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Nonlinear Anti-Synchronization of Two Hyperchaotic Systems

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Abstract

Based on the nonlinear control theory, the anti-synchronization between two different hyperchaotic systems is investigated. Through rigorous mathematical theory, the sufficient condition is drawn for the stability of the error dynamics, where the controllers are designed by using the sum of the relevant variables in hyperchaotic systems. Numerical simulations are performed for to demonstrate the effectiveness of the proposed control strategy.

Keywords: Nonlinear control; Anti-synchronization; Lyapunov direct method

1 Introduction

Chaos is an interesting phenomenon of nonlinear systems. A deterministic chaotic system has some remarkable dynamic characteristics [1], such as system evolution sensitive to the change in initial conditions and broad spectrum of Fourier transform. It can be treated as a carrier to modulate signals that have the random characteristics. It also has the overall stability. When we use a chaotic signal to drive two identical systems, the two systems or certain parts of them will have the synchronous behavior, which does well for confidential communication.

Chaos synchronization has attracted a great deal of attention ever since Pecora and Carroll [2] established a chaos synchronization scheme for two identical chaotic systems with different initial conditions. Various effective methods have been presented to synchronize various chaotic systems, for example, adaptive control [3, 4], linear and nonlinear feedback control [5, 6] active control [8], and so on. The concept of synchronization has been extended to the scope, such as generalized synchronization [9], phase synchronization [10], lag synchronization [11], and even anti-phase synchronization (APS) [12, 13]. APS can also be interpreted as anti-synchronization (AS), which is a phenomenon whereby the state vectors of the synchronized systems have the same amplitude but opposite signs as those of the driving system. Therefore, the sum of two signals are expected to converge to zero when either AS or APS appears.

The aim of this work is to further develop the state observer method for constructing anti-synchronized of the high dimensional system, since the aforementioned method is mainly concern with the synchronization of chaotic systems with low dimensional attractor which characterized by one positive Lyapunov exponent. This feature limits the complexity of the chaotic dynamics. It is believed that the chaotic systems with higher dimensional attractor have much wider application. In fact, the adoption of higher dimensional chaotic systems has been proposed for secure communication and the presence of more than one Lyapunov exponent clearly improves security of the communication scheme by generating more complex dynamics. Recently, hyperchaotic systems were also considered with quickly increasing interest. Hyperchaotic system is usually classified as a chaotic system with more than one positive Lyapunov exponent.

2 Design of controller via nonlinear control method

In this section we will propose a nonlinear control scheme to investigate the nonlinear anti-synchronization between two identical and different and hyperchaotic systems.
Consider the following system described by

$$\dot{x} = Ax + Bf(x) \tag{1}$$

where $x \in R^n$ is the state vector, $A \in R^{n\times n}, B \in R^{n\times n}$ are metrics and vectors of system parameters, and $f : R^n \longrightarrow R^n$ is a nonlinear function. Eq.(1) is considered as a drive system.

By introducing an additive control $U \in R^n$, then the controlled response system is given by

$$\dot{y} = A_1 y + B_1 g(y) + U \tag{2}$$

where $y \in R^n$ denotes the state vector of the response system, $A_1 \in R^{n\times n}, B_1 \in R^{n\times n}$ are metrics and vectors of this controlled response system parameters, and $g : R^n \longrightarrow R^n$ is a nonlinear function. $A = A_1, B = B_1$, for two identical chaotic systems, $A \neq A_1, B \neq B_1$, for two different chaotic systems. The anti-synchronization problem is to design a controller U which anti-synchronizes the states of both the drive and response systems. We add Eq.(2) to Eq.(1) and get

$$\dot{e} = A_1 y + B_1 g(y) + Ax + Bf(x) + U \tag{3}$$

where $e = x + y$, the aim of the anti-synchronization is to make $\lim_{t\to\infty} \|e(t)\| = 0$. Then let Lyapunov error function be $V(e) = \frac{1}{2}e^T e$, where $V(e)$ is a positive definite function. Assuming that the parameters of the drive and response systems are known and the states of both systems are measurable, we may achieve the anti-synchronization by selecting the controller U to make the first derivative of $V(e), i.e., \dot{V}(e) < 0$. Then the states of response and drive systems are anti-synchronized asymptotically globally.

3 Systems description

The hyperchaotic Lorenz system [14]. is described by

$$\begin{aligned} \dot{x} &= a(y - x) + w, \\ \dot{y} &= -xz + rx - y, \\ \dot{z} &= xy - bz, \\ \dot{w} &= -xz + dw. \end{aligned} \tag{4}$$

where a, b, r, and d are constants. When parameters $a = 10, r = 28, b = 8/3$ and $d = 1.3$, the system (4) has two positive Lyapunov exponents $\lambda_1 = 0.3985$ and $\lambda_2 = 0.2481$. Thus, the system (4) shows hyperchaotic behavior.

