An Algorithm for Solutions of Linear Partial Differential Equations via Lie Group of Transformations

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Abstract

Lie group of transformations associated with linear partial differential equations is derived using the invariance properties. Infinitesimals of Lie group of transformations with respect to independent and dependent variables along with invariance surface conditions are used to obtain generalized auxiliary equations. The main objective of this paper is to develop an easily applicable algorithm for linear partial differential equations, which can be used to solve the PDE or reduce the same to another PDE with fewer independent variables. The results are illustrated by considering suitable examples.

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1 Introduction

During the last three decades, there have been considerable developments in symmetry methods and much works have been carried out on Lie group of transformations for differential equations i.e. both ordinary differential equations (ODE’s) and partial differential equations (PDE’s), which arise in numerous applications in many different branches of physics and engineering.
Symmetry methods for differential equations, originally developed by Sophus Lie, are highly algorithmic. Often ingenious techniques for solving particular differential equations arise transparently from the group point of view [1].

The idea of representation of a Lie group plays an important role in the study of continuous symmetry in mathematics and theoretical physics. The basic tool in the study is the use of corresponding ‘infinitesimal’ representations of Lie algebras. By an expanded Lie group of transformations of partial differential equations we mean a continuous group of transformations acting on the expanded space of variables, which includes the equation parameters in addition to independent and dependent variables. Recent works on Lie group theory and its applications in different fields can be found in [2]-[9].

Radha & Sharma [10] exploited the Lie group method described in the works of Bluman Cole [3], Bluman Kumei [1] and Logan & Perez [11] to establish the entire class of self-similar solutions for converging shocks in a relaxing gas. The method enables to characterize the medium for which the problem is invariant and admits self-similar solutions. The method has been successively used to study the problem of shock wave propagation through a dusty gas mixture obeying the equation of state of Mie-Gruneisen type [12], solution of system of equations describing viscoelastic materials [13], interaction of a weak discontinuity wave with a bore in shallow water [14], and interaction of discontinuous waves in a relaxing gas [15].

Linear PDE’s include the PDE’s resulting from the description of shear flows, diffusion processes in inhomogeneous medium, acoustical disturbances in an inhomogeneous transmitting medium etc. Indeed, it contains many equations of mathematical physics as special cases; these include the telegraph equation, the diffusion equation, the Poisson equation, the Schrödinger equation, the Fokker-Plank equation etc. Here, with an aim to find a simpler approach for the usefulness of Lie group of transformations, we rewrote the single PDE to a system of first order PDE’s, applied the method adapted in [10] - [15] and used constantly conformally invariance of the system of equations. Interestingly, the method leads to a simpler algorithm which can be easily applied with less knowledge of Lie group of transformations.

The main objective in this paper is to develop an easily understood and applicable algorithm for linear partial differential equations, which can be used to solve the PDE or reduce the same to another PDE with fewer independent variables. In the following sections we used Lie group of transformations for second and third order linear PDE’s and used the same to obtain an algorithm for a PDE of any order. The algorithm was verified with suitable examples.
2 Lie Group of Transformations

We consider the second order linear partial differential equation with variable coefficients in time and space

\[
\tau_1 \frac{\partial u}{\partial t} + \tau_2 \frac{\partial^2 u}{\partial t^2} + Au + B \frac{\partial u}{\partial x} + C \frac{\partial u}{\partial y} + D \frac{\partial u}{\partial z} \\
+ E \frac{\partial^2 u}{\partial x^2} + F \frac{\partial^2 u}{\partial y^2} + G \frac{\partial^2 u}{\partial z^2} + H \frac{\partial^2 u}{\partial x \partial y} + I \frac{\partial^2 u}{\partial x \partial z} + J \frac{\partial^2 u}{\partial y \partial z} = 0,
\]

(1)

where \( \tau_1, \tau_2, A, B, C, D, E, F, G, H, I \) and \( J \) are known functions of \( t, x, y \) and \( z \).

