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Some Formulas for the Multiple Twisted (h,q)- Euler Polynomials and Numbers

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Abstract

By using p-adic q-deformed fermonic integral on Z_p, we define multiple the twisted (h,q)-Euler numbers of order α and polynomials of order α. After we obtain the multiplication formulae for the multiple twisted (h,q)-Euler polynomials. Also the multiple alternating sum obtained at the twisted (h,q)-Euler polynomials and the twisted (h,q)-Euler numbers.

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1 Introduction

Let p be a fixed odd prime number. Thoughout this paper $\mathbb{Z}_p$,$\mathbb{Q}_p$ and $\mathbb{C}_p$ are respectively; the ring of p-adic rational integers, the field of p-adic rational numbers and the p-adic completion of the algebraic closure of $\mathbb{Q}_p$. The p-adic absolute value in $\mathbb{C}_p$ is normalized so that $|p|_p = \frac{1}{p}$. When one talks about q-extension, q is variously considered as an indeterminate, a complex number, $q \in \mathbb{C}_p$ or a p-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}_p$, one normally assumes that $|q| < 1$. If $q \in \mathbb{C}_p$, one normally assumes that $|1-q|_p < p^{-\frac{1}{p-1}}$ so that $q^x = \exp(x \log q)$ for each $x \in \mathbb{Z}_p$. We use the notations

$$\begin{aligned} [x]_q &= \frac{1-q^x}{1-q}, \\ [x]_{-q} &= \frac{1-(-q)^x}{1+q} \end{aligned} \tag{1}$$

([1], [15]), for all $x \in \mathbb{Z}_p$. For a fixed odd positive integer d with $(p,d)=1$, set

$$\begin{aligned} X &= X_d = \varprojlim_n \frac{\mathbb{Z}}{dp^n\mathbb{Z}_p}, \ X_1 = \mathbb{Z}_p \\ X^* &= \bigcup_{\substack{0<a<dp \\ (a,p)=1}} (a + dp\mathbb{Z}_p), \\ a + dp^n\mathbb{Z}_p &= \{x \in X : x \equiv a \ (mod \ dp^n)\} \end{aligned}$$

where $a \in \mathbb{Z}_p$ lies in $0 < a < dp$. For any $n \in \mathbb{N}$

$$\mu_q(a + dp^n\mathbb{Z}_p) = \frac{q^a}{[dp^n]_q} \tag{2}$$

is known to be a distribution on X ([1], [15]). We say that f is uniformly differentiable function at a point $a \in \mathbb{Z}_p$ and denote this property by $f \in UD(\mathbb{Z}_p)$ if the difference quotients

$$F_f(x,y) = \frac{f(x) - f(y)}{xy} \tag{3}$$

have a limit $l = f'(a)$ as $(x,y) \longrightarrow (a,a)$ ([1], [3], [4], [8]). The p-adic q-integral of a function $f \in UD(\mathbb{Z}_p)$ was defined as

$$I_q(f) = \int_{\mathbb{Z}_p} f(x)d\mu_q(x) = \lim_{n\longrightarrow\infty} \frac{1}{[p^n]_q} \sum_{x=0}^{p^n-1} f(x)q^x \tag{4}$$

([1], [4], [8], [9], [11], [13]).

The q-deformed p-adic invariant on $\mathbb{Z}_p$, in the fermonic sense is defined by

$$I_{-q}(f) = \int_{\mathbb{Z}_p} f(x)d\mu_{-q}(x) = \lim_{n\longrightarrow\infty} \frac{1}{[p^n]_{-q}} \sum_{x=0}^{p^n-1} f(x)\,(-q)^x \tag{5}$$

([4], [13]) from (5), we write as

$$qI_{-q}(f_1) + I_{-q}(f) = [2]_q\, f(0)$$

where $f_1(x) = f(x+1)$.

