Abstract. Trigonometric interpolation equations for sparse prismatic arrays can be developed in several ways. This paper illustrates new expressions among data in the nine-point prismatic array. The new expressions permit the development of five alternative equations for the interpolation of data in the cited design.

Keywords: Operational calculus, trigonometric interpolation, prismatic array, data treatment

Introduction

Recent manuscripts have illustrated trigonometric equations for interpolating eight and nine data in rectangular prismatic arrays [1,2]. See Fig. 1. The equations are invariant under rotation of the designs but they are sensitive to translation of the data. The latter property is usually regarded as a disadvantage. However, it permits the development of interpolation equations with different properties by adding the same constant to each datum. That operation is followed by developing the interpolating equation for the new data and then subtracting the
constant from the equation. This approach has been described and illustrated
elsewhere [3].

**Alternative side-point formulas for the rectangular prism**

Equations for interpolating the missing measurement at the center of face
BDIG in Fig. 1 have been presented elsewhere [1-6]. The datum at that face-point
is also denoted by the four-letter combination BDIG. Two alternative expressions
for the missing datum at BDIG are Eqs. (1) and (3) [4,5]. Two-letter combinations
like CG represent the product of the data at vertices C and G in Fig. 1. The letter
E represents the datum at the center of the design.

\[(BDIG_1)^2 = \frac{E^2(B+I)(D+G)(Q_1)}{(1/2)}\]  \hspace{1cm} (1)

\[Q_1 = \frac{(H+I-A-B)(F+G-C-D)}{((A+B+H+I)(HI-AB)(C+D+F+G)(FG-CD))}\]  \hspace{1cm} (2)

\[(BDIG_2)^2 = \frac{(IB+DG-AH-CF)(I+B+D+G)^2}{2(I+B+D+G-A-H-F-C)(A+H+F+C+I+B+D+G))}\]  \hspace{1cm} (3)

Two expressions can be used to generate additional formulas for BDIG.
They appear as Eqs. (4) and (5) and are denoted P_1 and P_2, respectively. Three
alternative formulas for estimating the missing datum at face point BDIG are
denoted by BDIG_3, BDIG_4, and BDIG_5. Equations (1), (3), (6), (7) and (8) are
suitable for estimating the missing datum at the center of face BDIG in Fig. 1.

\[P_1 = \frac{(2E^2(CD-DF+CG+D^2-FG-G^2)+D(D^2G-DCF-G^3)+FCG^2)}{(2CD-2FG+D^2+C^2-F^2-G^2)}\] \hspace{1cm} (4)

\[P_2 = \frac{(2E^2(AB-BH+AI+B^2-HI-I^2)+B(B^2I-HBA-I^2)+AI^2H)}{(2AB-2HI+A^2+B^2-I^2-H^2)}\] \hspace{1cm} (5)

\[BDIG_3 = (P_1P_2)^{1/4}\] \hspace{1cm} (6)

\[BDIG_4 = ((P_1 + P_2)/2)^{1/2}\] \hspace{1cm} (7)

\[BDIG_5 = [(P_1)^{1/2} + (P_2)^{1/2}] / 2\] \hspace{1cm} (8)

The five numerical estimates of the datum at the center of face BDIG in
Fig. 1 are made in different ways so they are typically different. When applied
to the trilinear numbers 1 .. 9 as A .. I in Fig. 1, respectively, the five formulas
render BDIG=(11/2). The formulas are exact in this case, and they are exact when
the cited trilinear numbers are exponents in simple expressions like 2^x. They are
also exact when the trilinear numbers are substituted into sinh(x/4), cosh(x/4),
\(\sin(10x^\circ)\), and \(\cos(10x^\circ)\). The five formulas are sensitive to translation of the data.
Expressions for the centers of the remaining faces of the prism in Fig. 1 are obtained by rotating the prism and by substituting the appropriate data into the expressions for BDIG. The arguments of the circular or hyperbolic functions can then be numerically evaluated [1,2,4]. After the evaluations, the skeleton of the interpolating equation has nine remaining terms. Eight of them are three-member products of sines and cosines [1,2,4] as illustrated below. The coefficients of the eight, three-member products, and the additive term, remain to be determined. The nine unknowns are found from the data, their coordinates, and the set of nine simultaneous equations. They are found by solving the nine-member set or by the least-squares method.

