On the Modified Arithmetic Asian Option Equation
and its Analytical Solution

Zieneb Ali Elshegmani, Rokiah Rozita Ahmed
and Saiful Hafizah Jaaman

School of Mathematical Sciences
Faculty of Science and Technology
University Kebangsaan Malaysia
Bangi 43600 Selangor D. Ehsan, Malaysia
zelsheqmani@yahoo.com, rozy@ukm.my, shj@ukm.my

Abstract

Solving Black-Scholes PDE of the arithmetic Asian option is outstanding problem in Mathematics, because the PDE is a degenerate partial differential equation in three dimension. Since there is no analytical solution for the arithmetic Asian option is known yet, we are thinking about modification the Black-Scholes Asian option equation. Using general stochastic differential equation we derive a modified partial differential equation for arithmetic Asian option. We provide four different modification, which some of them can be transformed to the classical Black-Scholes PDE and then to a parabolic equation with constant coefficient, which can solved analytically using means of partial differential equations.

Keywords: stochastic differential equation, Partial differential equations, arithmetic Asian option

1 Introduction

Asian options are a contracts with payoffs depend on the average of the underlying stock price $S$ and fixed or floating strike price over a specific period of time. There are two different types of Asian options according to the way of computing the average, geometric Asian options and arithmetic Asian options. For the case of geometric average there is a closed-form solution to price these types of options see Barucci E. [1]. However, for arithmetic average there is no explicit formula to price them. In fact, this problem is concentrated on how we can describe the distribution of the sum of lognormals, since the
distribution of the sum of lognormals is not lognormally distribution. Numerous techniques have been investigated in the literature to address the problem of pricing arithmetic Asian options. Most recently, Geman and Yor [5] used Laplace transform in time of the Asian option price. However, this transform is applicable in some cases. Rogers and Shi [7] reduce the three dimension PDE of arithmetic Asian option to a parabolic equation in two variables. However, it is difficult to solve this equation analytically or numerically, and they derive lower-bound formulas for Asian options by computing the expectation based on some zero-mean Gaussian variable. Zhang [11] provide a theory of continuously-sampled Asian option pricing, he solves the PDE with perturbation approach, he give a pricing formula for geometric Asian options, and he show that the PDE for arithmetic Asian options cannot be transformed into a parabolic equation with constant coefficient. Thompson [8] derive an upper bound more accurate than that derived by Rogers and Shi [7]. Vecer [9] derive a one-dimensional PDE for Asian options. Vecer and Xu [10] use a special semimartingale process models for pricing Asian options. Francois and Tony [6] derived accurate and fast numerical methods to solve Rogers and Shi PDE. Chen K. and Lyuu Y. [3] develop the lower-bound pricing formulas of Rogers and Shi PDE [4] to include general maturities instead one year. Cruz-Baez and Gonzalez-Rodrigues [4] obtain the same solution of Geman and Yor for arithmetic Asian options using Partial differential equations, integral transforms, and Mathematica programming, instead Bessel processes. In order to transform Asian option PDE to the classical Black-Scholes PDE and get its analytical solution we modified arithmetic Asian option PDE using general stochastic differential equations. this procedure appear for the first time for pricing arithmetic Asian options.

2 Derivation of the PDE for continuous arithmetic Asian option

We begin by assuming that the spot price $S_t$ of the underlying asset of the Asian option satisfies the stochastic differential equation

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

(1)

Where $W_t$ is a standard Brownian motion, $\mu$ and $\sigma$ are constants. We now consider continuous arithmetic Asian option with the average rate defined by the running sum or arithmetic mean of the underlying asset price.
respective.

\[ A_t = \int_0^t S_u du \]  

(2)

\[ A_t = \frac{1}{t} \int_0^t S_u du \]  

(3)

The differential forms for the above two equations respectively are

\[ dA_t = S_t dt \]  

(4)

\[ dA_t = \frac{1}{t} (S_t - A_t) dt \]  

(5)

