Hydromagnetic Stability of Oscillating Hollow Jet

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Abstract

The hydromagnetic stability of a gas jet of negligible motion surrounded by an oscillating liquid has been discussed. A total second order integro-differential equation in the amplitude of the deflection wave has been derived from which a general dispersion relation is obtained and discussed. The oscillating liquid has stabilizing tendency. The axial magnetic fields pervaded in the liquid and gas jet regions have stabilizing effect, and this effect is true for all modes of perturbation. The capillary force is destabilizing only in a small axisymmetric domain while it is stabilizing in all other axisymmetric domains and all domains of non-axisymmetric perturbations. The destabilizing behavior of the model could be shrunked, reduced and suppressed, and then the stability sets in.

Keywords: Hydromagnetic stability, Hollow jet
1. Introduction

The classical studies of the capillary instability of a gas cylinder submerged into a liquid are given for first time by Chandrasekhar (1961) for axisymmetric perturbation. Drazin and Reid (1980) p. 16 gave the dispersion relation valid for all axisymmetric and non-axisymmetric modes. Cheng (1985) discussed the instability of a gas jet in an incompressible liquid for all modes of perturbation. However, we have to mention here that the results given by Cheng (1985), in Eqns. (4) and (5) are incorrect in the third term.

In fact the quantity \((1 - m^2 - k^2 R_0^2)\) must be in the numerator as it is clear from Eq. (3) there. See also equations (54)-(57) in the present work and Drazin's result (1980) p.16 and also Chandrasekhar; dispersion relation pp. 538 and pp. 540 [Eqns. (147) and (155) there]. Kendall (1986) performed experiments with modern equipment to check the breaking up of that model. Moreover, he (1986) attracted the attention for the importance of the stability and discussions of that model for its application in many domains of science. Concerning the hydrodynamic stability of a hollow jet endowed with surface tension we may refer to Chauhan et. al. (2000), Chen & Lin (2002), Cousin & Dumouchel (1966), Lee & Wang (1986) and (1989), Mehring & Sirignano (2000), Parthasrathy & Chiang (1998), Shen & Li (1996), Shi et al. (1999), Shukudov & Sisoev (1996) and Villermaux (1998). Soon afterwards a lot of researchers treated with the magnetodynamic stability of such model, see Radwan (1998), (2005) and (2009), analytically and numerically upon utilizing appropriate basic equations and boundary conditions.

In all foregoing works the liquid may be at rest or uniform streaming in the unperturbed state. Here we discuss the Hydromagnetic stability of an oscillating hollow jet (a gas jet surrounded by oscillating liquid).

2. Formulation of the problem

Consider a gas cylinder (of negligible motion) of radius \(R_0\) surrounded by an oscillating liquid with velocity

\[ \mathbf{u}_0 = (0, 0, U \cos \Omega t) \]  

(1)

where \(U\) and \(\Omega\) are the amplitude and oscillation frequency of the velocity. The interior cylinder is being a gas with constant pressure \(P_0^g\) and pervaded by the longitudinal magnetic field

\[ H_0^{gas} = (0, 0, \alpha H_0) \]  

(2)

The liquid is penetrated by the magnetic field

\[ H_0 = (0, 0, H_0) \]  

(3)

where \(H_0\) is the intensity of the magnetic field in the liquid and \(\alpha\) is parameter of \(H_0^{gas}\). The components of equations (1) – (3) are considered along the cylindrical coordinates \((r, \varphi, z)\) with the z-axis is coinciding with the axis of the hollow jet.
The model is acted by the inertia, pressure gradient, capillary and electromagnetic forces.

The hydromagnetic fundamental equations appropriate for studying the stability of the fluid model under consideration are the combination of the pure hydrodynamic equations and those of Maxwell concerning the electromagnetic theory. These equations may be given as follows.