The hyperchaotic Lü system [15] is described by

$$\begin{aligned} \dot{x} &= a(y - x) + w, \\ \dot{y} &= -xz + cy, \\ \dot{z} &= xy - bz, \\ \dot{w} &= xz + dw, \end{aligned} \tag{5}$$

where x, y, z, and w are state variables, and a, b, c, and d are real constants. When $a = 36, b = 3, c = 20, -1.03 \leq d \leq -0.46$, system (5) has periodic orbit, when $a = 36, b = 3, c = 20, -0.46 < d \leq -0.35$, system (5) has chaotic attractor, when $a = 36, b = 3, c = 20, -0.35 < d \leq 1.3$, system (5) has hyperchaotic attractor.

4 Anti-Synchronization of two different hyperchaotic systems

In order to achieve the behavior of the anti-synchronization between the hyperchaotic Lorenz system and the hyperchaotic Lü system by using the proposed method, assume that the hyperchaotic Lorenz system is the drive system whose variables are denoted by subscript 1 and the hyperchaotic Lü system is the response system whose variables are denoted by subscript 2. The drive and the response systems are described, respectively, by the following equations:

$$\begin{aligned} \dot{x_1} &= a_1(y_1 - x_1) + w_1, \\ \dot{y_1} &= -x_1 z_1 + r_1 x_1 - y_1, \\ \dot{z_1} &= x_1 y_1 - b_1 z_1, \\ \dot{w_1} &= -x_1 z_1 + d_1 w_1. \end{aligned} \tag{6}$$

and

$$\begin{aligned} \dot{x_2} &= a_2(y_2 - x_2) + w_2 + u_1, \\ \dot{y_2} &= -x_2 z_2 + c_2 y_2 + u_2, \\ \dot{z_2} &= x_2 y_2 - b_2 z_2 + u_3, \\ \dot{w_2} &= x_2 z_2 + d_2 w_2 + u_4. \end{aligned} \tag{7}$$

where $U = [u_1, u_2, u_3, , u_4]^T$ is the controller function. In the following, we design a nonlinear controller to achieve the anti-synchronization between the hyperchaotic Lorenz system and the hyperchaotic Lü system. By adding Eq. (7) to Eq. (6) yields the error dynamical system between Eq. (6) and (7):

$$\begin{aligned} \dot{e}_1 &= a_2(e_2 - e_1) + e_4 + (a_1 - a_2)(y_1 - x_1) + u_1, \\ \dot{e}_2 &= c_2 e_2 - (1 + c_2) y_1 + r_1 x_1 - x_2 z_2 - x_1 z_1 + u_2, \\ \dot{e}_3 &= -b_2 e_3 + (b_2 - b_1) z_1 + x_2 y_2 + x_1 y_1 + u_3, \\ \dot{e}_4 &= d_2 e_4 + (d_1 - d_2) w_1 + x_2 z_2 - x_1 z_1 + u_4. \end{aligned} \tag{8}$$

where $e_1 = x_2 + x_1$, $e_1 = y_2 + y_1$, $e_3 = z_2 + z_1$ and $e_4 = w_2 + w_1$. In order to determine the controller, let

$u_2 = u_{12} + u_{22}$, where $u_{22} = x_2 z_2 + x_1 z_1$
$u_3 = u_{13} + u_{23}$, where $u_{23} = -x_2 y_2 - x_1 y_1$
$u_4 = u_{14} + u_{24}$, where $u_{24} = -x_2 z_2 + x_1 z_1$

Then we rewrite (8) in the following form:

$$\begin{aligned} \dot{e_1} &= a_2(e_2 - e_1) + e_4 + (a_1 - a_2)(y_1 - x_1) + u_1, \\ \dot{e_2} &= c_2 e_2 - (1 + c_2) y_1 + r_1 x_1 + u_{12}, \\ \dot{e_3} &= -b_2 e_3 + (b_2 - b_1) z_1 + u_{13}, \\ \dot{e_4} &= d_2 e_4 + (d_1 - d_2) w_1 + u_{14}. \end{aligned} \tag{9}$$

By taking a Lyapunov function for Eq. (9) into consideration

$V(e) = \frac{1}{2} e^T e$

we get the first derivative of $V(e)$:

$$\begin{aligned}\dot{V} &= e_1\left[a_2e_2 - a_2e_1 + (a_1 - a_2)(y_1 - x_1) + e_4 + u_1\right] \\ &\quad + e_2\left[c_2e_2 + r_1x_1 - (1 + c_2)y_1 + u_{12}\right] \\ &\quad + e_3[-be_3 + (b_2 - b_1)z_1 + u_{13}] \\ &\quad + e_4[d_2e_4 + (d_1 - d_2)w_1 + u_{14}]\end{aligned} \tag{10}$$