Lemma 1.1 *(Multinomial identity [5. p. 28 theorem B]) If $x_1, x_2, \cdots, x_m$ are commuting elements of a ring ($\Leftrightarrow x_ix_j = x_jx_i$, $1 \le i \le j \le m$), then we have for all integers $n \ge 0$;*

$$(x_1 + x_2 + \cdots + x_m)^n = \sum_{\substack{a_1,a_2,\cdots,a_m\ge 0 \\ a_1+a_2+\cdots+a_m=n}} \binom{n}{a_1, a_2, \cdots, a_m} x_1^{a_1}x_2^{a_2}\cdots x_m^{a_m} \tag{6}$$

the last summation takes place over all positive or zero integers $a_i \geq 0$ *such that* $a_1+a_2+\cdots+a_m = n$ *where* $\binom{n}{a_1,a_2,\cdots,a_m} = \frac{n!}{a_1!a_2!\cdots a_m!}$ *are called multinomial coefficients defined by [5. p. 28].*

Lemma 1.2 *(Generalized Multinomial identity [5. p. 41] If* $x_1, x_2, \cdots, x_m$ *are commuting elements of a ring (*$\Leftrightarrow x_i x_j = x_j x_i$, $1 \leq i \leq j \leq m$*), then we have for all real or complex variable* α*,*

$$(1 + x_1 + x_2 + \cdots + x_m)^{\alpha} = \sum_{v_1,v_2,\cdots,v_m \geq 0} \binom{\alpha}{v_1, v_2, \cdots, v_m} x_1^{v_1} x_2^{v_2} \cdots x_m^{v_m} \quad (7)$$

the last summation takes place over all positive or zero integers $v_i \geq 0$ *where;*

$$\begin{aligned} \binom{\alpha}{v_1, v_2, \cdots, v_m} &= \frac{\{\alpha\}_{v_1+v_2+\cdots+v_m}}{v_1!v_2!\cdots v_m!} \\ &= \frac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-(v_1+v_2+\cdots+v_{m+1}))}{v_1!v_2!\cdots v_m!} \end{aligned}$$

are called generalized multinomial coefficients defined by [5. p. 27] where

$$\{a\}_k = a(a-1)(a-2)\cdots(a-k+1),\ a \in \mathbb{C}.$$

Jang in [5] defined the multiple twisted q-Euler numbers and polynomials on $\mathbb{Z}_p$. He obtained sums of consecutive multiple twisted q-Euler numbers. Also he constructed the multiple twisted Barnes' type q-Euler polynomials and multiple twisted Barnes' type q-Euler zeta functions. Özden et al in ([1], [2], [3], [4]) by using a p-adic q-Volkenborn integral. They constructed a new approach to generating functions of the (h, q)-Euler numbers and polynomials attached to a Dirichlet character χ. By applying the Mellin transformation and a derivative operator to these functions, they defined (h, q)-extensions of zeta functions. Min-Soo Kim et al in ([13], [14]) gave the existence of multiple twisted p-adic q-Euler ζ-functions and l-functions. Also they obtained some relations on these functions. Simsek in ([4], [15]) by using the fermonic p-adic q-integral and multinomial theorem, he consructed generating functions of the higher-order (h, q)-extension of Euler polynomials and numbers. Also he constructed Barnes' type multiple (h, q)-Euler zeta function.

T. Kim et al ([8], [9], [10], [11], [12]) constructed q-Volkenborn integration. He gave some theorems and relations on the twisted q-Euler numbers and polynomials.

In this work, we prove the multiplication formulae for the multiple twisted (h, q)-Euler polynomials. Moreover we give a formulae the multiple alternating sums between the twisted (h, q)-Euler polynomials.

2 The Twisted (h,q)-Euler Polynomials And Numbers

In this section, we assume that $q \in \mathbb{C}$ with $|1-q|_p < 1$. For $n \in \mathbb{N}$, by definition p-adic q-integral on $\mathbb{Z}_p$. We have

$$q^n I_{-q}(f_n) + (-1)^{n-1} I_{-q}(f) = [2]_q \sum_{x=0}^{n-1} (-1)^{n-1-x} q^x f(x) \tag{8}$$

where $f_n(x) = f(x+n)$.

