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Generalized Taylor MatrixMethod for Solving LinearIntegro-Fractional Differential Equations of Volterra Type

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Abstract

In this paper, a simple and important approximate technique is present for solving the variable coefficients linear VolterraIntegro-Fractional Differential Equation (VIFDE) of order $n\alpha$ for $0<\alpha\leq 1$and $n\in\mathbb{N}$. This technique is based on the Generalized Taylor matrix method. We convert this equation to a system of linear algebraic equations after using collocation points; finally apply Gaussian elimination method to determine the fractional Taylor coefficients. Hence, the truncated generalized Taylor series approach is obtained. Algorithm for solving VIFDEs using above process have been developed, in order to express these solutions, program is written in MatLab (V7.6). Finally,several illustrative examples are presented to show the effectiveness and accuracy of this method.

Keywords: Integro-Fractional Differential Equation, Generalized Taylor's Method, Collocation Points, Caputo Fractional Derivative.

1 Introduction

In this paper, we consider the high-order linear VolterraIntegro- Fractional Differential Equation (VIFDE) of order $n\alpha$ for $0<\alpha\leq 1$ and $n\in\mathbb{N}$, with variable coefficients:

$$ {}_a^C D_x^{n\alpha} y(x) + \sum_{i=1}^{n-1} P_i(x)\, {}_a^C D_x^{(n-i)\alpha} y(x) + P_n(x) y(x) = f(x) + \lambda \int_a^x \sum_{\ell=0}^{m} k_\ell(x,t)\, {}_a^C D_t^{(m-\ell)\alpha} y(t)\, dt \qquad \ldots(1) $$

together with initial conditions:

$$[D_x^k y(x)]_{x=e} = y_k \quad ; k = 0,1,\dots,\mu-1 \,, a \le e \le b \qquad \dots (2)$$

where $x \in [a,b] = I$; $\mu = \max\{\lceil n\alpha \rceil, \lceil m\alpha \rceil\}$ and $y_k \in \mathbb{R}$ for all k; as well as, $f, P_i: I \to \mathbb{R}$ and $k_\ell: \mathcal{S} \times \mathbb{R} \to \mathbb{R}$ with $(\mathcal{S} = \{(x,t): a \le t \le x \le b\})$ denotes the given continuous functions, $y(x)$ is the unknown function which is the solution of (1), and λ is a scalar parameter.

The consider integro-fractional differential equation of Volterra type have been found to be effective to describe some applied sciences such as polymer physics, thermodynamics, electrical networks and bioengineering. The other larger filed which requires the use of it is the unsaturated behavior of the Free Electron Laser (FEL) [4, 6, 9, 10].

Taylor methods to find the approximate solutions of integral and integro-differential equations have been presented in many papers [3, 7, 8, 11, 13]. During recent years, a new generalized Taylor's formula that involves Caputo fractional derivatives was presented to solve Bayley-Torvik equations [14].

In this paper, we discuss the numerical solution of equation (1) by a new algorithm based on the Taylor collocation method [1], generalized Taylor's formula [15] and Caputo fractional derivative [4]. Applying the collocation points transforms the given linear VIFDE with variable coefficients and initial conditions to matrix equation, including unknown fractional Taylor coefficients. The coefficients of generalized Taylor's formula can be computed by means of the matrix equation and the computer package program MatLab (V7.6).

2 Preliminaries

For completeness, this part introduces the necessary definitions and important properties of fractional calculus theory [2, 4, 5, 12], which are used throughout this paper.

Definition 2.1:

A real valued function y defined on $[a,b]$ be in the space $C_\gamma[a,b], \gamma \in \mathbb{R}$, if there exists a real number $p > \gamma$, such that $y(x) = (x-a)^p y_*(x)$, where $y_* \in C[a,b]$, and it is said to be in the space $C_\gamma^n[a,b]$ iff $y^{(n)} \in C_\gamma[a,b], n \in \mathbb{N}_0$.

Definition 2.2:

Let $y \in C_\gamma[a,b], \gamma \ge -1$ and $\alpha \in \mathbb{R}^+$. Then the Riemann-Liouville fractional integral operator ${}_aJ_x^\alpha$ of order α of a function y, is defined as:

$${}_aJ_x^\alpha y(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} y(t)\, dt \quad , \quad \alpha > 0$$

$${}_aJ_x^0 y(x) = Iy(x) = y(x)$$

Definition 2.3:

Let $\alpha \geq 0$ and $m = \lceil \alpha \rceil$, (where $\lceil . \rceil$ is the ceiling function), the Riemann-Liouville fractional derivative operator ${}_{a}^{R}D_{x}^{\alpha}$, of order α and $y \in C_{-1}^{m}[a,b]$, is defined as:

$$ {}_{a}^{R}D_{x}^{\alpha} y(x) = D_{x}^{m} \ {}_{a}J_{x}^{m-\alpha} y(x) $$

If $\alpha = m, m \in \mathbb{N}_0$, and $y \in C^m [a,b]$ we have

$$ {}_{a}^{R}D_{x}^{0} y(x) = y(x) \qquad ; \qquad {}_{a}^{R}D_{x}^{m} y(x) = y^{(m)}(x) $$

Definition 2.4:

The Caputo fractional derivative operator ${}_{a}^{C}D_{x}^{\alpha}$, of order $\alpha \in \mathbb{R}^{+}$of a function $y \in C_{-1}^{m}[a,b]$ and $m-1 < \alpha \leq m$ $(m \in \mathbb{N})$ is defined as:

$$ {}_{a}^{C}D_{x}^{\alpha} y(x) = \ {}_{a}J_{x}^{m-\alpha} D_{x}^{m} y(x) $$

Thus for $\alpha = m, m \in \mathbb{N}_0$, and $y \in C^m [a,b]$, we have for all $a \leq x \leq b$

$$ {}_{a}^{C}D_{x}^{0} y(x) = y(x) \quad ; \quad {}_{a}^{C}D_{x}^{m} y(x) = D_{x}^{m} y(x) = \frac{d^m y(x)}{dx^m} $$

The most common properties of the fractional operator are listed below:

(i) ${}_{a}^{R}D_{x}^{\alpha} y(x) = D_{x}^{m} \ {}_{a}J_{x}^{m-\alpha} y(x) \neq \ {}_{a}J_{x}^{m-\alpha} D_{x}^{m} y(x) = {}_{a}^{C}D_{x}^{\alpha} y(x)$; $m = \lceil \alpha \rceil$

(ii) ${}_{a}^{C}D_{x}^{\alpha} y(x) = {}_{a}^{R}D_{x}^{\alpha}\left[y(x) - T_{m-1}[y;a]\right]$; $m-1 < \alpha \leq m$ and $T_{m-1}[y;a]$denotes the normal Taylor polynomial of degree $m-1$ for the function y centered at a.