In the gas cylinder
\[ \nabla \wedge H^g = 0 \quad \text{(there is no current)} \quad (4) \]
\[ \nabla \cdot H^g = 0 \quad (5) \]

In the liquid region,
\[ \rho \left( \frac{\partial}{\partial t} + (u \cdot \nabla) \right) u = - \nabla P + \mu (I \wedge H) \quad (6) \]
\[ I = \nabla \wedge H \quad (7) \]
\[ \nabla \cdot u = 0 \quad (8) \]
\[ \nabla \cdot H = 0 \quad (9) \]
\[ \frac{\partial H}{\partial t} = \nabla \wedge (u \wedge H) \quad (10) \]

Along the gas – liquid interface,
\[ P_s = -T (\nabla \cdot N) \quad (11) \]

Here \( H \) and \( H^g \) are the magnetic field intensities in the gas and liquid regions, \( P_s \) the curvature pressure due to the capillary force, \( T \) the surface tension coefficient, \( N \) the outward unit vector normal to gas - liquid interface an indicates as \( r \) (the radial cylindrical coordinate) does, \( \mu \) the magnetic permeability coefficient, \( I \) the electric current density, while \( \rho \), \( u \) and \( P \) are the liquid mass density, velocity vector and kinetic pressure. Upon using some vector identities in order to represent the electromagnetic force \( \mu \left( \nabla \wedge H \right) \wedge H \) into magnetic pressure \( (\mu/2) \nabla \left( H \cdot H \right) \) and magnetic tension \( \mu \left( H \cdot \nabla \right) H : \) equation (6) may be written as
\[ \rho \left( \frac{\partial u}{\partial t} + (u \cdot \nabla) u \right) - \mu \left( H \cdot \nabla \right) H = - \nabla \Pi \quad (12) \]
with
\[ \Pi = P + \frac{\mu}{2} \left( H \cdot H \right) \quad (13) \]

where \( \Pi \) is the total hydromagnetic pressure which is the sum of the liquid kinetic pressure and magnetic pressure. Also, equation (10) could be rewritten as
\[ \frac{\partial H}{\partial t} + (u \cdot \nabla) H = \left( H \cdot \nabla \right) u \quad (14) \]

The basic equations (4) - (14) are solved, with taking into account equations (1) – (3), in the unperturbed state and we have obtained
\[ \Pi_o = P_o + (\mu/2) H^g_o^2 = \text{const.} \quad (15) \]
\[ P_{os} = -\frac{T}{R_o} \quad (16) \]

Upon applying the balance of the pressures at \( r = R_o \) we could determine the value of the constant in equation (15) and we have finally, obtained
\[ P_o = -\frac{T}{R_o} + P_o^g + \frac{\mu H^g_o^2}{2} (\alpha^2 - 1) \quad (17) \]
Here $P_o^\theta$ is the gas constant pressure in the initial state, \(-\frac{T}{R_o}\) the contribution of the capillary force, while the third term in the right side of equation (17) is being the net magnetic pressure due to the effect of electromagnetic forces acting in the gas and liquid regions.

One has to refer here that in absence of the magnetic field as:

(i) $H_o = 0$ or/and

(ii) $\alpha = 1$

the constant gas pressure $P_o^\theta$ must be greater than \(-\frac{T}{R_o}\) in order that $P_o > 0$, otherwise the model collapses and there will not be a gas pervades into the liquid region.

In the general case, such that $P_o \geq 0$, the gas kinetic pressure $P_o^\theta$ in the initial state must satisfy the restriction:

$$P_o^\theta \geq \frac{T}{R_o} + \frac{\mu H_o^2}{2} (1 - \alpha^2)$$ (18)

otherwise the model collapses and it will be a homogeneous liquid medium.

### 3. Perturbation Analysis

For a small departure from the unperturbed state, based on the normal mode analysis technique, every variable quantity $Q$ $(r, \varphi, z, t)$ could be expanded as

$$Q(r, \varphi, z, t) = Q_o(r) + Q_1(r, \varphi, z, t) + \ldots \ldots$$ (19)

Here $Q(r, \varphi, z, t)$ stands for $u, P, H, H^\theta, N$ and $P_s$ with $Q_o(r)$ is the value of $Q(r, \varphi, z, t)$ in the unperturbed state, while $Q_1(r, \varphi, z, t)$ is a small increment of $Q(r, \varphi, z, t)$ due to perturbation. Based on the expansion (19), the deformation in the cylindrical interface is given by

$$r = R_o + \bar{\eta}(\varphi, z, t)$$ (20)

with

$$\bar{\eta}(\varphi, z, t) = \eta(t) \exp(i(kz + m\varphi))$$ (21)

is the elevation of the surface wave measured from the unperturbed level. From the point of view of the expansions (19) – (21) for the basic equations (4) – (14), the relevant linearized perturbation equations are given as follows.