It may be noted that equation (1) can be reduced to many important equations of mathematical physics i.e. Heat equation, Wave equation, Laplace equation etc. on suitable selection of coefficients \( \tau_1, \tau_2, A, B, C, D, E, F, G, H, I \) and \( J \).

The second order PDE (1) can be reduced to a first order system of equations using the variables \( \psi, \eta, \zeta \) and \( \theta \) as

\[
\tau_1 \psi + \tau_2 \frac{\partial \psi}{\partial t} + Au + B \eta + C \zeta + D \theta + E \frac{\partial \eta}{\partial x} + F \frac{\partial \zeta}{\partial y} + G \frac{\partial \theta}{\partial z} + H \frac{\partial \eta}{\partial x \partial y} + I \frac{\partial \eta}{\partial x \partial z} + J \frac{\partial \zeta}{\partial y \partial z} = 0,
\]

\[\frac{\partial u}{\partial t} = \psi, \quad \frac{\partial u}{\partial x} = \eta, \quad \frac{\partial u}{\partial y} = \zeta, \quad \frac{\partial u}{\partial z} = \theta.\]

(2)

In order to use Lie group of transformations method for a similarity reduction of the system of equation (2), we seek a one-parameter infinitesimal group of transformations [10]

\[
x_j^* = x_j + \epsilon X_j(x_1, x_2, x_3, x_4, u_1, u_2, u_3, u_4, u_5),
\]

\[u_i^* = u_i + \epsilon U_i(x_1, x_2, x_3, x_4, u_1, u_2, u_3, u_4, u_5),\]

(3)

where \( j = 1, 2, 3, 4 \), \( i = 1, 2, 3, 4, 5 \), \( x_1 = t, x_2 = x, x_3 = y, x_4 = z, u_1 = u, u_2 = \psi, u_3 = \eta, u_4 = \zeta, u_5 = \theta \), \( \epsilon \) is the parameter of Lie group transformation, \( X_j \) and \( U_i \) are the infinitesimals of Lie group of transformations. We use \( p_j^i = \partial u_i / \partial x_j \) and rewrite the system of equations (2) in the form

\[F_k(x_j, u_i, p_j^i) = 0; \quad i, k = 1, 2, 3, 4, 5; \quad j = 1, 2, 3, 4.\]

(4)

The system of equations is said to be constantly conformally invariant under the infinitesimal group (3), if there exist constants \( \alpha_{rs} \) \( (r, s = 1, 2, 3, 4, 5) \) such that for all smooth surfaces, \( u_i = u_i(x_j) \), we have

\[\mathcal{L} F_k = \alpha_{kr} F_r,\]

(5)
where $\mathcal{L}$ is the Lie derivative in the direction of the extended vector field

$$\mathcal{L} = \xi^i_x \frac{\partial}{\partial x_j} + \xi^i_u \frac{\partial}{\partial u_i} + \xi^i_{p_j} \frac{\partial}{\partial p^i_j},$$

with $\xi^i_x = \mathcal{X}_j$, $\xi^i_u = \mathcal{U}_i$ and

$$\xi^i_{p_j} = \frac{\partial \xi^i_u}{\partial x_j} + \frac{\partial \xi^i_u}{\partial u_k} p^k_j - \frac{\partial \xi^i_x}{\partial x_j} p^i_l - \frac{\partial \xi^i_x}{\partial u_n} p^i_l p^n; \quad l = 1, 2, 3, 4; \quad n = 1, 2, 3, 4, 56$$

being the infinitesimals of the derivative transformation. The infinitesimals are to be determined in such a way that the partial differential equations (4) are constantly conformally invariant with respect to the transformations (3). Following the usual procedure outlined in [10] and [11], the invariance of equations (4) yield a system of determining equations for the infinitesimals $\xi^i_x$, $\xi^i_u$, and $\xi^i_{p_j}$, and the analysis of the said determining equations leads to following forms of the infinitesimals