(iii) ${}_{a}^{C,R}D_{x}^{\alpha}(c_1 f_1 \mp c_2 f_2)(x) = c_1 \, {}_{a}^{C,R}D_{x}^{\alpha} f_1(x) \mp c_2 \, {}_{a}^{C,R}D_{x}^{\alpha} f_2(x)$; c_1 and c_2 are constants .

(iv) ${}_{a}^{R}D_{x}^{\alpha} A = A \frac{(x-a)^{-\alpha}}{\Gamma(1-\alpha)}$and ${}_{a}^{C}D_{x}^{\alpha} A = 0$;$A$ is any constant and $\alpha \geq 0, \alpha \notin \mathbb{N}$.

(v) ${}_{a}^{C}D_{x}^{\alpha} \ {}_{a}J_{x}^{\alpha} y(x) = y(x)$ and

$$ {}_{a}J_{x}^{\alpha} \, {}_{a}^{C}D_{x}^{\alpha} y(x) = y(x) - \sum_{k=0}^{m-1} \frac{y^{(k)}(a)}{k!}(x-a)^k ; \ m = \lceil \alpha \rceil $$

We adopt Caputo's definition, which is a modification of the R-L definition and has the advantage of dealing properly with initial value problem, for the concept of the fractional derivative, [12].

Lemma 2.5, [4]**:**

Let $\alpha \geq 0$; $m = \lceil \alpha \rceil$ and for $y(x) = (x-a)^{\beta}$for some $\beta \geq 0$. Then:

$$ {}_{a}^{C}D_{x}^{\alpha} y(x) = \begin{cases} 0 & \text{if} \quad \beta \in \{0,1,2,\dots,m-1\} \\ \dfrac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}(x-a)^{\beta-\alpha} & \text{if} \quad \beta \in \mathbb{N} \text{ and } \beta \geq m \\ & \text{or} \quad \beta \notin \mathbb{N} \text{ and} \beta > m-1 \end{cases} $$

Theorem 2.6, [15]**:** (Generalized Taylor's Formula)

Suppose that ${}_{a}^{C}D_{x}^{k\alpha}y(x) \in C(a,b]$, for $k = 0,1,2,\dots,n+1$, where $0 < \alpha \leq 1$, then we have

$$y(x) = \sum_{i=0}^{n} \frac{(x-a)^{i\alpha}}{\Gamma(i\alpha+1)} \left[{}_{a}^{C}D_{x}^{i\alpha}y(x)\right]_{x=a} + R_{n}^{\alpha}(x,a)$$

with

$$R_{n}^{\alpha}(x,a) = \frac{(x-a)^{(n+1)\alpha}}{\Gamma\big((n+1)\alpha+1\big)} \left[{}_{a}^{C}D_{x}^{(n+1)\alpha}y(x)\right]_{x=\vartheta}, \vartheta \in [a,x], \forall x \in (a,b]$$

where

$${}_{a}^{C}D_{x}^{n\alpha} = {}_{a}^{C}D_{x}^{\alpha}\,{}_{a}^{C}D_{x}^{\alpha} \dots {}_{a}^{C}D_{x}^{\alpha}\,(n-\text{times})$$

3 Fundamental Matrix Relations

We assume that the solution of linear VIFDEs as formed in equation (1) is a truncated α-Caputo generalized Taylor's series. Let us first write equation (1) in the form:

$$\mathcal{D}_{*}^{\alpha}(x) = f(x) + \lambda V_{*}^{\alpha}(x) \qquad \dots (3)$$

where the sequential fractional differential part is

$$\mathcal{D}_{*}^{\alpha}(x) = \sum_{i=0}^{n} P_{i}(x)\,{}_{a}^{C}D_{x}^{(n-i)\alpha}\,y(x)\,;\; P_{0}(x) = 1 \qquad \dots (4)$$

and the Volterraintegro-fractional part is

$$V_{*}^{\alpha}(x) = \int_{a}^{x} \sum_{\ell=0}^{m} k_{\ell}(x,t)\,{}_{a}^{C}D_{t}^{(m-\ell)\alpha}\mathrm{y(t)}\,dt \qquad \dots (5)$$

Now we convert the solution $y(x)$ and its $k(\in \mathbb{Z}^{+})$-sequential α-Caputo fractional derivative ${}_{a}^{C}D_{x}^{k\alpha}y(x)$, parts $\mathcal{D}_{*}^{\alpha}$ and V_{*}^{α}, and the initial condition in equation (2) to matrix form.

3.1 Matrix Relations for $y(x)$ and ${}_{a}^{C}D_{x}^{k\alpha}y(x)$

We assume that the function $y(x)$ and k-th sequential α-Caputo fractional derivative ${}_{a}^{C}D_{x}^{k\alpha}y(x)$ can be expanded to α-Caputo generalized Taylor series about $x = \tau\ (a \leq \tau \leq b)$ as follows:

$$y(x) = \sum_{r=0}^{\infty} \Upsilon_{r}(x-\tau)^{r\alpha}\;;\; \Upsilon_{r} = \frac{1}{\Gamma(r\alpha+1)}\left[{}_{a}^{C}D_{x}^{r\alpha}y(x)\right]_{x=\tau} \qquad \dots (6)$$

and

$${}_{a}^{C}D_{x}^{k\alpha}y(x) = \sum_{r=0}^{\infty} \Upsilon_{r}^{(k)}(x-\tau)^{r\alpha} \qquad \dots (7)$$

where, for $k = 0$, ${}_{a}^{C}D_{x}^{0}y(x) = y(x)$ and $\Upsilon_{r}^{(0)} = \Upsilon_{r}$ for all $r = 0,1,2,\dots$.