Trigonometric interpolation equations for the nine-point prismatic array, developed as suggested in the preceding paragraph, are exact at all nine-vertices of the prism in Fig. 1. The W₁ .. W₆ factors previously described are not presently used [1,2]. The alternative approach therefore represents another way to generate trigonometric equations for the nine-point array in Fig. 1. For example, let the nine data be represented by the third powers of the first nine integers: [1³, 2³, 3³, .. 7³, 8³, 9³] as [A, B, C, .. G, H, I]. The five interpolating equations for the cited data are listed as Eqs. (9)-(13) below. They are based on BDGI₁, BDGI₂, BDGI₃, BDGI₄, and BDIG₅, respectively. In spite of appearances, the equations are sensitive to translation of the data. The coefficients in Eqs. (9)-(13) have been rounded. The letter R represents an interpolated number.

\[ R_1 = (108.1)\cosh(0.1981x)\cosh(0.3996y)\cosh(1.227z) + (123.7)\cosh(0.1981x)\cosh(0.3996y)\sinh(1.227z) + (123.3)\cosh(0.1981x)\sinh(0.3996y)\cosh(1.227z) + (115.0)\cosh(0.1981x)\sinh(0.3996y)\sinh(1.227z) + (121.6)\sinh(0.1981x)\cosh(0.3996y)\cosh(1.227z) + (111.7)\sinh(0.1981x)\cosh(0.3996y)\sinh(1.227z) + (99.04)\sinh(0.1981x)\sinh(0.3996y)\cosh(1.227z) + (58.84)\sinh(0.1981x)\sinh(0.3996y)\sinh(1.227z) + 16.88 \]  (9)

\[ R_2 = (193.1)\cos(0.4116x)\cosh(0.04820y)\cosh(1.141z) + (164.8)\cos(0.4116x)\cosh(0.04820y)\sinh(1.141z) + (1253)\cos(0.4116x)\sinh(0.04820y)\cosh(1.141z) + (1208)\cos(0.4116x)\sinh(0.04820y)\sinh(1.141z) + (70.19)\sin(0.4116x)\cosh(0.04820y)\cosh(1.141z) + (66.60)\sin(0.4116x)\cosh(0.04820y)\sinh(1.141z) + (450.8)\sin(0.4116x)\sinh(0.04820y)\cosh(1.141z) + (276.6)\sin(0.4116x)\sinh(0.04820y)\sinh(1.141z) - 68.05 \]  (10)
\[ R_3 = (124.9)\cosh(0.1637x)\cosh(0.3642y)\cosh(1.164z) + \\
(136.0)\cosh(0.1637x)\cosh(0.3642y)\sinh(1.164z) + \\
(144.0)\cosh(0.1637x)\sinh(0.3642y)\cosh(1.164z) + \\
(137.5)\cosh(0.1637x)\sinh(0.3642y)\sinh(1.164z) + \\
(157.3)\sinh(0.1637x)\cosh(0.3642y)\cosh(1.164z) + \\
(147.9)\sinh(0.1637x)\cosh(0.3642y)\sinh(1.164z) + \\
(139.4)\sinh(0.1637x)\sinh(0.3642y)\cosh(1.164z) + \\
(84.77)\sinh(0.1637x)\sinh(0.3642y)\sinh(1.164z) + 0.08021 \] (11)

\[ R_4 = (122.6)\cosh(0.1848x)\cosh(0.3736y)\cosh(1.166z) + \\
(134.6)\cosh(0.1848x)\cosh(0.3736y)\sinh(1.166z) + \\
(139.4)\cosh(0.1848x)\sinh(0.3736y)\cosh(1.166z) + \\
(133.0)\cosh(0.1848x)\sinh(0.3736y)\sinh(1.166z) + \\
(138.4)\sinh(0.1848x)\cosh(0.3736y)\cosh(1.166z) + \\
(130.0)\sinh(0.1848x)\cosh(0.3736y)\sinh(1.166z) + \\
(119.8)\sinh(0.1848x)\sinh(0.3736y)\cosh(1.166z) + \\
(72.80)\sinh(0.1848x)\sinh(0.3736y)\sinh(1.166z) + 2.401 \] (12)

\[ R_5 = (123.7)\cosh(0.1746x)\cosh(0.3689y)\cosh(1.165z) + \\
(135.3)\cosh(0.1746x)\cosh(0.3689y)\sinh(1.165z) + \\
(141.7)\cosh(0.1746x)\sinh(0.3689y)\cosh(1.165z) + \\
(135.2)\cosh(0.1746x)\sinh(0.3689y)\sinh(1.165z) + \\
(147.0)\sinh(0.1746x)\cosh(0.3689y)\cosh(1.165z) + \\
(138.1)\sinh(0.1746x)\cosh(0.3689y)\sinh(1.165z) + \\
(128.8)\sinh(0.1746x)\sinh(0.3689y)\cosh(1.165z) + \\
(78.25)\sinh(0.1746x)\sinh(0.3689y)\sinh(1.165z) + 1.251 \] (13)

Table 1 illustrates the sums of squares of deviations of the five trigonometric equations from the typical test functions. The entries in Table 1 herein can be compared to the corresponding entries in Table 1 of [2].