$$\begin{align*}
\xi^1_x &= \beta_1 x_1 + \gamma_1, \quad \xi^2_x = \beta_2 x_2 + \gamma_2, \\
\xi^3_x &= \beta_3 x_3 + \gamma_3, \quad \xi^4_x = \beta_4 x_4 + \gamma_4, \\
\xi^1_u &= \beta_u u_1 + \chi(x_1, x_2, x_3, x_4), \\
\xi^2_u &= \alpha_{22} u_2 + \frac{\partial \chi}{\partial x_1}, \quad \xi^3_u = \alpha_{33} u_3 + \frac{\partial \chi}{\partial x_2}, \\
\xi^4_u &= \alpha_{44} u_4 + \frac{\partial \chi}{\partial x_3}, \quad \xi^5_u = \alpha_{55} u_5 + \frac{\partial \chi}{\partial x_4},
\end{align*}$$

(7)

where $\beta_j, \gamma_j (j = 1, 2, 3, 4)$ are arbitrary constants, $\chi$ is an arbitrary function of its arguments and

$$\beta_* = \alpha_{22} + \beta_1 = \alpha_{33} + \beta_2 = \alpha_{44} + \beta_3 = \alpha_{55} + \beta_4.$$  

(8)

The conditions to be satisfied by the infinitesimals are as per the following:

$$\begin{align*}
\xi^i_x \frac{\partial A}{\partial x_1} + \xi^2_x \frac{\partial A}{\partial x_2} + \xi^3_x \frac{\partial A}{\partial x_3} + \xi^4_x \frac{\partial A}{\partial x_4} &= (\alpha_{11} - \beta_*) A, \\
\xi^i_x \frac{\partial \tau_1}{\partial x_1} + \xi^2_x \frac{\partial \tau_1}{\partial x_2} + \xi^3_x \frac{\partial \tau_1}{\partial x_3} + \xi^4_x \frac{\partial \tau_1}{\partial x_4} &= (\alpha_{11} - \alpha_{22}) \tau_1, \\
\xi^i_x \frac{\partial B}{\partial x_1} + \xi^2_x \frac{\partial B}{\partial x_2} + \xi^3_x \frac{\partial B}{\partial x_3} + \xi^4_x \frac{\partial B}{\partial x_4} &= (\alpha_{11} - \alpha_{33}) B, \\
\xi^i_x \frac{\partial C}{\partial x_1} + \xi^2_x \frac{\partial C}{\partial x_2} + \xi^3_x \frac{\partial C}{\partial x_3} + \xi^4_x \frac{\partial C}{\partial x_4} &= (\alpha_{11} - \alpha_{44}) C, \\
\xi^i_x \frac{\partial D}{\partial x_1} + \xi^2_x \frac{\partial D}{\partial x_2} + \xi^3_x \frac{\partial D}{\partial x_3} + \xi^4_x \frac{\partial D}{\partial x_4} &= (\alpha_{11} - \alpha_{55}) D,
\end{align*}$$