Let Let $T_p = \bigcup_{n\geq 1} C_{p^n} = \lim_{n\longrightarrow\infty} C_{p^n} = C_{p^\infty}$ be the locally constant space, where $C_{p^n} = \left\{w : w^{p^n} = 1\right\}$ is the cyclic group of order p^n. For $w \in T_p$, we denote the locally constant function by

$$\Phi_w : \mathbb{Z}_p \longrightarrow \mathbb{C}_p,\ x \longrightarrow w^x$$

([1], [4], [9]). If we take $f_w(x,t) = w^x q^{hx} e^{tx}$ in (5), then we write as

$$wq^h e^t I_{-1}(w^x q^{hx} e^{tx}) + I_{-1}(w^x q^{hx} e^{tx}) = 2. \tag{9}$$

Using the above equation, define the twisted (h,q) extension of Euler numbers, $E_{n,w}^{(h)}(q)$ by means of the following generating function

$$I_{-1}(w^x q^{hx} e^{tx}) = \frac{2}{w^x q^{hx} e^x + 1} = \sum_{n=0}^{\infty} E_{n,w}^{(h)}(q) \frac{t^n}{n!}. \tag{10}$$

Remark 1 *If $w \to 1$ and $q \to 1$, then the equation (10) reduces to classical Euler number*

$$\frac{2}{e^t+1} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!}.$$

Similarly, the twisted (h,q) extension of Euler polynomials are defined as

$$\frac{2}{wq^h e^t + 1} e^{xt} = \sum_{n=0}^{\infty} E_{n,w}^{(h)}(q) \frac{t^n}{n!}. \tag{11}$$

Definition 2.1 *The twisted (h,q) extension of Euler numbers order α and the twisted (h,q) extension of Euler polynomials order α are defined as respectively*

$$\sum_{n=0}^{\infty} E_{n,w}^{(h,\alpha)}(q) \frac{t^n}{n!} = \left(\frac{2}{wq^h e^t + 1}\right)^{\alpha}, \tag{12}$$

$$\sum_{n=0}^{\infty} E_{n,w}^{(h,\alpha)}(x,q) \frac{t^n}{n!} = \left(\frac{2}{wq^h e^t + 1}\right)^{\alpha} e^{xt}. \tag{13}$$

Theorem 2.2 *$m \in \mathbb{N}$, $n \in \mathbb{N}_0$, $q,\alpha \in \mathbb{C}$, the twisted (h,q)-Euler polynomials order α are satisfed the following multiplication formulae*

$$E_{n,w}^{(h,\alpha)}(mx,q) = \sum_{v_1,v_2,\cdots,v_{m-1}\geq 0} \binom{\alpha}{v_1, v_2, \cdots, v_{m-1}} \left(-wq^h\right)^r E_{n,w^m}^{(h^m,\alpha)}\left(x+\frac{r}{m}\right) \tag{14}$$

where $r = v_1 + 2v_2 + \cdots + (m-1)v_{m-1}$.

Proof. It is easy to observe that

$$\frac{1}{wq^he^t+1} = -\frac{\sum_{k=0}^{m-1}\left(-q^hwe^t\right)^k}{\left(-q^hwe^t\right)^m+1} \tag{15}$$

for m odd, by equation (7, 13, 14).

$$\begin{aligned}
\sum_{n=0}^{\infty} E_{n,w}^{(h,\alpha)}(mx,q)\frac{t^n}{n!} &= \left(\frac{2}{wq^he^t+1}\right)^{\alpha} e^{mxt} \\
&= \left(\frac{2}{\left(wq^he^t\right)^m+1}\right)^{\alpha}\left(\sum_{k=0}^{m-1}\left(-q^hwe^t\right)^k\right)^{\alpha} e^{mxt} \\
&= \sum_{v_1,v_2,\cdots,v_{m-1}\geq 0}\binom{\alpha}{v_1, v_2, \cdots, v_{m-1}}\left(-wq^h\right)^r \\
&\quad \times\left(\frac{2}{\left(wq^h\right)^m e^{mt}+1}\right)^{\alpha} e^{mt(x+\frac{r}{m})} \\
&= \sum_{v_1,v_2,\cdots,v_{m-1}\geq 0}\left(m^n\binom{\alpha}{v_1, v_2, \cdots, v_{m-1}}\left(-wq^h\right)^r\right. \\
&\quad \left.\times E_{n,w^n}^{(h^m,\alpha)}(x+\frac{r}{m})\right)\frac{t^n}{n!}
\end{aligned}$$