First, we take α-Caputo derivative for equation (7) with respect to x, using lemma 2.5 and definition of sequential fractional derivative:

$$
\begin{aligned}
{}_{a}^{C}D_{x}^{(k+1)\alpha}y(x) = {}_{a}^{C}D_{x}^{\alpha}\left({}_{a}^{C}D_{x}^{k\alpha}y(x)\right) &= \sum_{r=0}^{\infty}\Upsilon_r^{(k)}\,{}_{a}^{C}D_{x}^{\alpha}(x-\tau)^{r\alpha} \\
&= \sum_{r=1}^{\infty}\Upsilon_r^{(k)}\frac{\Gamma(r\alpha+1)}{\Gamma((r-1)\alpha+1)}(x-\tau)^{(r-1)\alpha} \quad ; \ (r=r-1) \\
&= \sum_{r=0}^{\infty}\Upsilon_r^{(k)}\frac{\Gamma((r+1)\alpha+1)}{\Gamma(r\alpha+1)}(x-\tau)^{r\alpha} \qquad \ldots(8)
\end{aligned}
$$

from expression (7), we can see that

$$
{}_{a}^{C}D_{x}^{(k+1)\alpha}y(x) = \sum_{r=0}^{\infty}\Upsilon_r^{(k+1)}(x-\tau)^{r\alpha} \qquad \ldots(9)
$$

To get the recurrence relation between the fractional Taylor's coefficients $\Upsilon_r^{(k)}$ and $\Upsilon_r^{(k+1)}$ of ${}_{a}^{C}D_{x}^{k\alpha}y(x)$ and ${}_{a}^{C}D_{x}^{(k+1)\alpha}y(x)$ respectively, we use the equality between the relations (8) and (9). Thus

$$
\Upsilon_r^{(k+1)} = \Upsilon_r^{(k)}\frac{\Gamma\big((r+1)\alpha+1\big)}{\Gamma(r\alpha+1)} \ ; \quad r,k = 0,1,2,\ldots \qquad \ldots(10)
$$

we now take $r = 0,1,\ldots,N$, and assume $\Upsilon_r^{(k)} = 0$ for $r > N$. Then we can put the recurrence relation (10) in the matrix form:

$$
\mathcal{A}^{(k+1)} = \mathcal{M}\mathcal{A}^{(k)} \ , \quad k = 0,1,2,\ldots \qquad \ldots(11)
$$

where

$$
\mathcal{A}^{(k)} = \begin{bmatrix}\Upsilon_0^{(k)} & \Upsilon_1^{(k)} & \Upsilon_2^{(k)} & \cdots & \Upsilon_N^{(k)}\end{bmatrix}^t
$$

and

$$
\mathcal{M} = \begin{bmatrix}
0 & \Gamma(\alpha+1) & 0 & & 0 \\
0 & 0 & \dfrac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)} & \cdots & 0 \\
 & \vdots & & \ddots & \vdots \\
0 & 0 & 0 & \cdots & \dfrac{\Gamma(N\alpha+1)}{\Gamma\big((N-1)\alpha+1\big)} \\
0 & 0 & 0 & & 0
\end{bmatrix}_{(N+1)\times(N+1)} \qquad \ldots(12)
$$

By substituting $k = 0,1,2,\ldots$ into equation (11) we have the following matrix relations:

$$
\begin{aligned}
\mathcal{A}^{(1)} &= \mathcal{M}\mathcal{A}^{(0)} = M\mathcal{A} \\
\mathcal{A}^{(2)} &= \mathcal{M}\mathcal{A}^{(1)} = \mathcal{M}(\mathcal{M}\mathcal{A}) = \mathcal{M}^2\mathcal{A} \\
&\ \vdots \\
\mathcal{A}^{(k)} &= \mathcal{M}^k\mathcal{A} \qquad \ldots(13)
\end{aligned}
$$

Which is the recurrence relation matrices between the fractional Taylor coefficient matrix $\mathcal{A}$ of $y(x)$ and the fractional Taylor coefficient matrix $\mathcal{A}^{(k)}$ of the k-th sequential α-Caputo derivative of $y(x)$, ${}_a^C D_x^{k\alpha} y(x)$. Clearly that $\mathcal{A}^{(0)} = \mathcal{A} = [\Upsilon_0 \quad \Upsilon_1 \quad \cdots \quad \Upsilon_N]^t$.

Using matrix relation (13) expresses the k-th sequential α-Caputo derivative of $y(x)$ in equation (7) in the following matrix form:

$$ {}_a^C D_x^{k\alpha} y(x) = X^\alpha \mathcal{A}^{(k)} = X^\alpha \mathcal{M}^k \mathcal{A} \ ; \ \mathcal{M}^0 = I \qquad \dots(14)$$

where

$$X^\alpha = [1 \quad (x-\tau)^\alpha \quad (x-\tau)^{2\alpha} \quad \cdots \quad (x-\tau)^{N\alpha}]_{(N+1)}$$

Then substituting the collocation points defined by

$$x_i = x_0 + ih \, , i = \overline{0:N} \, , h = (b-a)/N \ ; \ x_0 = a \, , x_N = b \qquad \dots(15)$$

From matrix expression (14) and using (15), we obtain the matrix forms:

$$[{}_a^C D_x^{k\alpha} y(x)]_{x=x_i} = X_{x_i}^\alpha \mathcal{M}^k \mathcal{A} \ ; k = 0,1,\dots,N \qquad \dots(16)$$

Thus, we get a new matrix form

$$Y^{[k\alpha]} = \mathcal{C}^\alpha \mathcal{M}^k \mathcal{A} \qquad \dots(17)$$

where

$$Y^{[k\alpha]} = \left[[{}_a^C D_x^{k\alpha} y(x)]_{x=x_0} \quad [{}_a^C D_x^{k\alpha} y(x)]_{x=x_1} \quad \cdots \quad [{}_a^C D_x^{k\alpha} y(x)]_{x=x_N} \right]^t \qquad \dots(18)$$

$$\mathcal{C}^\alpha = \begin{bmatrix} X_{x_0}^\alpha \\ X_{x_1}^\alpha \\ \vdots \\ X_{x_N}^\alpha \end{bmatrix} = \begin{bmatrix} 1 & (x_0-\tau)^\alpha & (x_0-\tau)^{2\alpha} & \cdots & (x_0-\tau)^{N\alpha} \\ 1 & (x_1-\tau)^\alpha & (x_1-\tau)^{2\alpha} & & (x_1-\tau)^{N\alpha} \\ & \vdots & & \ddots & \vdots \\ 1 & (x_N-\tau)^\alpha & (x_N-\tau)^{2\alpha} & \cdots & (x_N-\tau)^{N\alpha} \end{bmatrix}_{(N+1)\times(N+1)} . (19)$$

In addition, from matrix equation (14), putting $k = 0$ and using the α-Caputo properties, we can write the matrix relation solution form as:

$$y(x) = X^\alpha \mathcal{A} \qquad \dots(20)$$

3.2 Matrix Relation for α-Caputo Differential Part $\mathcal{D}_*^\alpha(x)$

To derive a matrix form of $\mathcal{D}_*^\alpha(x)$ in the relation (4), first we substitute the collocation points (15) into α-Caputo fractional differential part to obtain the system:

$$\mathcal{D}_*^\alpha(x_k) = \sum_{i=0}^{n} P_i(x_k) \left[{}_a^C D_x^{(n-i)\alpha} y(x) \right]_{x=x_k} \ ; k = 0,1,\dots,N \qquad \dots(21)$$

