Some Formulas for the Multiple Twisted $(h,q)$-Euler Polynomials and Numbers

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Abstract

By using $p$-adic $q$-deformed fermionic integral on $\mathbb{Z}_p$, we define multiple the twisted $(h,q)$-Euler numbers of order $\alpha$ and polynomials of order $\alpha$. After we obtain the multiplication formulae for the multiple twisted $(h,q)$-Euler polynomials. Also the multiple alternating sum obtained at the twisted $(h,q)$-Euler polynomials and the twisted $(h,q)$-Euler numbers.

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1 Introduction

Let $p$ be a fixed odd prime number. Throughout this paper $\mathbb{Z}_p,\mathbb{Q}_p$ and $\mathbb{C}_p$ are respectively; the ring of $p$-adic rational integers, the field of $p$-adic rational numbers and the $p$-adic completion of the algebraic closure of $\mathbb{Q}_p$. The $p$-adic absolute value in $\mathbb{C}_p$ is normalized so that $|p|_p = \frac{1}{p}$. When one talks about $q$-extension, $q$ is variously considered as an indeterminate, a complex number, $q \in \mathbb{C}_p$ or a $p$-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}_p$, one normally assumes that $|q| < 1$. If $q \in \mathbb{C}_p$, one normally assumes that $|1 - q|_p < p^{-\frac{1}{p-1}}$ so that $q^x = \exp(x \log q)$ for each $x \in \mathbb{Z}_p$. We use the notations

$$[x]_q = \frac{1 - q^x}{1 - q},$$

$$[x]_{-q} = \frac{1 - (-q)^x}{1 + q} \quad (1)$$

$$[x]_{-q} = \frac{1 - (-q)^x}{1 + q}$$
For a fixed odd positive integer \(d\) with \((p, d) = 1\), set
\[
X^* = \bigcup_{0 < a < dp \atop (a, p) = 1} (a + dp\mathbb{Z}_p),
\]
where \(a \in \mathbb{Z}_p\) lies in \(0 < a < dp\). For any \(n \in \mathbb{N}\)
\[
\mu_q(a + dp^n\mathbb{Z}_p) = \frac{q^a}{[dp^n]_q}
\]
is known to be a distribution on \(X\) ([1], [15]). We say that \(f\) is uniformly differentiable function at a point \(a \in \mathbb{Z}_p\) and denote this property by \(f \in UD(\mathbb{Z}_p)\) if the difference quotients
\[
F_f(x, y) = \frac{f(x) - f(y)}{xy}
\]
have a limit \(l = f'(a)\) as \((x, y) \to (a, a)\) ([1], [3], [4], [8]). The \(p\)-adic \(q\)-integral of a function \(f \in UD(\mathbb{Z}_p)\) was defined as
\[
I_q(f) = \int_{\mathbb{Z}_p} f(x) d\mu_q(x) = \lim_{n \to \infty} \frac{1}{[p^n]_q} \sum_{x=0}^{p^n-1} f(x)q^x
\]
([1], [4], [8], [9], [11], [13]).

The \(q\)-deformed \(p\)-adic invariant on \(\mathbb{Z}_p\), in the fermonic sense is defined by
\[
I^{-q}(f) = \int_{\mathbb{Z}_p} f(x) d\mu^{-q}(x) = \lim_{n \to \infty} \frac{1}{[p^n]_{-q}} \sum_{x=0}^{p^n-1} f(x)(-q)^x
\]
([4], [13]) from (5), we write as
\[
qI^{-q}(f_1) + I^{-q}(f) = [2]_q f(0)
\]
where \(f_1(x) = f(x + 1)\).

**Lemma 1.1** *(Multinomial identity [5. p. 28 theorem B])* If \(x_1, x_2, \ldots, x_m\) are commuting elements of a ring \((\leftrightarrow x_i x_j = x_j x_i, 1 \leq i \leq j \leq m)\), then we have for all integers \(n \geq 0\);
\[
(x_1 + x_2 + \cdots + x_m)^n = \sum_{a_1, a_2, \ldots, a_m \geq 0 \atop a_1 + a_2 + \cdots + a_m = n} \binom{n}{a_1, a_2, \ldots, a_m} x_1^{a_1} x_2^{a_2} \cdots x_m^{a_m}
\]
the last summation takes place over all positive or zero integers $a_i \geq 0$ such that $a_1 + a_2 + \cdots + a_m = n$ where \( \binom{n}{a_1,a_2,\ldots,a_m} = \frac{n!}{a_1!a_2!\cdots a_m!} \) are called multinomial coefficients defined by [5. p. 28].