Suppose an Asian option has payoff function \( \psi(S_T, A_T) \) at expiration date \( T \). The value of the Arithmetic Asian option at time \( t \) is

\[ V(t, S_t, A_t) = e^{-r(T-t)} E(\psi(S_T, A_T) | F_t) \]  

(6)

By multi-dimension of It’s Lemma we have

\[ dV = \left( \frac{\partial V}{\partial t} + \frac{\sigma^2 S^2}{2} \frac{\partial^2 V}{\partial S^2} + \mu S \frac{\partial V}{\partial S} \right) dt + \frac{\partial V}{\partial A} dA + \sigma S \frac{\partial V}{\partial S} dW \]

from Eq. (2.4) we have

\[ dV = \left( \frac{\partial V}{\partial t} + \frac{\sigma^2 S^2}{2} \frac{\partial^2 V}{\partial S^2} + \mu S \frac{\partial V}{\partial S} + S \frac{\partial V}{\partial A} \right) dt + \sigma S \frac{\partial V}{\partial S} dW \]  

(7)

\[ dV = h(t, S, A) dt + g(t, S, A) dW \]  

(8)

where

\[ h(t, S, A) = \frac{\partial V}{\partial t} + \frac{\sigma^2 S^2}{2} \frac{\partial^2 V}{\partial S^2} + \mu S \frac{\partial V}{\partial S} + S \frac{\partial V}{\partial A} \]  

(9)

and

\[ g(t, S, A) = \sigma S \frac{\partial V}{\partial S} \]  

(10)

Or using Eq. (2.5)

\[ dV = \left( \frac{\partial V}{\partial t} + \frac{\sigma^2 S^2}{2} \frac{\partial^2 V}{\partial S^2} + \mu S \frac{\partial V}{\partial S} + \frac{1}{t} (S - A) \frac{\partial V}{\partial A} \right) dt + \sigma S \frac{\partial V}{\partial S} dW \]  

(11)
Set up a portfolio of two Asian options $V_1$ and $\Delta V_2$ with different maturities $T_1$ and $T_2$ respectively, where $\Delta$ is a number of second Asian options, and an underlying asset $C$. The value of this portfolio is:

$$\Pi = V_1(t, S_t, A_t) - \Delta V_2(t, S_t, A_t) + C$$

(12)

from Eq. (2.8) we can rewrite the above equation as following

$$\Pi = (h_1 - \Delta h_2) + (g_1 - \Delta g_2) + C$$

(13)

And the change in the value of this portfolio is:

$$d\Pi = dV_1(t, S_t, A_t) - \Delta dV_2(t, S_t, A_t) + rCdt$$

(14)

$$d\Pi = (h_1 - \Delta h_2) dt + (g_1 - \Delta g_2) dW + rCdt$$

(15)

To get rid from the stochastic term of the risk free portfolio, choose $\Delta = \frac{g_1}{g_2}$

$$d\Pi = \left( h_1 - \frac{g_1}{g_2} h_2 + rC \right) dt$$

(16)

from the above equation it is seen that, the portfolio is risk-less over time $dt$, so the return of the portfolio must equal to the return of other risk-free securities. the change in the value of the portfolio with riskless asset is riskless asset is

$$d\Pi = r\Pi dt$$

(17)

From Eq. (2.16) and (2.17) we have

$$d\Pi = \left( h_1 - \frac{g_1}{g_2} h_2 + rC \right) dt = r\Pi dt$$

$$\left( h_1 - \frac{g_1}{g_2} h_2 + rC \right) dt = r \left( V_1 - \frac{g_1}{g_2} V_2 + C \right) dt$$

$$\left( h_1 - \frac{g_1}{g_2} h_2 \right) = r \left( V_1 - \frac{g_1}{g_2} V_2 \right)$$

$$h_1 - \frac{g_1}{g_2} h_2 = rV_1 - \frac{g_1}{g_2} rV_2$$

$$h_1 - rV_1 = \frac{g_1}{g_2} (h_2 - rV_2)$$
Modified arithmetic Asian option equation

\[
\frac{h_1 - rV_1}{g_1} = \frac{h_2 - rV_2}{g_2}
\] (18)