The system (21) can be written in the matrix form:

$$\boldsymbol{D}_*^\alpha = \sum_{i=0}^{n} P_i Y^{[(n-i)\alpha]} \qquad \dots(22)$$

where

$$\left.\begin{array}{l} P_i = diag[P_i(x_0) \quad P_i(x_1) \quad \cdots \quad P_i(x_N)]_{(N+1)} \\ \boldsymbol{D}_*^{\alpha} = [\mathcal{D}_*^{\alpha}(x_0) \quad \mathcal{D}_*^{\alpha}(x_1) \quad \cdots \quad \mathcal{D}_*^{\alpha}(x_N)]^t \end{array}\right\} \qquad ...(23)$$

and $Y^{[k\alpha]}$; $k = \overline{0:n}$, are defined in equation (18). Then from equations (22 and 17), we obtain the matrix relation:

$$\boldsymbol{D}_*^{\alpha} = \left\{\sum_{i=0}^{n} P_i \mathcal{C}^{\alpha} \mathcal{M}^{n-i}\right\} \mathcal{A} \qquad ...(24)$$

3.3 Matrix Relation for α -Volterra Integro-Fractional Differential Part $V_*^{\alpha}(x)$

First: the kernel functions $k_{\ell}(x,t)$ in the relation (5) can be approximated by a truncated normal Taylor series of degree N_1 about $x = \tau$, $t = \tau$ $(a \leq \tau \leq b)$ in the form:

$$k_{\ell}(x,t) = \sum_{r=0}^{N_1}\sum_{p=0}^{N_1} \mathcal{K}_{rp}^{\ell}(x-\tau)^r (t-\tau)^p \quad , \ell = 0,1,\dots,m \qquad ...(25)$$

where

$$\mathcal{K}_{rp}^{\ell} = \frac{1}{r!\,p!}\frac{\partial^{r+p} k_{\ell}(x,t)}{\partial x^r \partial t^p}\bigg|_{(x=\tau,t=\tau)} \quad ; \; r,p = 0,1,\dots,N_1 \qquad ...(26)$$

The expression (25) can be put in the matrix form

$$[k_{\ell}(x,t)] = X K_{\ell} T^t \quad ; \quad \ell = 0,1,\dots,m \qquad ...(27)$$

where

$$X = [1 \quad (x-\tau) \quad (x-\tau)^2 \quad \cdots \quad (x-\tau)^{N_1}]_{(N_1+1)}$$
$$T = [1 \quad (t-\tau) \quad (t-\tau)^2 \quad \cdots \quad (t-\tau)^{N_1}]_{(N_1+1)}$$

and

$$K_{\ell} = \left[\mathcal{K}_{rp}^{\ell}\right] = \begin{bmatrix} \mathcal{K}_{00}^{\ell} & \mathcal{K}_{01}^{\ell} & \cdots & \mathcal{K}_{0N_1}^{\ell} \\ \mathcal{K}_{10}^{\ell} & \mathcal{K}_{11}^{\ell} & & \mathcal{K}_{1N_1}^{\ell} \\ \vdots & & \ddots & \vdots \\ \mathcal{K}_{N_1 0}^{\ell} & \mathcal{K}_{N_1 1}^{\ell} & \cdots & \mathcal{K}_{N_1 N_1}^{\ell} \end{bmatrix}_{(N_1+1)\times(N_1+1)} \qquad ...(28)$$

Second: substituting the matrix forms (14) and (27) corresponding to the functions ${}_{a}^{C}D_t^{(m-\ell)\alpha} y(t)$ and $k_{\ell}(x,t)$ into the equation (5), $V_*^{\alpha}(x)$, we have the matrix relation:

$$\begin{aligned} [V_*^{\alpha}(x)] &= \int_a^x \left\{X \sum_{\ell=0}^{m} K_{\ell} T^t T^{\alpha} \mathcal{M}^{m-\ell} \mathcal{A}\right\} dt \\ &= X\left(\sum_{\ell=0}^{m} K_{\ell} \mathcal{H}(x) \mathcal{M}^{m-\ell}\right) \mathcal{A} \qquad ...(28) \end{aligned}$$

where

$$T^{\alpha} = [1 \quad (t-\tau)^{\alpha} \quad (t-\tau)^{2\alpha} \quad \cdots \quad (t-\tau)^{N\alpha}]^{t}$$

$$\mathcal{H}(x) = [h_{i,j}(x)] = \int_a^x T^t T^{\alpha}\, dt \ ;$$

$$h_{i,j}(x) = \left.\frac{(x-\tau)^{i+j\alpha+1} - (a-\tau)^{i+j\alpha+1}}{i+j\alpha+1}\right|_{\substack{i=0,1,\ldots,N_1 \\ j=0,1,\ldots,N}} \qquad \ldots (29)$$

At last, we use the collocation points in equation (15), to obtain the matrix forms:

$$\boldsymbol{V}_{*}^{\alpha} = \boldsymbol{\mathcal{C}}\left(\sum_{\ell=0}^{m} \boldsymbol{K}_{\ell}\mathcal{H}\boldsymbol{\mathcal{M}}^{m-\ell}\right)\mathcal{A} \qquad \ldots (30)$$

where $\boldsymbol{\mathcal{C}}, \boldsymbol{K}_{\ell}$, $\mathcal{H}$ and $\boldsymbol{\mathcal{M}}^{m-\ell}$ matrices and can be written by blocked matrices as follows:

$$\mathcal{H} = diag[\mathcal{H}(x_0) \quad \mathcal{H}(x_1) \quad \cdots \quad \mathcal{H}(x_N)]_{(N+1)} \qquad \ldots (31)$$

$$\boldsymbol{\mathcal{M}}^{\vartheta} = [\mathcal{M}^{\vartheta} \quad \mathcal{M}^{\vartheta} \quad \ldots \quad \mathcal{M}^{\vartheta}]_{(N+1)}^{t} \ , \ \ \vartheta = m-\ell$$

$$\boldsymbol{K}_{\ell} = diag[K_{\ell} \quad K_{\ell} \quad \cdots \quad K_{\ell}]_{(N+1)}$$

$$\boldsymbol{\mathcal{C}} = diag\left[X_{x_0} \quad X_{x_1} \quad \cdots \quad X_{x_N}\right]_{(N+1)}$$

where

$$X_{x_i} = [1 \quad (x_i-\tau) \quad (x_i-\tau)^2 \quad \cdots \quad (x_i-\tau)^{N_1}]_{(N_1+1)}$$

3.4 Matrix Relation for the Initial Conditions

Here we can use the matrix relation (14) to obtain the corresponding matrix form for initial condition (2). Using equation (14) with definition 2.4 and specially putting $x = e, (a \le e \le b)$:

$$[y(x)|_{x=e}] = [1 \quad (e-\tau)^{\alpha} \quad (e-\tau)^{2\alpha} \quad \cdots \quad (e-\tau)^{N\alpha}]\mathcal{M}^0\mathcal{A} = E\mathcal{M}^0\mathcal{A}$$