Therefore, if we choose Uas follows:

$$\begin{aligned}u_1 &= -e_4 - (a_1 - a_2)(y_1 - x_1), \\ u_{12} &= -a_2e_1 - 2c_2e_2 - r_1x_1 + (1 + c_2)y_1, \\ u_{13} &= -(b_2 - b_1)z_1, \\ u_{14} &= -2d_2e_4 - (d_1 - d_2)w_1.\end{aligned} \tag{11}$$

then

$$\dot{V} = -a_2e_1^2 - c_2e_2^2 - b_2e_3^2 - d_2e_4^2. \tag{12}$$

where $\dot{V}(e) < 0$ is satisfied. Since $\dot{V}(e)$ is a negative-definite function, the error states

$\lim_{t\to\infty} \|e(t)\| = 0.$

Therefore, this choice will lead the error states e_1, e_2, e_3, e_4 to converge to zero as time t tends to infinity and hence the anti-synchronization of two different hyperchaotic systems is achieved.

4.1 Numerical simulation results

The fourth-order Runge-Kutta integration method is used to solve the systems of differential equations (6) and (7). In addition, a time step size 0.001 is employed. We select the parameters of the hyperchaotic Lorenz system as $a = 10, b = 8/3, r = 28, d = 1.3$, and for the hyperchaotic Lü as $a = 36, b = 3, c = 20, d = 1.3$, so that these systems exhibits a hyperchaotic behavior. The initial values of the drive and response systems are $x_1(0) = 5, y_1(0) = 8, z_1(0) = -1, w_1(0) = -3$ and $x_2(0) = 3, y_2(0) = 4, z_2(0) = 5, w_2(0) = 5$, respectively. A mirroring effect of the anti-synchronization is illustrated in Fig. (1) and the error dynamics shown in Fig. (2) dictates the effectiveness of this anti-synchronization scheme. Fig. (1)–(2) visually shows the effects of the hyperchaotic Lü system being controlled to be hyperchaotic Lorenz.

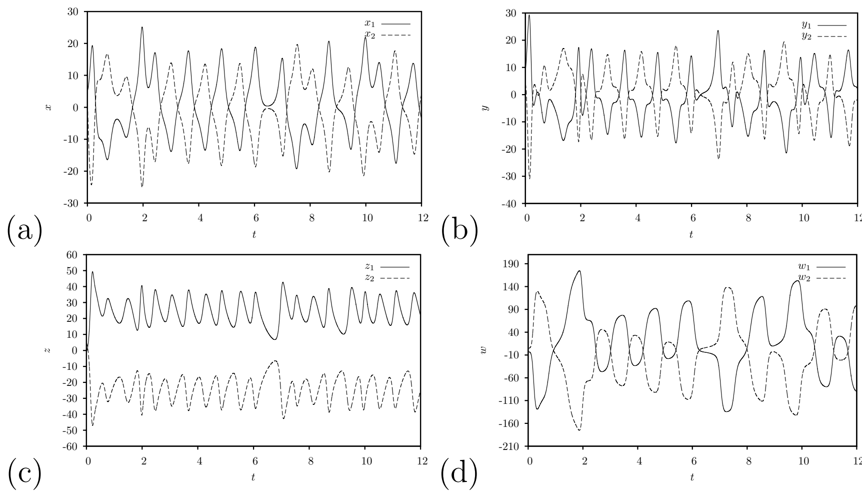

Figure 1: The time response of states for drive system (hyperchaotic Lorenz) and the response system (hyperchaotic Lü) via nonlinear control (**a**) signals x_1 and x_2; (**b**) signals y_1 and y_2; (**c**) signals z_1 and z_2 (**d**) signals w_1 and w_2.

5 Conclusion

Based on the Lyapunov stability theory, we propose a nonlinear control method to anti-synchronize two different hyperchaotic systems. The simulation results show that the two different hyperchaotic systems, the hyperchaotic Lü system is controlled to trace the hyperchaotic Lorenz system and the states of two systems become the same finally. This shows that our proposed method has strong robustness.