**Lemma 1.2** *(Generalized Multinomial identity [5. p. 41]*) If $x_1, x_2, \ldots, x_m$ are commuting elements of a ring \( \Leftrightarrow x_ix_j = x_jx_i, 1 \leq i \leq j \leq m \), then we have for all real or complex variable $\alpha$,

\[
(1 + x_1 + x_2 + \cdots + x_m)^\alpha = \sum_{v_1,v_2,\ldots,v_m \geq 0} \binom{\alpha}{v_1,v_2,\ldots,v_m} x_1^{v_1} x_2^{v_2} \cdots x_m^{v_m} \tag{7}
\]

the last summation takes place over all positive or zero integers $v_i \geq 0$ where;

\[
\binom{\alpha}{v_1,v_2,\ldots,v_m} = \frac{\{\alpha\}_{v_1+v_2+\cdots+v_m}}{v_1!v_2!\cdots v_m!} = \frac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-(v_1+v_2+\cdots+v_{m+1}))}{v_1!v_2!\cdots v_m!}
\]

are called generalized multinomial coefficients defined by [5. p. 27] where

\[
\{a\}_k = a(a-1)(a-2)\cdots(a-k+1), \quad a \in \mathbb{C}.
\]

Jang in [5] defined the multiple twisted $q$-Euler numbers and polynomials on $\mathbb{Z}_p$. He obtained sums of consecutive multiple twisted $q$-Euler numbers. Also he constructed the multiple twisted Barnes’ type $q$-Euler polynomials and multiple twisted Barnes’ type $q$-Euler zeta functions. Özden et al in ([1], [2], [3], [4]) by using a $p$-adic $q$-Volkenborn integral. They constructed a new approach to generating functions of the $(h, q)$-Euler numbers and polynomials attached to a Dirichlet character $\chi$. By applying the Mellin transformation and a derivative operator to these functions, they defined $(h, q)$-extensions of zeta functions. Min-Soo Kim et al in ([13], [14]) gave the existence of multiple twisted $p$-adic $q$-Euler $\zeta$-functions and $l$-functions. Also they obtained some relations on these functions. Simsek in ([4], [15]) by using the fermonic $p$-adic $q$-integral and multinomial theorem, he constructed generating functions of the higher-order $(h, q)$-extension of Euler polynomials and numbers. Also he constructed Barnes’ type multiple $(h, q)$-Euler zeta function.

T. Kim et al ([8], [9], [10], [11], [12]) constructed $q$-Volkenborn integration. He gave some theorems and relations on the twisted $q$-Euler numbers and polynomials.

In this work, we prove the multiplication formulae for the multiple twisted $(h, q)$-Euler polynomials. Moreover we give a formulae the multiple alternating sums between the twisted $(h, q)$-Euler polynomials.
2 The Twisted $(h, q)$-Euler Polynomials And Numbers

In this section, we assume that $q \in \mathbb{C}$ with $|1 - q|_p < 1$. For $n \in \mathbb{N}$, by definition $p$-adic $q$-integral on $\mathbb{Z}_p$. We have

$$q^n I_q(f_n) + (-1)^{n-1} I_q(f) = [2]_q \sum_{x=0}^{n-1} (-1)^{n-1-x} q^x f(x)$$

(8)

where $f_n(x) = f(x + n)$.

Let $T_p = \bigcup_{n \geq 1} C_{p^n} = \lim_{n \to \infty} C_{p^n} = C_{p^\infty}$ be the locally constant space, where $C_{p^n} = \{w : w^{p^n} = 1\}$ is the cyclic group of order $p^n$. For $w \in T_p$, we denote the locally constant function by

$$\Phi_w : \mathbb{Z}_p \to \mathbb{C}_p, x \to w x$$

([1], [4], [9]). If we take $f_w(x, t) = w^x q^{hx e^{tx}}$ in (5), then we write as

$$w q^h e^t I_{-1}(w^x q^{hx e^{tx}}) + I_{-1}(w^x q^{hx e^{tx}}) = 2.$$  (9)

Using the above equation, define the twisted $(h, q)$ extension of Euler numbers, $E_{n,w}(q)$ by means of the following generating function

$$I_{-1}(w^x q^{hx e^{tx}}) = \frac{2}{w^x q^{hx e^t} + 1} = \sum_{n=0}^{\infty} E_{n,w}(q) \frac{t^n}{n!}.$$  (10)

**Remark 1** If $w \to 1$ and $q \to 1$, then the equation (10) reduces to classical Euler number

$$\frac{2}{e^t + 1} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!}.$$  

Similarly, the twisted $(h, q)$ extension of Euler polynomials are defined as

$$\frac{2}{w q^h e^t + 1} e^{xt} = \sum_{n=0}^{\infty} E_{n,w}(q) \frac{t^n}{n!}.$$  (11)

**Definition 2.1** The twisted $(h, q)$ extension of Euler numbers order $\alpha$ and the twisted $(h, q)$ extension of Euler polynomials order $\alpha$ are defined as respectively

$$\sum_{n=0}^{\infty} E_{n,w}(q) \frac{t^n}{n!} = \left(\frac{2}{w q^h e^t + 1}\right)^\alpha,$$