Therefore, for $x = e$, the k-sequential α-Caputo fractional derivative, ${}_{a}^{C}D_x^{k\alpha}y(x)$, can be given in the matrix form:

$$[{}_{a}^{C}D_x^{k\alpha}y(x)|_{x=e}] = E\mathcal{M}^k\mathcal{A} \qquad \ldots (32)$$

Substituting (32) into (2), to obtain

$$E\mathcal{M}^k\mathcal{A} = y_k \ ; \quad k = 0,1,\ldots,\mu-1$$

Now, taking

$$S_k = E\mathcal{M}^k = [s_{k0} \quad s_{k1} \quad \cdots \quad s_{kN}] \qquad \ldots (33)$$

Finally, condition (2) becomes

$$S_k\mathcal{A} = [y_k] \quad or \quad [S_k; y_k] \qquad \ldots (34)$$

$$k = 0,1,\ldots,\mu-1 \ ; \ \mu = \max\{\lceil n\alpha \rceil, \lceil m\alpha \rceil\}$$

4 The Method

In this section, first we construct the fundamental matrix equation corresponding to high-order linear VIFDE with variable coefficients (1), and secondly we addthe initial conditions (2) to this matrix, and then solve this linear algebra equation to find the fractional generalized Taylor's coefficients which is the solution of equations (1-2) after putting init equation (20).

Inserting the matrix relations (24) and (30) into equation (3), we obtain the fundamental matrix equation:

$$\left\{\sum_{i=0}^{n} P_i \mathcal{C}^{\alpha} \mathcal{M}^{n-i} - \lambda \boldsymbol{\mathcal{C}} \sum_{\ell=0}^{m} \boldsymbol{K}_{\ell} \mathcal{H} \boldsymbol{\mathcal{M}}^{m-\ell}\right\} \mathcal{A} = \mathcal{F} \qquad \ldots(35)$$

where

$$\mathcal{F} = [f(x_0) \quad f(x_1) \quad \cdots \quad f(x_N)]^t \qquad \ldots(36)$$

and$P_i, \mathcal{C}^{\alpha}, \mathcal{M}, K_{\ell}(\ell = 0,1,\ldots,m)$ and $\mathcal{H}$ are defined in equations (23,19,12,28,31) respectively. Briefly we can write (35) in the form:

$$\mathcal{R}\mathcal{A} = \mathcal{F} \quad or \quad [\mathcal{R};\mathcal{F}] \qquad \ldots(37)$$

that corresponds to a linear algebraic system of $(N+1)$ equations with the $(N+1)$ unknown fractional generalized Taylor coefficients $(\Upsilon_0, \Upsilon_1, \ldots, \Upsilon_N)$ where

$$\mathcal{R} = \left[R_{pq}\right]_{(p,q=\overline{0:N})} = \sum_{i=0}^{n} P_i \mathcal{C}^{\alpha} \mathcal{M}^{n-i} - \lambda \boldsymbol{\mathcal{C}} \sum_{\ell=0}^{m} \boldsymbol{K}_{\ell} \mathcal{H} \boldsymbol{\mathcal{M}}^{m-\ell}$$

Finally, to find the unknown fractional coefficient in truncated generalized Taylor formula (20) which is the approximate solution of problem (1) with condition (2), by replacing the rows of matrix (34) by the last μ-rows of the matrix (37), we have the required augmented matrix

$$[\tilde{\mathcal{R}};\tilde{\mathcal{F}}] = \begin{bmatrix} R_{00} & R_{01} & & R_{0N} & ; & f(x_0) \\ R_{10} & R_{11} & \cdots & R_{1N} & ; & f(x_1) \\ \vdots & \vdots & & \vdots & \vdots & \vdots \\ R_{N-\mu,0} & R_{N-\mu,1} & \cdots & R_{N-\mu,N} & ; & f(x_{N-\mu}) \\ s_{00} & s_{01} & & s_{0N} & ; & y_0 \\ s_{10} & s_{11} & \cdots & s_{1N} & ; & y_1 \\ \vdots & \vdots & & \vdots & \vdots & \vdots \\ s_{\mu-1,0} & s_{\mu-1,1} & \cdots & s_{\mu-1,N} & ; & y_{\mu-1} \end{bmatrix} \qquad \ldots(38)$$

or, the corresponding matrix equation

$$\tilde{\mathcal{R}}\mathcal{A} = \tilde{\mathcal{F}} \qquad \ldots(39)$$

If $rank\, \tilde{\mathcal{R}} = rank\, [\tilde{\mathcal{R}};\tilde{\mathcal{F}}] = N+1$ then by Gaussian elimination the coefficients $\Upsilon_r, r = 0,1,\ldots,N$ in matrix $\mathcal{A}$ are uniquely determined by equation (38). Thus the linear VIFDE (1) with initial conditions (2) has a unique solution, which is given by the truncated fractional generalized Taylor series

$$\tilde{y}(x) = \sum_{r=0}^{N} \Upsilon_r (x-\tau)^{r\alpha} + R_N^{\alpha}(x,\tau); \quad a \le x, \tau \le b, N \ge \max\{n,m\}$$

Also, if $rank\,\tilde{\mathcal{R}} = rank\,[\tilde{\mathcal{R}};\tilde{\mathcal{F}}] < N+1$, then the proposed method fails to provide a solution, but in this case, the number of collocation points (or equivalently the dimension of the matrix $\tilde{\mathcal{R}}$) can be increased to find the particular or general solution.

The Algorithm [GTM-V]:

Step 1:

a. Input the number of truncated generalized Taylor series N such that $(\geq \max\{n,m\})$and truncated normal Taylor series N_1.

b. Assume $h = (b-a)/N$, $(N \in \mathbb{N})$.

c. Put y_k initial conditions, $k = 0,1,\dots,\mu-1$; $\mu = \max\{\lceil n\alpha \rceil, \lceil m\alpha \rceil\}$.

Step 2: Determine the matrices $\mathcal{M}$ and $K_\ell(\ell = \overline{0:m})$ from the equations(12) and (28) respectively.

Step 3:

a. Set the collocation points $x_k = x_0 + kh, k = \overline{0:N}, x_0 = a, x_N = b$

b. Evaluate the matrices $P_i(i = 0,1,\dots,n), \mathcal{F}$ and $\mathcal{C}^\alpha$ from the equations (23), (36) and (19) respectively.

c. Compute the matrix $\mathcal{H}(x) = [h_{i,j}(x)]$ from equation (29) for all $x = x_k, (k = \overline{0:N})$ and each $i = \overline{0:N_1}$; $j = \overline{0:N}$.