**Discussion**

The eight-point prismatic array has traditionally been represented by the trilinear equation. Straight lines are very useful but there is no compelling reason to limit the interpretation of the eight-point array to the trilinear equation. The eight-point array can be represented by polynomial equations with second-order or second and third-order terms, or by exponential equations, or by trigonometric
Alternative trigonometric equations

equations. There seems to be no compelling evidence that the trilinear equation has merits surpassing those of all other representations.

Similar arguments apply to the nine-point prismatic array in Fig. 1. Textbooks commonly give the impression that the nine-point array has little interest or seldom merits consideration. They typically do this by ignoring that array. It presently appears that there are more operational equations for the nine-point array than there are for the similar eight-point array. The equations illustrated herein, and in recent papers, make interesting comparisons [1-6].

It seems to be a tradition that trigonometric representations of experimental data be limited to the sine and the cosine. Equations (9)-(13) use those functions but the tradition need not be restrictive. To make the subject clear, Eq. (14) represents the eight-point prism with 1, 8, 27, 64 at vertices A, B, C, D, respectively, and 216, 343, 512, 729 at vertices F, G, H, I, respectively. Equation (14) applies the secant and the tangent instead of the sine and the cosine. In exploratory work, foreknowledge of the best representation of experimental data is seldom available. If laboratory experience supports a secant-tangent representation, tradition alone need not eliminate it. The references are limited to the shifting operator as applied to the development of interpolating equations for geometric data arrays.

\[ R = (116.4)\sec(0.1968x)\sec(0.3894y)\sec(1.000z) + \\
(121.6)\tan(0.1968x)\sec(0.3894y)\sec(1.000z) + \\
(123.3)\sec(0.1968x)\tan(0.3894y)\sec(1.000z) + \\
(123.7)\sec(0.1968x)\sec(0.3894y)\tan(1.000z) + \\
(99.04)\tan(0.1968x)\tan(0.3894y)\sec(1.000z) + \\
(111.7)\tan(0.1968x)\sec(0.3894y)\tan(1.000z) + \\
(115.0)\sec(0.1968x)\tan(0.3894y)\tan(1.000z) + \\
(58.84)\tan(0.1968x)\tan(0.3894y)\tan(1.000z) \]  

\[ (14) \]
Table 1. Sums of squares of deviations of five operational, trigonometric interpolation equations from typical surfaces [1-6]. The side-point formula used in each equation appears on the first line of each column. The entries are rounded.

<table>
<thead>
<tr>
<th>Function*</th>
<th>BDIG_1</th>
<th>BDIG_2</th>
<th>BDIG_3</th>
<th>BDIG_4</th>
<th>BDIG_5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M^2$</td>
<td>16.1</td>
<td>12.7</td>
<td>14.8</td>
<td>15.2</td>
<td>15.0</td>
</tr>
<tr>
<td>$M^3$</td>
<td>2571</td>
<td>2702</td>
<td>1857</td>
<td>1964</td>
<td>1910</td>
</tr>
<tr>
<td>$2^M$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>sinh($M/2$)</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>tan($9M^2$)</td>
<td>0.433</td>
<td>0.531</td>
<td>0.433</td>
<td>0.400</td>
<td>0.416</td>
</tr>
<tr>
<td>cosh($M/2$)</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$M + \cosh(M/2)$</td>
<td>1.33</td>
<td>0.905</td>
<td>1.22</td>
<td>1.20</td>
<td>1.21</td>
</tr>
<tr>
<td>$(M)\cosh(M/2)$</td>
<td>36.0</td>
<td>33.0</td>
<td>17.9</td>
<td>18.4</td>
<td>18.1</td>
</tr>
<tr>
<td>$1000/2^M$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$100/M^2$</td>
<td>175</td>
<td>73.6</td>
<td>138</td>
<td>89.6</td>
<td>110</td>
</tr>
<tr>
<td>$(5)\sin(10M^3)$ + cos$(10M^3)$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

*M = $(5 + x/2 + y + 5z/2)$

Fig. 1. The nine-point rectangular prismatic array with center point denoted E.
References


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