The above equation is independent of both maturities \( T_1 \) and \( T_2 \), and depends only on \( t, S, A \), so we can assume that

\[
\frac{h - rV}{g} = \varphi (t, S, A) \quad (19)
\]

\[
h = rV + g\varphi (t, S, A) \quad (20)
\]

From Eq. (2.10) we have

\[
h = rV + \sigma S \frac{\partial V}{\partial S} \varphi (t, S, A) \quad (21)
\]

And from Eq. (2.9)

\[
\frac{\partial V}{\partial t} + \frac{\sigma^2 S^2}{2} \frac{\partial^2 V}{\partial S^2} + \mu S \frac{\partial V}{\partial S} + S \frac{\partial V}{\partial A} = rV + \sigma S \frac{\partial V}{\partial S} \varphi (t, S, A) \quad (22)
\]

\[
\frac{\partial V}{\partial t} + \frac{\sigma^2 S^2}{2} \frac{\partial^2 V}{\partial S^2} + (\mu S - \sigma S \varphi (t, S, A)) \frac{\partial V}{\partial S} + S \frac{\partial V}{\partial A} - rV = 0 \quad (23)
\]

Or

\[
\frac{\partial V}{\partial t} + \frac{\sigma^2 S^2}{2} \frac{\partial^2 V}{\partial S^2} + (\mu S - \sigma S \varphi (t, S, A)) \frac{\partial V}{\partial S} + \frac{1}{t}(S - A) \frac{\partial V}{\partial A} - rV = 0 \quad (24)
\]

The above two equations are the modified arithmetic Asian options equations. Now we will show how the modified PDE of the Asian options changes according to the value of the function \( \varphi (t, S, A) \)

- The first case: when \( \varphi (t, S, A) = \frac{\mu - r}{\sigma} \) Then Eq. (2.23) and (2.24) are reduced to the classical arithmetic Asian option respectively

\[
\begin{align*}
\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} + S \frac{\partial V}{\partial A} - rV &= 0 \\
V(T, S, A) &= \psi \left( S, \frac{A}{T} \right)
\end{align*}
\]

\[
\begin{align*}
\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} + \frac{1}{t}(S - A) \frac{\partial V}{\partial A} - rV &= 0 \\
V(T, S, A) &= \psi(S, A)
\end{align*}
\]
• The second case: when \( \varphi(t, S, A) = \frac{A}{S} \) then Eq. (2.23), (2.24) are reduced respectively to

\[
\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + S \frac{\partial V}{\partial A} - rV = 0 \tag{25}
\]

\[
\frac{\partial \tilde{V}}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 \tilde{V}}{\partial S^2} + \frac{1}{t} (S - A) \frac{\partial \tilde{V}}{\partial A} - r\tilde{V} = 0 \tag{26}
\]

To solve equation (2.25) make the following change of variables

\[
V(t, S, A) = e^{rt} \tilde{V}(t, S, A)
\]

\[
\frac{\partial \tilde{V}}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 \tilde{V}}{\partial S^2} + S \frac{\partial \tilde{V}}{\partial A} = 0 \tag{27}
\]

\[
\tilde{V}(T, S, A) = e^{-rT} \psi(S, \frac{A}{T})
\]

Assume \( \tau = T - t \)

\[
\frac{\partial \tilde{V}}{\partial \tau} = \frac{\sigma^2 S^2}{2} \frac{\partial^2 \tilde{V}}{\partial S^2} + S \frac{\partial \tilde{V}}{\partial A} \tag{28}
\]

Now make the following changes

\[
\tilde{V}(\tau, S, A) = f(\tau, x), x = \frac{A}{S} - \tau, x > 0
\]

Equation (2.28) becomes

\[
\frac{\partial f}{\partial \tau} = \sigma^2(x - \tau) \frac{\partial f}{\partial x} + \frac{\sigma^2}{2} (x - \tau)^2 \frac{\partial^2 f}{\partial x^2} \tag{29}
\]

\[
f(0, x) = e^{(-rT)} \psi(x)
\]

To get rid from the derivation respect to the time and have ODE instead of PDE we will apply Laplace transform in time.