Step 4: Construct the conditional μ-row matrix $S_k(k = 0,1,\dots,\mu-1)$ from equation (33).

Step 5: Construct the matrices $\tilde{\mathcal{R}}$ and $\tilde{\mathcal{F}}$ which are represented in equ. (38)

Step 6: Solve the system (39) for fractional generalized Taylorcoefficients $\Upsilon_r\ (r = \overline{0:N})$using Gaussian elimination method.

Step 7: Substituting all Υ_r ‘s into truncated generalized Taylor series (3) toobtain the approximate solution $\tilde{y}(x)$ of $y(x)$.

5 Numerical Experiment

In this section, we select some examples in which the exact solution already exists to show the accuracy, efficiency and effectiveness of the proposed algorithm [GTM-V]. All of them were performed on the computer using a program written in MatLab (V7.6). The least square errors in tables are the values of $\sum_{k=0}^{M}[y(x_k) - \tilde{y}_N(x_k)]^2, M \in \mathbb{N}$ at M-selected points x_k.

Example 1:

We first consider a high-order linear VIFDE with variable coefficients

$$ {}_0^C D_x^{0.4} y(x) + x\,{}_0^C D_x^{0.2} y(x) - 2y(x) = f(x) + \int_0^x [(x-2t^2)\,{}_0^C D_t^{0.2} y(t) + (tx^2-1)y(t)]dt $$

$$f(x) = \frac{2}{3}x^5 - \frac{1}{2}x^4 - x^2 + 5x - 2 - \frac{100}{\Gamma(0.8)}\left(\frac{1}{76}x^2 - \frac{1}{72}x - \frac{1}{40}\right)x^{1.8} - \frac{10}{3\Gamma(0.6)}x^{0.6}$$

together with initial condition: $y(0) = 1$; $0 \le x \le 1$.

Now we try to find the approximate solution $\tilde{y}(x)$ by truncated $(N = 5)$ generalized Taylor series around $x = \tau = 0$:

$$\tilde{y}(x) = \sum_{r=0}^{5} \Upsilon_r x^{r\alpha} \quad ; \quad \Upsilon_r = \frac{1}{\Gamma(r\alpha + 1)} \left[{}_a^C D_x^{r\alpha} y(x)\right]_{x=0}$$

where $\alpha = 0.2$, $n = 2$, $m = 1$, $a = 0$, $b = 1$, $e = 0$, $\lambda = 1$; and

$$P_0(x) = 1 , \quad P_1(x) = x , \quad P_2(x) = -2$$

$$k_0(x,t) = x - 2t^2, \quad k_1(x,t) = tx^2 - 1$$

Then, for $N = 5$ and $N_1 = 3$, the matrix equation (35) became:

$$\{P_0\mathcal{C}^{\alpha}\mathcal{M}^2 + P_1\mathcal{C}^{\alpha}\mathcal{M}^1 + P_2\mathcal{C}^{\alpha}\mathcal{M}^0 - \boldsymbol{\mathcal{C}}(\boldsymbol{K}_0\mathcal{H}\boldsymbol{\mathcal{M}}^1 + \boldsymbol{K}_1\mathcal{H}\boldsymbol{\mathcal{M}}^0)\}\,\mathcal{A} = \mathcal{F}$$

where $P_0, P_1, P_2; \mathcal{C}^{\alpha}, \mathcal{M}$ are matrices of order (6×6) defined by:

$$P_0 = I, P_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.8 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1.0 \end{bmatrix}, P_2 = \begin{bmatrix} -2 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 \end{bmatrix}$$

$$\mathcal{C}^{\alpha} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & (0.2)^{\alpha} & (0.2)^{2\alpha} & (0.2)^{3\alpha} & (0.2)^{4\alpha} & (0.2)^{5\alpha} \\ 1 & (0.4)^{\alpha} & (0.4)^{2\alpha} & (0.4)^{3\alpha} & (0.4)^{4\alpha} & (0.4)^{5\alpha} \\ 1 & (0.6)^{\alpha} & (0.6)^{2\alpha} & (0.6)^{3\alpha} & (0.6)^{4\alpha} & (0.6)^{5\alpha} \\ 1 & (0.8)^{\alpha} & (0.8)^{2\alpha} & (0.8)^{3\alpha} & (0.8)^{4\alpha} & (0.8)^{5\alpha} \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

$$\mathcal{M} = \begin{bmatrix} 0 & \Gamma(\alpha+1) & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\Gamma(3\alpha+1)}{\Gamma(2\alpha+1)} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{\Gamma(4\alpha+1)}{\Gamma(3\alpha+1)} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{\Gamma(5\alpha+1)}{\Gamma(4\alpha+1)} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and $\mathcal{F}$ is a vector of order (1×6) defined by:

$$\mathcal{F} = [-2 \quad -2.000633 \quad -1.813402 \quad -1.753459 \quad -1.866603 \quad -2.156238]^t$$

Also, $\boldsymbol{\mathcal{C}}, \boldsymbol{K}_{\ell}, \mathcal{H}$ and $\boldsymbol{\mathcal{M}}$ are block matrices defined by equations (31) respectively, and here

$$\boldsymbol{C} = diag\begin{bmatrix} X_{x_0=0} & X_{x_1=0.2} & X_{x_2=0.4} & X_{x_3=0.6} & X_{x_4=0.8} & X_{x_5=1} \end{bmatrix}$$
$$\boldsymbol{K}_\ell = \begin{bmatrix} K_0 & 0 \\ 0 & K_1 \end{bmatrix}$$
$$\boldsymbol{\mathcal{M}}^{\vartheta} = [\mathcal{M}^{\vartheta} \quad \mathcal{M}^{\vartheta} \quad \mathcal{M}^{\vartheta} \quad \mathcal{M}^{\vartheta} \quad \mathcal{M}^{\vartheta} \quad \mathcal{M}^{\vartheta}]^{t}, \qquad \vartheta = 0 \; and \; 1$$

where

$$X_{x_0=0} = [1 \quad 0 \quad 0 \quad 0]$$
$$X_{x_1=0.2} = [1 \quad 0.2 \quad 0.04 \quad 0.008]$$
$$X_{x_2=0.4} = [1 \quad 0.4 \quad 0.16 \quad 0.064]$$
$$X_{x_3=0.6} = [1 \quad 0.6 \quad 0.36 \quad 0.216]$$
$$X_{x_4=0.8} = [1 \quad 0.8 \quad 0.64 \quad 0.512]$$
$$X_{x_1=1} = [1 \quad 1 \quad 1 \quad 1]$$