(9)
\[\begin{align*}
\xi_1^0 \partial \tau_2 + \xi_2^0 \partial \tau_2 + \xi_3^0 \partial \tau_2 + \xi_4^0 \partial \tau_2 &= (\alpha_{11} - \alpha_{22} + \beta_1) \tau_2, \\
\xi_1^0 \partial E + \xi_2^0 \partial E + \xi_3^0 \partial E + \xi_4^0 \partial E &= (\alpha_{11} - \alpha_{33} + \beta_2) E, \\
\xi_1^0 \partial F + \xi_2^0 \partial F + \xi_3^0 \partial F + \xi_4^0 \partial F &= (\alpha_{11} - \alpha_{44} + \beta_3) F, \\
\xi_1^0 \partial G + \xi_2^0 \partial G + \xi_3^0 \partial G + \xi_4^0 \partial G &= (\alpha_{11} - \alpha_{55} + \beta_4) G, \\
\xi_1^0 \partial H + \xi_2^0 \partial H + \xi_3^0 \partial H + \xi_4^0 \partial H &= (\alpha_{11} - \alpha_{33} + \beta_5) H, \\
\xi_1^0 \partial I + \xi_2^0 \partial I + \xi_3^0 \partial I + \xi_4^0 \partial I &= (\alpha_{11} - \alpha_{55} + \beta_4) I, \\
\xi_1^0 \partial J + \xi_2^0 \partial J + \xi_3^0 \partial J + \xi_4^0 \partial J &= (\alpha_{11} - \alpha_{44} + \beta_4) J,
\end{align*}\]
and
\[\begin{align*}
\tau_1 \partial x_1 + \tau_2 \partial^2 x_2 + A \partial x_2 + B \partial x_2 + C \partial x_3 + D \partial x_4 + E \partial^2 x_2 + F \partial^2 x_3 + G \partial^2 x_3 + H \partial^2 x_3 + I \partial^2 x_3 + J \partial^2 x_3 &= 0. (10)
\end{align*}\]

It may be noted that \(\chi(x_1, x_2, x_3, x_4)\) in equation (10) corresponds to the trivial infinite - parameter Lie group and hence can be assumed to be zero [1]. This is in agreement with the characteristic of any linear PDE that, it always admits a "trivial" infinite - parameter Lie group of transformations. The relations between parameters \(\alpha_{ii}, \beta_j\) and \(\gamma_j\) satisfying equations (8) and (9) can be used to find the forms of \(\xi_x^i\) in (7), whereas \(\xi_u^i\) in (7) admit the following forms
\[
\begin{align*}
\xi_u^1 &= \beta_4 u_1, \quad \xi_u^2 = \alpha_{22} u_2, \quad \xi_u^3 = \alpha_{33} u_3, \\
\xi_u^4 &= \alpha_{44} u_4, \quad \xi_u^5 = \alpha_{55} u_5.
\end{align*}
\]

The invariant surface conditions (auxiliary equations for \(u_i\)) are given by
\[
\begin{align*}
\frac{dx_1}{\beta_1 x_1 + \gamma_1} &= \frac{dx_2}{\beta_2 x_2 + \gamma_2} = \frac{dx_3}{\beta_3 x_3 + \gamma_3} = \frac{dx_4}{\beta_4 x_4 + \gamma_4}, \\
\frac{du_1}{\beta_4 u_1} &= \frac{du_2}{\alpha_{22} u_2} = \frac{du_3}{\alpha_{33} u_3} = \frac{du_4}{\alpha_{44} u_4} = \frac{du_5}{\alpha_{55} u_5}.
\end{align*}
\]

It may be noted that the parts of the auxiliary equations
\[
\begin{align*}
\frac{dx_1}{\beta_1 x_1 + \gamma_1} &= \frac{dx_2}{\beta_2 x_2 + \gamma_2} = \frac{dx_3}{\beta_3 x_3 + \gamma_3} = \frac{dx_4}{\beta_4 x_4 + \gamma_4} = \frac{du_1}{\beta_4 u_1},
\end{align*}
\]
are sufficient to determine the form of the solution \(u_1\); however, the rest of the conditions in (11) are helpful in cross checking or evaluating the relationship between the parameters \(\alpha_{ii}, \beta_j\) and \(\gamma_j\).
3 Extension to Third Order PDEs

In this section, we intend to extend the theory to third order equations by considering the following PDE in two independent variables

\[
\tau_1 \frac{\partial u}{\partial t} + \tau_2 \frac{\partial^2 u}{\partial t^2} + Au + B \frac{\partial u}{\partial x} + C \frac{\partial^2 u}{\partial x^2} + D \frac{\partial^3 u}{\partial x^3} = 0,
\]