\[
L f(\tau) = f^*(\xi) = \int_0^\infty f(\tau) e^{-\tau \xi} d\tau
\]

\[
L^{-1} \{f^*(\xi)\} = f(\tau) = \frac{1}{2\pi i} \int_0^\infty f^*(\xi) e^{\tau \xi} d\xi
\]
\[
L \left\{ \frac{\partial f}{\partial \tau} \right\} = \xi f^* (\xi) - f(0)
\]

\[
L \{ \tau \} = \int_0^{\infty} \tau e^{-\xi \tau} d\tau = \frac{1}{\xi^2}
\]

After using the above transforms Eq. (2.29) becomes

\[
\xi f^* - e^{-rT} \psi(x) = \sigma^2 \left( x - \frac{1}{\xi^2} \right) \frac{\partial f^*}{\partial x} + \frac{\sigma^2}{2} \left( x - \frac{1}{\xi^2} \right)^2 \frac{\partial^2 f^*}{\partial x^2}
\]  

(30)

Assume

\[
f^*(\xi, x) = f^*(\xi, k), \quad k = x - \frac{1}{\xi^2}
\]

\[
\xi f^* - e^{-rT} \psi(k + \frac{1}{\xi^2}) = \sigma^2 k \frac{\partial f^*}{\partial k} + \frac{\sigma^2}{2} k^2 \frac{\partial^2 f^*}{\partial k^2}
\]  

(31)

Applying inverse Laplace in \(\xi\)

\[
L^{-1} \{ \xi f^*(\xi) \} = \frac{\partial f(\tau)}{\partial \tau}
\]

(32)

\[
\frac{\partial f}{\partial \tau} - e^{-rT} \psi(k) = \sigma^2 k \frac{\partial f}{\partial k} + \frac{\sigma^2}{2} k^2 \frac{\partial^2 f}{\partial k^2}
\]

We have \(k > 0\), so we can assume \(z = \ln k\)

\[
\frac{\partial f}{\partial \tau} - e^{rT} \psi (e^z) = \frac{\sigma^2}{2} \frac{\partial f}{\partial z} + \frac{\sigma^2}{2} \frac{\partial^2 f}{\partial z^2}
\]  

(33)

Make the following changes

\[
g(\tau, \eta) = f(\tau, Z = \eta + \frac{\sigma^2}{2} \tau)
\]

\[
\frac{\partial g}{\partial \tau} - e^{rT} \psi (e^\eta) = \frac{\sigma^2}{2} \frac{\partial^2 g}{\partial \eta^2}
\]  

(34)

\[
g(0, x) = e^{rT} \psi (e^\eta) = \psi (\eta)
\]
This is an inhomogeneous parabolic equation. After using Fourier transform in $\eta$ we have

$$\frac{\partial \hat{g}}{\partial \tau} - \psi(\omega) = -\frac{1}{2}\sigma^2 \omega^2 \hat{g}$$

(35)

$$\hat{g}(0, \omega) = \psi(\omega)$$

$$\hat{g}(\tau, \omega) = \int \psi(\omega) e^{\frac{1}{2} \sigma^2 \omega^2 \tau} d\tau + c$$

$$e^{\frac{1}{2} \sigma^2 \omega^2 \tau} \left[ -\int_0^T \psi(\omega) e^{\frac{1}{2} \sigma^2 \omega^2 \tau} d\tau + \psi(\omega) \right]$$

(36)

$$\hat{g}(\tau, \omega) = \frac{e^{\frac{1}{2} \sigma^2 \omega^2 \tau}}{\frac{1}{2} \sigma^2 \omega^2} \psi(\omega) \left[ 2 - e^{\frac{1}{2} \sigma^2 \omega^2 T} \right]$$

To solve equation (2.24) we make the following change of variable

$$V(t, S, A) = e^{rt} f(t, x), x = \frac{tA}{S}$$

$$\frac{\partial f}{\partial t} + \frac{\sigma^2}{2} x^2 \frac{\partial^2 f}{\partial x^2} + (\sigma^2 x + 1) \frac{\partial f}{\partial x} = 0$$