In the gas region

$$\nabla \times H_1^\theta = 0$$ (22)

$$\nabla \cdot H_1^\theta = 0$$ (23)

In the liquid region

$$\rho \left( \frac{\partial u_1}{\partial t} + (u_5 \cdot \nabla)u_1 \right) - \mu \left( H_\varphi \cdot \nabla \right) H_3 = - \nabla \Pi_1$$ (24)

$$\Pi_1 = P_1 + \mu \left( H_\varphi \cdot H_3 \right)$$

(25)

$$\nabla \cdot u_1 = 0$$ (26)

$$\nabla \cdot H_3 = 0$$ (27)

$$\frac{\partial H_1}{\partial t} = \left( H_\varphi \cdot \nabla \right) u_1 - \left( u_\varphi \cdot \nabla \right) H_3$$ (28)

Along the gas – liquid interface

$$P_{1s} = \frac{T}{R_o^2} \left( \bar{\eta} + \frac{\partial^2 \bar{\eta}}{\partial \varphi^2} + R_o^2 \frac{\partial^2 \bar{\eta}}{\partial z^2} \right)$$ (29)
Based on the expansions (19) – (21), every fluctuating quantity \( Q_1(\rho, \phi, z, t) \) could be written as

\[
Q_1(\rho, \phi, z, t) = Q_1(r) \eta(t) \exp(i(kz + m\phi))
\]

(30)

where \( \eta(t) \) is the amplitude of the perturbed sinusoidal wave equation

\[
H_1 = i k H_o \left( \frac{\partial \eta}{\partial t} + i k \eta \frac{U}{\cos \Omega t} \right) \nu_1
\]

(31)

Since the liquid is assumed to be non–dissipative and irrotational, the velocity \( \nu_1 \) could be derived from a scalar function \( \phi_1 = (r, \phi, z, t) \) such that

\[
\nu_1 = \nabla \phi_1
\]

(32)

Combining equations (26) and (32), we get

\[
\nabla^2 \phi_1 = 0
\]

(33)

Equation (22) means that the magnetic field \( H_1^\theta \) could be derived from a scalar function, say \( \psi_1^\theta \), such that

\[
H_1^\theta = \nabla \psi_1^\theta
\]

(34)

Upon utilizing equation (23) with equation (34), we have

\[
\nabla^2 \psi_1^\theta = 0
\]

(35)

Substituting for \( \nu_o \) and \( H_o \) and acting by the divergence operator on the equation (24), we get

\[
\nabla^2 \Pi_1 = 0
\]

(36)

Now, we may see that the system of perturbed equations (22) – (29) could be solved, as Laplace's equations (33), (35) and (36) are solved.

For this task, we may write the expansion

\[
\xi_1(r, \phi, z, t) = \xi_1(r) \eta(t) \exp(i(kz + m\phi))
\]

(37)

for \( \phi_1(r, \phi, z, t) \), \( \psi_1^\theta(r, \phi, z, t) \) and \( \Pi_1(r, \phi, z, t) \). Substituting from (37) into equations (33), (35) and (36), we obtain the total second order differential equation of Bessel. For the problem under consideration, the non–singular solutions of (33), (35) and (36) are given by

\[
\phi_1(r, \phi, z, t) = c_1 \eta(t) K_m(k r) \exp(i(kz + m\phi))
\]

(38)

\[
\psi_1^\theta(r, \phi, z, t) = c_2 \eta(t) I_m(k r) \exp(i(kz + m\phi))
\]

(39)

\[
\Pi_1(r, \phi, z, t) = c_3 \eta(t) K_m(k r) \exp(i(kz + m\phi))
\]

(40)

where \( c_1, c_2 \) and \( c_3 \) are constants of integration to be determined, while \( I_m(k r) \) and \( K_m(k r) \) are the modified Bessel functions of the first and second kind of order \( m \).