(12)

which can be written in a similar form of system of equations as in (4)

\[
F_1 = \tau_1 u_2 + \tau_2 p_1^2 + Au_1 + Bu_3 + Cu_4 + D p_2^4 = 0,
\]

\[
F_2 = p_1^1 - u_2,
\]

\[
F_3 = p_2^1 - u_3,
\]

\[
F_4 = p_2^1 - u_4.
\]

(13)

In a similar way as in section - 2, it can be verified that

\[
\xi_1^1 x = \beta_1 x_1 + \gamma_1, \quad \xi_2^2 = \beta_2 x_2 + \gamma_2,
\]

\[
\xi_1^3 = \beta_2 u_1, \quad \xi_2^3 = \alpha_{33} u_3, \quad \xi_4^4 = \alpha_{44} u_4,
\]

(14)

where \(\beta_\ast, \beta_j, \gamma_j, j = 1, 2\) and \(\alpha_{rr}, r = 1, 2, 3, 4\) are arbitrary constants.

The invariant surface conditions (auxiliary equations) are given by

\[
\frac{dx_1}{\beta_1 x_1 + \gamma_1} = \frac{dx_2}{\beta_2 x_2 + \gamma_2} = \frac{du_1}{\beta_\ast u_1} = \frac{du_2}{\alpha_{22} u_2} = \frac{du_3}{\alpha_{33} u_3} = \frac{du_4}{\alpha_{44} u_4},
\]

(15)

where the relations between the coefficients \(\beta_\ast, \beta_j, \gamma_j, j = 1, 2\) and \(\alpha_{rr}, r = 1, 2, 3, 4\) can be obtained from the following conditions

\[
\xi_1^1 \frac{\partial A}{\partial x_1} + \xi_2^2 \frac{\partial A}{\partial x_2} = (\alpha_{11} - \beta_\ast) A,
\]

\[
\xi_1^1 \frac{\partial \tau_1}{\partial x_1} + \xi_2^2 \frac{\partial \tau_1}{\partial x_2} = (\alpha_{11} - \alpha_{22}) \tau_1,
\]

\[
\xi_1^1 \frac{\partial B}{\partial x_1} + \xi_2^2 \frac{\partial B}{\partial x_2} = (\alpha_{11} - \alpha_{33}) B,
\]

\[
\xi_1^1 \frac{\partial C}{\partial x_1} + \xi_2^2 \frac{\partial C}{\partial x_2} = (\alpha_{11} - \alpha_{44}) C,
\]

\[
\xi_1^1 \frac{\partial \tau_2}{\partial x_1} + \xi_2^2 \frac{\partial \tau_2}{\partial x_2} = (\alpha_{11} - \alpha_{22} + \beta_1) \tau_2,
\]

\[
\xi_1^1 \frac{\partial D}{\partial x_1} + \xi_2^2 \frac{\partial D}{\partial x_2} = (\alpha_{11} - \alpha_{44} + \beta_2) D,
\]

\[
\beta_\ast = \alpha_{22} + \beta_1 = \alpha_{33} + \beta_2.
\]

(16)
To demonstrate the applicabilities of the above discussion, we consider the following example of third order PDE with two independent variables

$$\frac{\partial u}{\partial t} + a \frac{\partial^3 u}{\partial x^3} = 0,$$

(17)

where $a$ is a constant. The equation turns out to be the linearized Korteweg de Vries (KdV) equation when $a = 1$ [16].

The above equation has the auxiliary equations (15), and the solution

$$u = (x_1 + \gamma_1^*)^{\beta_1/\beta_1} U(\xi), \quad \xi = \frac{(x_2 + \gamma_2^*)}{(x_1 + \gamma_1^*)^{1/3}},$$

(18)

where $\gamma_1^* = \gamma_1/\beta_1$, $\gamma_2^* = \gamma_2/\beta_2$. Using (18), equation (17) reduces to the following ODE

$$a \frac{d^3 U}{d\xi^3} - \frac{\xi dU}{3 \frac{d\xi}{U}} + U = 0.$$

(19)

4 Algorithm for Reduction of Independent Variables for Any Order Linear PDE’s

From the experiences gained from previous sections we propose the following algorithm for linear PDE’s of any order.

