[
\theta_f = k(\alpha - \theta) + \theta + \beta
\]

(21)

Therefore, the feedback control law is given in (17) with \( \theta_f \) given by (21).

**Remark 3.1.** Notice that probably don’t achieved the desired results in interval boundaries and therefore, the interval boundaries requires exactly examination.

### 3.1 Capture zone

**Definition 3.2.** Assume \( \varepsilon \geq 0 \). If \( |x_e|, |y_e| \leq \varepsilon \), we say that \( \varepsilon \)-capture zone is obtained, i.e. evader be located within the \( \varepsilon \)-neighbourhood.

Also, we defined \( M \) as;

\[
M = \{(\theta_f, x_e, y_e) : |x_e| \leq \varepsilon, |y_e| \leq \varepsilon\}.
\]

Therefore, the triple \((\theta_f, x_e, y_e)\) is called capture zone, such that evader reached \( \varepsilon \)-neighbourhood.

We consider the initial position of pursuer and evader as \((x_{p0}, y_{p0}) = (6, 4)\) and \((x_{e0}, y_{e0}) = (5, 4.5)\). Simulation results related to above argument are expressed in the table (3).

The results show that by given limited \( \varepsilon \)-neighbourhood, the range of final angle is small and final time \( t_f \) increases. In the sequel, to express exactly this contents.
Table 3: results simulation for $(x_{p_0}, y_{p_0}) = (6, 4)$ and $(x_{e_0}, y_{e_0}) = (5, 4.5)$

<table>
<thead>
<tr>
<th></th>
<th>$(0, 2)$</th>
<th>$(0, 1.5)$</th>
<th>$(0, 1)$</th>
<th>$(0, 0.5)$</th>
<th>$(0, 0)$</th>
<th>$(0.5, 0)$</th>
<th>$(1.0, 0)$</th>
<th>$(1.5, 0)$</th>
<th>$(2, 0)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$t_f$</td>
<td>5.9</td>
<td>6.25</td>
<td>6.55</td>
<td>6.95</td>
<td>7.25</td>
<td>6.95</td>
<td>6.65</td>
<td>6.4</td>
<td>6.2</td>
</tr>
<tr>
<td>$\theta_f$</td>
<td>45.7</td>
<td>50.4</td>
<td>54.4</td>
<td>57.8</td>
<td>60.7</td>
<td>65.6</td>
<td>71.2</td>
<td>77.4</td>
<td>84.2</td>
</tr>
</tbody>
</table>

**Theorem 3.3.** Let $0 \leq \varepsilon_2 < \varepsilon_1$. Also assumed that

$$M_i = \{(\theta_f, x_e, y_e) : |x_e| \leq \varepsilon_i \text{ and } |y_e| \leq \varepsilon_i \}, \quad i = 1, 2\}.$$  

Therefore

$$M_2 \subseteq M_1$$

**proof.** Let $(\theta_f, x_e, y_e) \in M_2$. According to definition, $|x_e|, |y_e| \leq \varepsilon_2$. Due to hypothesis, $\varepsilon_2 < \varepsilon_1$ that gives $|x_e|, |y_e| \leq \varepsilon_1$ and therefore $(\theta_f, x_e, y_e) \in M_1$. Now, it is enough to show that $M_1 \nsubseteq M_2$. Let $M_1 \subseteq M_2$ and $(\theta_f, x_e, y_e) \in M_1$ that in which case, $(\theta_f, x_e, y_e) \in M_2$. Therefore $\varepsilon_1 \leq \varepsilon_2$ that contradicts the hypothesis and completes the proof.

### 3.2 Example

As example, a herding dog and sheep problem is studied where the agent dog is considered the control action for moving the agent sheep to a fixed location using the dynamics of their interaction. The dynamics of the dog and sheep is given by (1)-(4). In this pursuit evasion game, the aim of the dog is to drive sheep to a $\varepsilon$-neighbourhood of origin with $\varepsilon = 0.5$, such that a norm characterizing distance traveled by both the dog and the sheep is minimized. We have chosen the normalized velocity of the sheep to be 1 unit and distance between the two agents is always the same and equals $r_0$. Based on objects presented in previous sections, optimal control law is given by (17). In figure (5) the trajectories of sheep and dog from different positions are shown and its results are expressed in the table (4).

<table>
<thead>
<tr>
<th></th>
<th>$(x_p, y_p)$</th>
<th>$(x_e, y_e)$</th>
<th>$r_0$</th>
<th>$\theta_0$</th>
<th>$\alpha_0$</th>
<th>$\theta_f$</th>
<th>$t_f$</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>(2.5, 2.5)</td>
<td>(3, 3)</td>
<td>0.71</td>
<td>-45°</td>
<td>45°</td>
<td>69°</td>
<td>4.5</td>
</tr>
<tr>
<td>b</td>
<td>(2.5, 3.5)</td>
<td>(3, 3)</td>
<td>0.71</td>
<td>135°</td>
<td>45°</td>
<td>17°</td>
<td>4.5</td>
</tr>
<tr>
<td>c</td>
<td>(3.5, 3.3)</td>
<td>(3, 3)</td>
<td>0.58</td>
<td>31°</td>
<td>45°</td>
<td>45°</td>
<td>3.95</td>
</tr>
<tr>
<td>d</td>
<td>(2.5, 2.5)</td>
<td>(3, 3)</td>
<td>0.71</td>
<td>225°</td>
<td>45°</td>
<td>16°</td>
<td>6</td>
</tr>
</tbody>
</table>

Table 4: results related to figure (5)

**Remark 3.4.** This example can be considered for two robots with a little tolerance, because robots have limited additional in dynamics that should be applied in the model.
4 Conclusion

In this paper the class of pursuit evasion games is considered that the goal of
the pursuer is to drive the evader to a certain location in the x-y grid. Based
on mentioned model was shown that the optimal control policy was proven to
be dependent only on the space variables and therefore a feedback control law
and optimal trajectories was earned. Since there are several difficulties such
as the non-linear nature of the differential equation obtained, the unknown
final angle, $\theta_f$, and the final time $t_f$, the simulation is done and linear approxi-
mation is obtained for final angle $\theta_f$. Moreover, the concept of capture zone is
introduced and is shown that with limited $\varepsilon$-neighbourhood, the range of final
angle is small and final time $t_f$ increases.

References

[1] T. Basar and G.J. Olsder, Dynamic Noncooperative Game Theory, Aca-


Received: September, 2010