Along the gas – liquid cylindrical fluid interface, the surface pressure in the perturbed state due to the capillary force, in view of equations (21) and (29), is given by

\[
P_{1s} = \frac{\tau}{R_o} (1 - m^2 - x^2) \eta(t) \exp(i(kz + m\phi))
\]

(41)

where \( x = k R_o \) is the longitudinal dimensionless wave number.
4. Boundary Conditions

The solution of the basic equations (4) – (14) in the unperturbed state given by equations (1) – (3) & (15) – (18) and in the perturbed state given by equations (31) – (41) must satisfy the boundary conditions of the problem. These boundary conditions are given as follows.

(i) The magnetodynamic condition

This condition is being "the normal component of the magnetic field must be continuous across the gas - liquid interface (20) at r = R_o". Mathematically this condition is given by

\[ N_0 \cdot H_1 + N_1 \cdot H_0 = N_0 \cdot H_1^0 + N_1 \cdot H_0^0 \]  

(42)

with

\[ N_0 = (1, 0, 0), \quad N_1 = \left(0, \frac{-i m}{R_o}, -i k\right) \eta \]  

(43)

Substituting from equations (2), (3), (31), (32), (33), (38), (39) and (43) into the condition (42), we get

\[ c_2 = \frac{i \times H_0}{l_m^0(x)} \]  

(44)

(ii) Kinematic conditions

The first condition states that "the normal component of the velocity \( \mathbf{u} \) of the liquid must be compatible with the velocity of the perturbed boundary gas - liquid (20) at r = R_o". This condition is being

\[ \frac{d f (r, \varphi, z, t)}{d t} = 0 \quad \text{i.e.} \quad \frac{d f}{d t} + (\mathbf{u} \cdot \nabla) f = 0 \]  

(45)

with

\[ f (r, \varphi, z, t) = r - R_o - \eta(t) \exp \left(i (k z + m \varphi)\right), \]  

(46)

from which, we have

\[ u_{1r} = \frac{d \eta}{d t} + i k U \eta \cos \Omega t \]  

(47)

Combining equations (47) and

\[ u_{1r} = \frac{\partial \phi_1}{\partial r} = c_1 k \eta(t) I_m^l(k r) \exp \left(i (k z + m \varphi)\right) \]  

(48)

we obtain

\[ c_1 = \frac{1}{\eta k I_m^l(x)} \left(\frac{d \eta}{d t} + i k U \eta \cos \Omega t\right) \]  

(49)

The second boundary condition could be described and utilized as follows. From equations (24) and (31), we have

\[ \rho \left(\frac{\partial u_{1r}}{\partial t} + U \cos \Omega t \frac{\partial u_{1r}}{\partial z}\right) + \frac{\mu H_0^2 k^2}{\eta k m(x)} \frac{\eta}{k} - \frac{\partial \eta}{\partial t} + i k U \eta \cos \Omega t - \frac{\partial u_{1r}}{\partial r} \]  

(50)

from which we get

\[ c_3 = \frac{-\rho}{\eta k m(x)} \left[\frac{d^2 \eta}{d t^2} + (2ikU \cos \Omega t)\frac{\eta}{d t} + \frac{\mu H_0^2 x^2}{R_o^2} \eta - (ikU \Omega \sin \Omega t + k^2 U^2 \cos^2 \Omega t) \eta\right] \]  

(51)
Stresses condition

The jump of the normal component of the stresses in the gas and liquid regions must be discontinuous by the surface pressure $P_{1s}$ across the cylindrical gas - liquid interface (20) at $r = R_o$. This condition is being

$$\Pi_1 + \tilde{\eta} \frac{\partial \Pi_0}{\partial r} - \left( \mu \left( H_0^g \cdot H_0^q \right) + \frac{\mu}{2} \tilde{\eta} \frac{\partial \left( 2H_0^g \cdot H_0^q \right)}{\partial r} \right) = P_{1s}$$