**Step - I:** Write the given linear PDE in ”$n$” independent variables $x_1, x_2...x_n$ and dependent variable $u_1$ as a system of ”$m$” equations as per the following:

$$F_1 = A u_1 + B_{2} u_2 + B_{3} u_3 + ... + B_m u_m$$
$$+ C_{11} \frac{\partial u_1}{\partial x_1} + C_{12} \frac{\partial u_1}{\partial x_2} + ... + C_{1n} \frac{\partial u_1}{\partial x_n}$$
$$+ C_{21} \frac{\partial u_2}{\partial x_1} + C_{22} \frac{\partial u_2}{\partial x_2} + ... + C_{2n} \frac{\partial u_2}{\partial x_n}$$
$$+ ... + ... + ... + ...$$
$$+ C_{m1} \frac{\partial u_m}{\partial x_1} + C_{m2} \frac{\partial u_m}{\partial x_2} + ... + C_{mn} \frac{\partial u_m}{\partial x_n},$$

and

$$F_i = u_i - \frac{\partial u_i}{\partial x_k},$$

(20)

where $l \neq i$ takes a particular value from 1 to $m$, $k$ takes a particular value from 1 to $n$ and $i$ varies from 2 to $m$. 
Step - II: Use following conditions to establish relationship between
\( \beta, \beta_i, \gamma_j, (j = 1, 2, ..., n) \) and \( \alpha_{rr}, (r = 1, 2, ..., m) \), required in step -III.

\[
\sum_{j=1}^{n} (\beta_j x_j + \gamma_j) \frac{\partial A}{\partial x_j} = (\alpha_{11} - \beta_*) A
\]
\[
\sum_{j=1}^{n} (\beta_j x_j + \gamma_j) \frac{\partial B_i}{\partial x_j} = (\alpha_{11} - \alpha_{ii}) B_i, \quad i = 2, 3, ... m,
\]
\[
\sum_{j=1}^{n} (\beta_j x_j + \gamma_j) \frac{\partial C_{rs}}{\partial x_j} = (\alpha_{11} - \alpha_{rr} + \beta_s) C_{rs}, \quad r = 2, 3, ... m, \quad s = 1, 2, ... n.
\]

\( \beta_* = \alpha_{22} + \beta_1 = \alpha_{33} + \beta_2 = ... = \alpha_{n+1,n+1} + \beta_n. \)

It may be noted that \( m \geq n + 1 \) and some of \( C_{rs} \) may be zero.

Step - III: Use the following invariant surface conditions i.e. auxiliary
equations to find the similarity variables and forms of solutions

\[
\frac{dx_1}{\beta_1 x_1 + \gamma_1} = \frac{dx_2}{\beta_2 x_2 + \gamma_2} = ... = \frac{dx_n}{\beta_n x_n + \gamma_n} = \frac{du_1}{\beta_* u_1} = \frac{du_2}{\beta_* u_2} = ... = \frac{du_m}{\alpha_{mm} u_m}, \quad (22)
\]

It may be noted that the conditions

\[
\frac{dx_1}{\beta_1 x_1 + \gamma_1} = \frac{dx_2}{\beta_2 x_2 + \gamma_2} = ... = \frac{dx_n}{\beta_n x_n + \gamma_n} = \frac{du_1}{\beta_* u_1}, \quad (23)
\]

are sufficient to reduce the given PDE to another PDE with fewer independent
variables. The rest of the conditions in the axillary equations (22) are helpful in cross checking or evaluating the relationship between different constants i.e. \( \beta, \beta_i, \gamma_j, j = 1, 2, ... n \) and \( \alpha_{rr}, r = 1, 2, ... m. \)

The above algorithm will reduce a given linear PDE to another linear PDE
with fewer independent variables and continuing the process may lead to the
solution of the PDE. The process can be continued till \( \xi_x = (\xi_{x1}, \xi_{x2}, ..., \xi_{xn}) \neq 0. \)

4.1 Example:

We consider the following fourth order equation

\[
\frac{\partial^2 u}{\partial t^2} + a^2 \frac{\partial^4 u}{\partial x^4} = 0, \quad (24)
\]

which is the evolution equation of transverse vibration of elastic rods [16].