$$K_0 = \begin{bmatrix} 0 & 0 & -2 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad K_1 = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathcal{H}(x_0) = 0_{4\times 6}$$
$$\mathcal{H}(x_1) = \begin{bmatrix} 0.2 & 0.120797 & 0.075044 & 0.047591 & 0.030661 & 0.02 \\ 0.02 & 0.013178 & 0.008755 & 0.005857 & 0.003942 & 0.002667 \\ 0.002667 & 0.001812 & 0.001236 & 0.000846 & 0.000581 & 0.0004 \\ 0.0004 & 0.000276 & 0.000191 & 0.000132 & 9.198198 & 6.4e-05 \end{bmatrix}$$
$$\mathcal{H}(x_2) = \begin{bmatrix} 0.4 & 0.277518 & 0.198041 & 0.14427 & 0.106767 & 0.08 \\ 0.08 & 0.060549 & 0.04621 & 0.035513 & 0.027454 & 0.021333 \\ 0.021333 & 0.016651 & 0.013047 & 0.010259 & 0.008092 & 0.0064 \\ 0.0064 & 0.005075 & 0.004033 & 0.003212 & 0.002562 & 0.002048 \end{bmatrix}$$
$$\mathcal{H}(x_3) = \begin{bmatrix} 0.6 & 0.451440 & 0.349368 & 0.276008 & 0.221513 & 0.18 \\ 0.18 & 0.147744 & 0.122279 & 0.101911 & 0.085441 & 0.072 \\ 0.072 & 0.060944 & 0.051789 & 0.044161 & 0.037774 & 0.0324 \\ 0.0324 & 0.027860 & 0.024011 & 0.020737 & 0.017943 & 0.015552 \end{bmatrix}$$
$$\mathcal{H}(x_4) = \begin{bmatrix} 0.8 & 0.637568 & 0.522634 & 0.437345 & 0.371783 & 0.32 \\ 0.32 & 0.278212 & 0.243896 & 0.215308 & 0.191203 & 0.170667 \\ 0.170667 & 0.153016 & 0.137729 & 0.124400 & 0.112709 & 0.1024 \\ 0.1024 & 0.093267 & 0.085142 & 0.077885 & 0.071382 & 0.065536 \end{bmatrix}$$
$$\mathcal{H}(x_5) = \begin{bmatrix} 1 & 0.833333 & 0.714286 & 0.625 & 0.555556 & 0.5 \\ 0.5 & 0.454545 & 0.416667 & 0.384615 & 0.357143 & 0.333333 \\ 0.333333 & 0.3125 & 0.294117 & 0.277778 & 0.263158 & 0.25 \\ 0.25 & 0.238095 & 0.227273 & 0.217391 & 0.208333 & 0.2 \end{bmatrix}$$

From (34), the matrices for conditions are computed as:

$$[S_0; y_0] = [1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad ; \quad 1]$$

Substituting the above matrices for fundamental equation, we have the augmented matrix based on condition which is:

$$[\tilde{\mathcal{R}}; \tilde{\mathcal{F}}] = \begin{bmatrix} -2 & 0 & 0.887264 & 0 & 0 & 0 & ; & -2 \\ -1.8008 & -1.177486 & 0.031579 & 0.084391 & 0.101253 & 0.099916 & ; & -2.000633 \\ -1.6128 & -1.137741 & -0.061656 & 0.020342 & 0.070936 & 0.100301 & ; & -1.813402 \\ -1.4648 & -1.054932 & -0.058241 & 0.031685 & 0.097176 & 0.1443296 & ; & -1.753459 \\ -1.4048 & -0.992884 & -0.033240 & 0.074032 & 0.160529 & 0.230902 & ; & -1.866603 \\ 1 & 0 & 0 & 0 & 0 & 0 & ; & 1 \end{bmatrix}$$

Solving this system, generalized Taylor coefficients are obtained

$$\mathcal{A} = [1.0 \quad -0.883929\mathrm{e}-14 \quad -0.783885\mathrm{e}-15 \quad 0.451208\mathrm{e}-12 \quad -.952065\mathrm{e}-12 \quad -2.0]^t$$

Substituting the elements $\Upsilon_r\left(r = \overline{0{:}5}\right)$ for truncated equation (3), we get the approximate solution $\tilde{y}(x)$ of $y(x)$:

$$\tilde{y}(x) = 1 - 2x$$

This coincides with the exact solution.

Example 2:

Let us consider the linear VIFDE on $0 \le x \le 1$:

$$ {}_0^C D_x^{2\alpha} y(x) - \frac{1}{2}\, {}_0^C D_x^{\alpha} y(x) + (1 + x^2) y(x) = f(x) + \int_0^x [xt\, {}_0^C D_t^{2\alpha} y(t) + (x^2 - t)\, {}_0^C D_t^{\alpha} y(t) + e^{x+t} y(t)]\, dt $$

$$ f(x) = x^5 + x^3 - x^2 - 1 - 7e^x - e^{2x}(x^3 - 3x^2 + 6x - 7) + \frac{6}{\Gamma(4-2\alpha)}\left(1 - \frac{1}{5-2\alpha}x^3\right)x^{3-2\alpha} + \frac{3}{\Gamma(4-\alpha)}\left(\frac{2}{5-\alpha}x^2 - 1\right)x^{3-\alpha} - \frac{6}{\Gamma(5-\alpha)}x^{6-\alpha} $$

with initial conditions: $\begin{cases} if \quad 0 < \alpha < 0.5; \; y(0) = -1 \\ if \; 0.5 < \alpha \le 1; \quad y(0) = -1\,, y'(0) = 0 \end{cases}$

The exact solution of this problem is known $y(x) = x^3 - 1$.

Apply the algorithm [GTM-V], for $(N = 5 \; and \; N_1 = 5)$, obtain the fundamental matrix relation for $\alpha = 0.6$:

$$ \left\{ P_0 \mathcal{C}^\alpha \mathcal{M}^2 + P_1 \mathcal{C}^\alpha \mathcal{M}^1 + P_2 \mathcal{C}^\alpha \mathcal{M}^0 - \boldsymbol{\mathcal{C}}(\boldsymbol{K}_0 \mathcal{H} \boldsymbol{\mathcal{M}}^2 + \boldsymbol{K}_1 \mathcal{H} \boldsymbol{\mathcal{M}}^1 + \boldsymbol{K}_2 \mathcal{H} \boldsymbol{\mathcal{M}}^0) \right\} \mathcal{A} = \mathcal{F} $$