Substituting from the foregoing analysis about $\Pi_0, \Pi_1, H_0^g, H_0^q, \tilde{\eta}$ and $P_{1s}$ in the condition (53), yields

$$\frac{d^2 \eta}{dt^2} + (2i k U \cos \Omega t) \frac{d \eta}{dt} - (i k U \Omega \sin \Omega t + k^2 U^2 \cos^2 \Omega t) \eta = - \frac{T}{\rho R_o^2} \left( \frac{xK_m'(x)}{K_m(x)} \right) (1 - m^2 - x^2) + \frac{\mu H_o^2}{\rho R_o^2} \left[ -x^2 + \alpha^2 \frac{K_m(x)l_m(x)}{l_m'(x)K_m(x)} \right]$$

(53)

5. Results

Equation (53) is a total second order integro – differential equation in the amplitude $\eta$ of the deflection of the perturbation. Equation (53) relates $\eta$ with the modified Bessel functions $I_m(x), K_m(x)$ and their derivatives, the wave numbers $m$ and $x$, the amplitude $U$ of the streaming velocity, $\Omega$ the oscillation frequency of the oscillating streaming, $\alpha$ the parameter of the magnetic field in the gas cylinder and with the parameters $T, \rho, R_o, \mu$ and $H_o$ of the problem. One has to mention here that the relation (53) contains $(T/\rho R_o^2)$ as well as $(\mu H_o^2/\rho R_o^2)$ as unit of $(\text{time})^{-2}$ and this fact is vital in formulating (53) in dimensionless form.

If we neglect the secular term in equation (54) and rewrite it in Mathieu type equation (cf. McLachlan (1947) and also see Kelly (1965)), we may prove that the oscillating streaming of the liquid has stabilizing tendency.

If we assume that $\eta \sim \exp (\sigma t)$ with $\sigma$ is the growth rate, the general equation (53), yields

$$\sigma^2 + (2i k U \cos \Omega t) \sigma - (i k U \Omega \sin \Omega t + k^2 U^2 \cos^2 \Omega t) = - \frac{T}{\rho R_o^2} \left( \frac{xK_m'(x)}{K_m(x)} \right) (1 - m^2 - x^2) + \frac{\mu H_o^2}{\rho R_o^2} \left[ -x^2 + \alpha^2 \frac{K_m(x)l_m(x)}{l_m'(x)K_m(x)} \right]$$

(54)

This is the general dispersion relation of the present model.

If we assume that $\Omega = 0$, the dispersion relation (54) yields

$$(\sigma + i k U)^2 = - \frac{T}{\rho R_o^2} \left( \frac{xK_m'(x)}{K_m(x)} \right) (1 - m^2 - x^2) + \frac{\mu H_o^2}{\rho R_o^2} \left[ -x^2 + \alpha^2 \frac{K_m(x)l_m(x)}{l_m'(x)K_m(x)} \right]$$

(55)

The discussion of this equation reveal that the uniform streaming of the liquid has destabilizing effect, and that effect is valid not only in the axisymmetric mode ($m=0$) of perturbation but also in those of the non-axisymmetric modes ($m \geq 1$).

If we assume that $U = 0, \ \Omega = 0$ and $m \geq 0$, equation (54) degenerates to

$$\sigma^2 = \frac{T}{\rho R_o^2} \left( \frac{xK_m'(x)}{K_m(x)} \right) (m^2 + x^2 - 1) + \frac{\mu H_o^2}{\rho R_o^2} \left[ -x^2 + \alpha^2 \frac{K_m(x)l_m(x)}{l_m'(x)K_m(x)} \right]$$

(56)
The discussion of equation (56) as a global, show that the problem under consideration is stable if and only if
\[
(H_o/H_T)^2 \geq \frac{xK_m(x)K_m(x)(m^2+x^2-1)}{x^2I_m(x)K_m(x)-\alpha^2K_m(x)} (57)
\]
where \(H_T = (T/(\mu R_o))^{1/2}\) has a unit of magnetic field.

In order to examine the effects of the capillary and magnetodynamic forces on the stability of the present model, we have to write down about some properties of the modified Bessel functions.