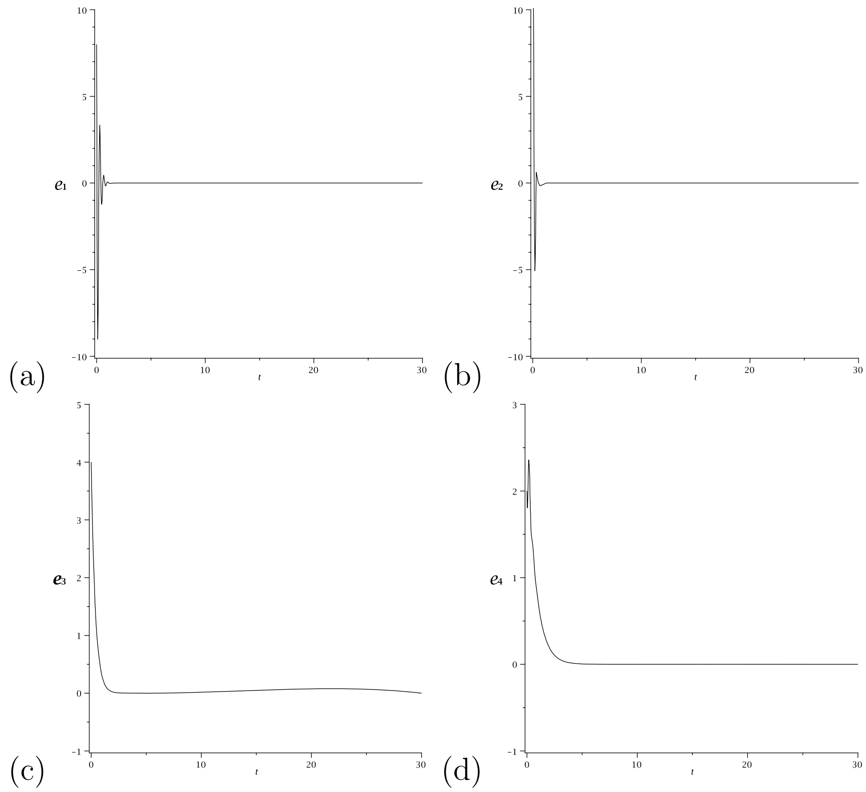

Figure 2: Anti-synchronization error in hyperchaotic Lorenz system and hyperchaotic Lü system via nonlinear control (**a**) signals x_1 and x_2; (**b**) signals y_1 and y_2; (**c**) signals z_1 and z_2; (**d**) signals w_1 and w_2.

References

[1] G. Chen,X. Dong, From chaos to order. Singapore: World Scientific; 1998.

[2] L.M. Pecora, T.L. Carroll, Synchronization in chaotic systems, Phys. Rev. Lett. 64 (1990) 821-824.

[3] M.M. Al-Sawalha, M.S.M. Noorani, Anti-synchronization of chaotic systems with uncertain parameters via adaptive control, Phys. Lett. A 373 (32) (2009) 2852–2857.

[4] M.M. Al-Sawalha, M.S.M. Noorani, M.M. Al-dlalah, Adaptive anti-synchronization of chaotic systems with fully unknown parameters, Computers and Mathematics with Applications 59 (2010) 3234–3244

[5] J.Lü, J. Lu, Controlling uncertain Lü system using linear feedback. Chaos, Solitons & Fractals. 17 (2003) 127-133.

[6] H.K. Chen, Global chaos synchronization of new chaotic systems via nonlinear control. Solitons & Fractals 23 (2005) 1245-1251.

[7] Q. Zhang, J.n. Lu, Chaos synchronization of a new chaotic system via nonlinear control. Chaos, Solitons & Fractals 37 (2008) 175-179.

[8] M.M. Al-Sawalha, M.S.M. Noorani, Anti-synchronization of chaotic systems with uncertain parameters via adaptive control, Phys. Lett. A 373 (32) (2009) 2852–2857.

[9] S.S. Yang, K. Duan, Generalized synchronization in chaotic systems. Chaos, Solitons & Fractals 10 (1998) 1703-1707.

[10] G.M. Rosenblum, S.A. Pikovsky, J. Kurths, Phase synchronization of chaotic oscillators. Phys Rev Lett 76 (1996) 1804-1807.

[11] S. Taherion1, Y.C. Lai, Observability of lag synchronization of coupled chaotic oscillators. Phys Rev E 59 (1999) 6247-6250.

[12] Y. Zhang, J. Sun, Chaotic synchronization and anti-synchronization based on suitable separation. Phys Lett A 330 (2004) 442-447.

[13] J.B. Liu, C.F.Ye, S.J.Zhang, W.T. Song, Anti-phase synchronization in coupled map lattices. Phys Lett A 274 (2000)27-29.

[14] J.Qiang, Hyperchaos generated from the Lorenz chaotic system and its control,Phy Lett A 366 (2007) 217-222.

[15] A. Chen, J. Lu, Lü, S. YuS. Generating hyperchaotic Lü attractor via state feedback control. Physica A 364 (2006) 103–110

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