where

$$ P_0 \mathcal{C}^\alpha \mathcal{M}^2 + P_1 \mathcal{C}^\alpha \mathcal{M}^1 + P_2 \mathcal{C}^\alpha \mathcal{M}^0 = \begin{bmatrix} 1 & -0.446758 & 1.101803 & 0 & 0 & 0 \\ 1.04 & -0.050798 & 1.017815 & 0.661475 & 0.364998 & 0.184692 \\ 1.16 & 0.222655 & 1.132306 & 1.052335 & 0.858850 & 0.650431 \\ 1.36 & 0.554232 & 1.384755 & 1.511109 & 1.510388 & 1.425435 \\ 1.64 & 0.987733 & 1.817243 & 2.156601 & 2.435090 & 2.645676 \\ 2 & 1.553242 & 2.485248 & 3.115492 & 3.816633 & 4.572600 \end{bmatrix} $$

$$ \boldsymbol{\mathcal{C}}(\boldsymbol{K}_0 \mathcal{H} \boldsymbol{\mathcal{M}}^2 + \boldsymbol{K}_1 \mathcal{H} \boldsymbol{\mathcal{M}}^1 + \boldsymbol{K}_2 \mathcal{H} \boldsymbol{\mathcal{M}}^0) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0.270422 & 0.055105 & 0.018018 & 0.005827 & 0.001991 & 0.000705 \\ 0.733713 & 0.262428 & 0.139329 & 0.071227 & 0.038025 & 0.020845 \\ 1.497929 & 0.768137 & 0.526144 & 0.346863 & 0.237335 & 0.166157 \\ 2.726886 & 1.796470 & 1.451631 & 1.137877 & 0.922699 & 0.764367 \\ 4.667384 & 3.688934 & 3.36302 & 3.003534 & 2.769401 & 2.606586 \end{bmatrix} $$

and

$$\mathcal{F} = [-1 \quad -0.585590 \quad 0.203303 \quad 1.397271 \quad 2.968683 \quad 4.635446]^t$$

From equations (33 and 34), the matrix forms for initial conditions are:

$$S_k\mathcal{A} = [y_k] \quad or \quad [S_k; y_k]\,; \; k = 0 \; and \; 1$$

or clearly

$$[S_0; y_0] = \quad [1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad ; \quad -1]$$

$$[S_1; y_1] = \quad [0 \quad 0.893515 \quad 0 \quad 0 \quad 0 \quad 0 \quad ; \quad 0]$$

After the system of the augmented matrices and condition are computed, we get the new augmented matrix in the form

$$[\tilde{\mathcal{R}}; \tilde{\mathcal{F}}] = \begin{bmatrix} 1 & -0.446758 & 1.101803 & 0 & 0 & 0 & ; & -1 \\ 0.769578 & -0.105903 & 0.999797 & 0.655648 & 0.363007 & 0.183988 & ; & -0.585590 \\ 0.4262873 & -0.039773 & 0.992977 & 0.981108 & 0.820825 & 0.629587 & ; & 0.203303 \\ -0.137929 & -0.213905 & 0.858611 & 1.164246 & 1.273054 & 1.259278 & ; & 1.397271 \\ 1 & 0 & 0 & 0 & 0 & 0 & ; & -1 \\ 0 & 0.893515 & 0 & 0 & 0 & 0 & ; & 0 \end{bmatrix}$$

This system has the solution

$$\mathcal{A} = [-1.00004 \quad -0.103101\text{e}-3 \quad -0.194346\text{e}-4 \quad 0.271458\text{e}-2 \quad -0.665339\text{e}-2 \quad 1.00419]^t$$

and $\tilde{y}(x)$ is evaluated as:

$$\tilde{y}(x) = -1.00004 - 0.000103101\, x^{3/5} - 0.0000194346\, x^{6/5} + 0.00271458\, x^{9/5} - 0.00665339\, x^{12/5} + 1.00419 x^3$$

For $\alpha = 0.4$, we apply the algorithm [GTM-V] by the previous procedure to obtain the approximate function $\tilde{y}(x)$ for the solution of consider problem, take $N = N_1 = 6$;

$$\tilde{y}(x) = -1.00 - 0.0225457\, x^{2/5} - 0.0107382\, x^{4/5} - 0.721801\, x^{6/5} + 3.48758\, x^{8/5} - 6.13334 x^2 + 4.35416\, x^{12/5}$$

Table (1), for $\alpha = 0.6$, presents a comparison between the exact and approximate solution which depends on the least square error and running time with different values of $N \; and \; N_1$.

Table (1)

x	Exact Solution	Present Method for $N = 5$			
		$N_1 = 5$	$N_1 = 7$	$N_1 = 10$	$N_1 = 15$
0.0	-1	-1.00004	-1	-1	-1
0.1	-0.999	-0.999046	-0.9990002	-0.999	-0.999
0.2	-0.992	-0.992039	-0.9920001	-0.992	-0.992
0.3	-0.973	-0.973041	-0.9730002	-0.973	-0.973
0.4	-0.936	-0.936054	-0.9360006	-0.936	-0.936
0.5	-0.875	-0.875074	-0.8750012	-0.87500001	-0.875
0.6	-0.784	-0.784092	-0.7840019	-0.78400002	-0.784
0.7	-0.657	-0.657097	-0.6570026	-0.65700003	-0.657
0.8	-0.488	-0.488078	-0.4880030	-0.48800004	-0.488
0.9	-0.271	-0.271021	-0.2710029	-0.27100005	-0.271
1.0	0.0	$8.86e-5$	$-2.31e-6$	$-7.22e-8$	$-1.2e-13$
L.S.E		$0.414783e-006$	$0.336713e-009$	$0.816289e-013$	$0.241e-024$
R.T/Sec		0.460625	0.660062	1.081343	1.845121

For $\alpha = 0.4$, comparisons of numerical result with the exact solution are showed in table (2) for different values of N and N_1.

Table (2)

(N, N_1)	$N = 6$			$N = 9$		
	$N_1 = 6$	$N_1 = 8$	$N_1 = 10$	$N_1 = 6$	$N_1 = 8$	$N_1 = 10$
Error	0.60252 $e-01$	0.59104 $e-01$	0.59063 $e-01$	0.37327 $e-07$	0.17733 $e-07$	0.79683 $e-08$
R.T/Sec	0.57448	0.82439	1.03503	0.62802	0.85083	1.08504

6 Conclusion

To find analytically the exact solutions of high-order linear VIFDEs with variable coefficients are usually difficult and mostly impossible. For this purpose, we introduced a new numerical method for approximating the solution of such a problem, in which a generalized Taylor collocation method is applied in matrix form.

A considerable advantage of the method is that the generalized Taylor coefficients of the solution are found very easily by using computer programs. For this reason, this technique is much faster than the other methods.

Several examples are included for illustration and good results are achieved. We concluded that the numerical results show that the accuracy improves with increasing the truncation limit N, hence for better results, using large number N is recommended. Also, the truncation limit N_1 (the degree of the approximating kernel) must be chosen to be large enough.

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