where $r = v_1 + 2v_2 + \cdots + (m-1)v_{m-1}$. By comparing the coefficient $\frac{t^n}{n!}$ in the both sides of the above equation. We easily arrive at (14). ■

We define the multiple alternating sums as

$$Z_k^l(m;wq^h) = \sum_{\substack{0\leq v_1,\cdots,v_m\leq l\\ v_1+\cdots+v_m=l}} (-1)^l\binom{l}{v_1,\cdots,v_m}(-qw)^{v_1+\cdots+mv_m}\left(v_1+\cdots+mv_m\right)^k. \tag{16}$$

Theorem 2.3 *For m, n, $l \in \mathbb{N}$, $\alpha \in \mathbb{C}$ the following recursive formulae for multiple twisted (h,q)-Euler polynomials order α is satisfying*

$$Z_n^l(m;wq^h) = (wq^h)2^{-l}\sum_{j=0}^{l}\binom{l}{j}(-1)^{(m+1)j}\left(q^h w\right)^{mj} \tag{17}$$

$$\times\sum_{k=0}^{n}\binom{n}{k}E_{k,w}^{(h,j)}(mj+l;q)\,E_{n-k}^{(hq,l-j)}(q). \tag{18}$$

Proof. By equation (14, 16)

$$\begin{aligned}
&\sum_{n=0}^{\infty} Z_n^l(m;wq^h)\frac{t^n}{n!}\\
=\ &\sum_{n=0}^{\infty}(-1)^l\sum_{\substack{0\le v_1,v_2,\cdots,v_m\le l\\ v_1+v_2+\cdots+v_m=l}}\binom{l}{v_1,v_2,\cdots,v_m}\\
&\times(-q^h w)^{v_1+2v_2+\cdots+mv_m}\left(v_1+2v_2+\cdots+mv_m\right)\frac{t^n}{n!}\\
=\ &(-1)^l\sum_{\substack{0\le v_1,v_2,\cdots,v_m\le l\\ v_1+v_2+\cdots+v_m=l}}\binom{l}{v_1,v_2,\cdots,v_m}(-q^h we^t)^{v_1+\cdots+mv_m}\\
=\ &\left(wq^he^t-w^2q^{2h}e^{2t}+\cdots+(-1)^{m+1}(q^hw)^me^{mt}\right)^l\\
=\ &\left(\frac{wq^he^t+(-1)^{m+1}(q^hwe^t)^{m+1}}{wq^he^t+1}\right)\\
=\ &\sum_{j=0}^{l}\binom{l}{j}\left(\frac{(-1)^{m+1}(q^hw)^{m+1}e^{(m+1)t}}{wq^he^t+1}\right)^j\left(\frac{wq^he^t}{wq^he^t+1}\right)^{l-j}\\
=\ &\sum_{n=0}^{\infty}\left(\left(wq^h\right)^l2^{-l}\sum_{j=0}^{l}\binom{l}{j}(-1)^{(m+1)j}\left(wq^h\right)^{mj}\right.\\
&\left.\times\sum_{k=0}^{n}\binom{n}{k}E_{k,w}^{(h,j)}(mj+l;q)\,E_{n-k}^{(hq,l-j)}(q)\right)\frac{t^n}{n!}.
\end{aligned}$$

By comparing the coefficient $\frac{t^n}{n!}$ in the both sides of the above equation. We easily arrive at (17). ■

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