Consider the recurrence relation (cf. Abramowitz and Stegun (1970))
\[
\frac{I_m(x)}{x^2} = 0.5 \left( I_{m-1}(x) + I_{m+1}(x) \right) \quad (58)
\]
\[
\frac{K_m(x)}{x} = 0.5 \left( -K_{m-1}(x) - K_{m+1}(x) \right) \quad (59)
\]
For each non-zero value of \(x \neq 0\), we know that \(I_m(x)\) is positive definite and monotonically increasing
\[
I_m(x) > 0 \quad (60)
\]
while \(K_m(x)\) is monotonically decreasing but never negative
\[
K_m(x) > 0 \quad (61)
\]
Utilizing the inequalities (60) and (61) for the recurrence relations (58) and (59), we see for each non-zero value of \(x\) that
\[
\frac{I_m(x)}{x^2} > 0 \quad (62)
\]
\[
\frac{K_m(x)}{x} < 0 \quad (63)
\]

Now, let us return to the general dispersion relation (54) to discuss the capillary instability and magnetodynamic stability of the present model.

As \(U = 0, \Omega = 0, H_o = 0\) and \(m = 0\), the relation (54) yields
\[
\sigma^2 = \frac{T}{\rho R_o^2} \left( \frac{xK_j(x)}{K_0(x)} \right) \left( 1 - x^2 \right) \quad (64)
\]
The relation (65) has been given for the first time by (Nobel Prize winner (1986)) Chandrasekhar (1981), which is the mirror case of a full liquid cylinder ambient in a gas medium.

As \(U = 0, \Omega = 0, H_o = 0\) and \(m \geq 0\), the general dispersion relation (54) becomes
\[
\sigma^2 = \frac{T}{\rho R_o^2} \left( \frac{xK_m'(x)}{K_m(x)} \right) (m^2 + x^2 - 1) \quad (65)
\]
This relation has been given by Drazin and Reid (1980) p. 16, which is valid for all axisymmetric and non-axisymmetric modes (\(m=0\) and \(m \geq 1\)) of perturbation.

The discussion of the dispersion relations (64) and (65) reveal that
\[
\frac{\sigma^2}{T/\rho R_o^2} > 0 \quad \text{as } 0 < x < 1 \quad \text{in the mode } m = 0 \quad (66)
\]
\[
\frac{\sigma^2}{T/\rho R_o^2} \leq 0 \quad \text{as } 1 \leq x < 0 \quad \text{in m = 0 mode} \quad \text{and as } 0 < x < \infty \quad \text{in m \geq 1 modes}. \quad (67)
\]
This means that the hollow gas jet is capillary unstable only in the axisymmetric mode \(m = 0\) in the small domain \(0 < x < 1\). While it is capillary stable in the domains \(1 \leq x < 0\) in the axisymmetric perturbation \(m = 0\) and all domains \(0 < x < \infty\) in non-axisymmetric perturbation modes \(m \geq 1\).

As \(U = 0, \Omega = 0, T = 0\) and \(m \geq 0\), the general dispersion relation (54) becomes
\[ \sigma^2 = \frac{\mu H_0^2}{\rho R_0^2} \left[ -x^2 + \alpha^2 \frac{K_m^I(x)Y_m(x)}{l_m^I(x)K_m(x)} \right] \] (68)

The axial magnetic field pervaded in the liquid is represented by the term \( \frac{\mu H_0^2}{\rho R_0^2} (-x^2) \). It has strong stabilizing effect and that effect is independent of the perturbed modes \( m = 0 \) and \( m \geq 1 \). The effect of the magnetic field pervaded in the gas cylinder is represented by the term \( \alpha^2 \frac{K_m^I(x)Y_m(x)}{l_m^I(x)K_m(x)} \) followed by \( \frac{\mu H_0^2}{\rho R_0^2} \). It has (see the inequalities (60) – (63)) strong stabilizing effect and this effect is valid for all axisymmetric mode \( m = 0 \) and non-axisymmetric modes \( m \geq 1 \).

So, we see that the present model is purely stabilizing under the acting electromagnetic forces in gas and liquid regions.

Therefore, we conclude that the stabilizing effects of the oscillating streaming and magnetodynamic force could be reduced, shrunk and suppressed the instability of the capillary force and then the stability sets in.

References


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