(12)

$$\sum_{n=0}^{\infty} E_{n,w}(x, q) \frac{t^n}{n!} = \left(\frac{2}{w q^h e^t + 1}\right)^\alpha e^{xt}.$$  (13)
Theorem 2.2 \( m \in \mathbb{N}, \ n \in \mathbb{N}_0, \ q, \alpha \in \mathbb{C} \), the twisted \( (h,q) \)-Euler polynomials order \( \alpha \) are satisfied the following multiplication formulae

\[
E_{n,w}^{(h,\alpha)}(mx, q) = \sum_{v_1,v_2,\cdots,v_{m-1} \geq 0} \binom{\alpha}{v_1,v_2,\cdots,v_{m-1}} (-wq^h)^r E_{n,w}^{(h^m,\alpha)}(x + \frac{r}{m})
\]

(14)

where \( r = v_1 + 2v_2 + \cdots + (m - 1)v_{m-1} \).

**Proof.** It is easy to observe that

\[
\frac{1}{wq^h e^t + 1} = - \frac{\sum_{k=0}^{m-1} (-q^h w e^t)^k}{(-q^h w e^t)^m + 1}
\]

(15)

for \( m \) odd, by equation (7, 13, 14).

\[
\sum_{n=0}^{\infty} E_{n,w}^{(h,\alpha)}(mx, q) \frac{t^n}{n!} = \left( \frac{2}{wq^h e^t + 1} \right)^\alpha e^{mt(x + \frac{r}{m})}
\]

where \( r = v_1 + 2v_2 + \cdots + (m - 1)v_{m-1} \). By comparing the coefficient \( \frac{t^n}{n!} \) in the both sides of the above equation. We easily arrive at (14). \( \blacksquare \)

We define the multiple alternating sums as

\[
Z_k^l(m; wq^h) = \sum_{0 \leq v_1,\cdots,v_m \leq l \atop v_1 + \cdots + v_m = l} (-1)^l \binom{l}{v_1,\cdots,v_m} (-qw)^{v_1 + \cdots + mv_m} (v_1 + \cdots + mv_m)^k.
\]

(16)
Theorem 2.3 For $m, n, l \in \mathbb{N}, \alpha \in \mathbb{C}$ the following recursive formulae for multiple twisted $(h, q)$-Euler polynomials order $\alpha$ is satisfying

$$Z_n^l(m; wq^h) = (wq^h)^{2^{-l}} \sum_{j=0}^{l} \left( \begin{array}{c} l \\ j \end{array} \right) (-1)^{(m+1)j} (q^h w^m j)$$

$$\times \sum_{k=0}^{n} \left( \begin{array}{c} n \\ k \end{array} \right) E_{k,w}^{(h,j)} (mj + l; q) E_{n-k}^{(h, l-j)} (q).$$

Proof. By equation (14, 16)

$$\sum_{n=0}^{\infty} Z_n^l(m; wq^h) \frac{t^n}{n!} = \sum_{n=0}^{\infty} (-1)^l \sum_{v_1, v_2, \cdots, v_m \leq l} \left( \begin{array}{c} l \\ v_1, v_2, \cdots, v_m \end{array} \right) (wq^h)^{v_1 + 2v_2 + \cdots + mv_m} \frac{t^n}{n!}$$

$$= (-1)^l \sum_{v_1, v_2, \cdots, v_m \leq l} \left( \begin{array}{c} l \\ v_1, v_2, \cdots, v_m \end{array} \right) (-q^h w^{v_1 + 2v_2 + \cdots + mv_m}) \frac{t^n}{n!}$$

$$= (wq^h e^t - w^2 q^{2h} e^{2t} + \cdots + (-1)^{m+1}(q^h w^m e^{mt})^l)$$

$$= \left( \frac{wq^h e^t + (-1)^{m+1}(q^h w^m e^{mt})^l}{wq^h e^t + 1} \right)$$

$$= \sum_{j=0}^{l} \left( \begin{array}{c} l \\ j \end{array} \right) \left( \frac{(-1)^{m+1}(q^h w^m e^{(m+1)t})^j}{wq^h e^t + 1} \right) \left( \frac{wq^h e^t}{wq^h e^t + 1} \right)^{l-j}$$

$$= \sum_{n=0}^{\infty} \left( \begin{array}{c} n \\ l \end{array} \right) 2^{-l} \sum_{j=0}^{l} \left( \begin{array}{c} l \\ j \end{array} \right) (-1)^{(m+1)j} (wq^h)^{mj}$$

$$\times \sum_{k=0}^{n} \left( \begin{array}{c} n \\ k \end{array} \right) E_{k,w}^{(h,j)} (mj + l; q) E_{n-k}^{(h, l-j)} (q) \frac{t^n}{n!}.$$

By comparing the coefficient $\frac{t^n}{n!}$ in the both sides of the above equation. We easily arrive at (17).

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Some formulas for the multiple twisted $(h,q)$-Euler polynomials

References


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