Step - I: The equation (24) can be written as the following system of
equations

\[
F_1 = C_{21} \frac{\partial u_2}{\partial x_1} + C_{52} \frac{\partial u_5}{\partial x_2},
\]
\[ F_2 = u_2 - \frac{\partial u_1}{\partial x_1}, \quad F_3 = u_3 - \frac{\partial u_2}{\partial x_2}, \quad (25) \]
\[ F_2 = u_4 - \frac{\partial u_3}{\partial x_2}, \quad F_2 = u_5 - \frac{\partial u_4}{\partial x_2}, \]

where \( C_{21} = 1 \) and \( C_{25} = a^2 \).

**Step - II:**

From the forms of \( C_{21}, C_{25} \) and equations (21) we obtain

\[ \alpha_{11} - \alpha_{22} + \beta_1 = 0, \quad \alpha_{11} - \alpha_{55} + \beta_2 = 0, \]
\[ \beta_* = \alpha_{22} + \beta_1 = \alpha_{33} + \beta_2. \quad (26) \]

**Step - III:**

The auxiliary equations or invariant surface conditions (22) are given by

\[ \frac{dx_1}{\beta_1 x_1 + \gamma_1} = \frac{dx_2}{\beta_2 x_2 + \gamma_2} = \frac{du_1}{\beta_* u_1} = \frac{du_2}{\alpha_{22} u_2} = \frac{du_3}{\alpha_{33} u_3} = \frac{du_4}{\alpha_{44} u_4} = \frac{du_5}{\alpha_{55} u_5}. \quad (27) \]

It may be verified from equations (26) and (27) that \( \beta_1 = 2\beta_2 \). Thus, the solution \( u_1 \) and the similarity variable \( \xi \) are given by

\[ u_1 = (x_1 + \gamma_1^*)^{\beta_*/\beta_1} U(\xi), \quad \xi = \frac{(x_2 + \gamma_2^*)}{(x_1 + \gamma_1^*)^{1/2}}, \quad (28) \]

where \( \gamma_1^* = \gamma_1/\beta_1 \) and \( \gamma_2^* = \gamma_2/\beta_2 \). The equation (24) now reduced to

\[ a^2 \frac{d^4 U}{d\xi^4} + \frac{\xi^2}{2} \frac{d^2 U}{d\xi^2} - \xi \frac{dU}{d\xi} + U = 0. \quad (29) \]

**5 Conclusion**

In the present paper we considered generalized second order linear partial differential equation, which can be reduced to many important equations of interest in Mathematics and Mathematical Physics after suitable choice of its coefficients. We employed the method of Lie group of transformations after transforming the equation to a system of first order partial differential equations. We considered constantly conformally invariance of the system of equations and obtained a generalized system of auxiliary equations called invariant surface conditions to find the solution of the PDE or reduce the same to another PDE with fewer independent variables (or to an ODE) subject to condition that the Lie group of transformation has nonzero infinitesimals with respect to the independent variables. In fact, the infinitesimals with respect to
the independent variables satisfy a system of determining equations. We also considered a third order PDE in order to generalize the procedure to higher orders. We proposed an algorithm, which is capable of reducing a linear PDE of any order and with any number of independent variables to a linear PDE with fewer independent variables or to an ODE. The procedure suggested are verified by suitable examples.

References


[8] Norbert, E., Transformation Properties of $\ddot{x} + f_1(t)\dot{x} + f_2(t)x + f_3(t)x^n = 0$, *Nonlinear Mathematical Physics*, 4 N 3-4 (1997) 310 - 337.


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