(37)

$$f(T, x) = \pi(x)$$

This equation is similar to the Rogers and Shi equation [7], and if we assume $z = x - t$

$$\frac{\partial f}{\partial t} + \frac{\sigma^2}{2} (z + t)^2 \frac{\partial^2 f}{\partial z^2} + \sigma^2 (z + t) \frac{\partial f}{\partial z} = 0$$

(38)

$$f(T, x) = \pi(x)$$

The equation becomes similar to Francois and Tony [6] equation for transform Asian option, so we can solve it by the same method.
• The third case: when \( \varphi(t, S, A) = \frac{S}{\sigma A} \). Equation (2.23) is reduced to

\[
\frac{\partial V}{\partial t} + \frac{\sigma^2 S^2}{2} \frac{\partial^2 V}{\partial S^2} + \left( \mu S + \frac{S^2}{tA} \right) \frac{\partial V}{\partial S} + S \frac{\partial V}{\partial A} - rV = 0
\]  

(39)

To solve this equation use the following change of variables

\[ V(t, S, A) = f(t, x = \frac{A}{S}) \]

\[
\frac{\partial f}{\partial t} + \frac{\sigma^2}{2} x^2 \frac{\partial^2 f}{\partial x^2} + \left( \sigma^2 - \mu \right) x \frac{\partial f}{\partial x} - rf = 0, x > 0
\]

(40)

This equation is similar to the classical Black-Scholes PDE, so one can easily transform it to a parabolic equation using the following change of variable

\[ f(t, x) = e^{rt} f(t, z), z = \ln x, x > 0 \]

And taking into account that

\[ x^2 \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 f}{\partial z^2} - \frac{\partial f}{\partial z} \]

Then Eq. (2.40) becomes:

\[
\frac{\partial f}{\partial t} + \frac{\sigma^2}{2} \frac{\partial^2 f}{\partial z^2} \quad \frac{\sigma^2}{2} \frac{\partial f}{\partial z} - \mu \frac{\partial f}{\partial z} = 0
\]

(41)

\[
\frac{\partial f}{\partial t} + \frac{\sigma^2}{2} \frac{\partial^2 f}{\partial z^2} + \left( \frac{\sigma^2}{2} - \mu \right) \frac{\partial f}{\partial z} = 0
\]

(42)

Now applying the following changes of the variables

\[ f(t, z) = h(t, \xi = z - \left( \frac{\sigma^2}{2} - \mu \right) t) \]

\[
\frac{\partial h}{\partial t} + \frac{\sigma^2}{2} \frac{\partial^2 h}{\partial \xi^2} = 0
\]

(43)

Let \( \tau = T - t \)

\[
\frac{\partial h}{\partial \tau} = \frac{\sigma^2}{2} \frac{\partial^2 h}{\partial \xi^2}
\]

(44)

Introducing further a new independent variable \( \eta \) by \( \xi = \sqrt{\sigma} \eta \)

\[
\frac{\partial h}{\partial \tau} = \frac{\partial^2 h}{\partial \eta^2}
\]

(45)

This is a simpler parabolic equation, for full details of its solution see [2].
• The fourth case: when $\varphi(t, S, A) = -\frac{S}{\sigma t A}$ then equation (2.24) becomes
\[
\frac{\partial V}{\partial t} + \frac{\sigma^2 S^2}{2} \frac{\partial^2 V}{\partial S^2} + \left(\mu S + \frac{S^2}{t A}\right) \frac{\partial V}{\partial S} + \frac{1}{t} (S - A) \frac{\partial V}{\partial A} - rV = 0 \quad (46)
\]
To solve the above equation we will use the following change of variables
\[
V(t, S, A) = f(t, x), \quad x = \frac{t A}{S}
\]
\[
\frac{\partial f}{\partial t} + \frac{\sigma^2}{2} x^2 \frac{\partial^2 f}{\partial x^2} + (\sigma^2 - \mu) x \frac{\partial f}{\partial x} - rf = 0, \quad x \geq 0 \quad (47)
\]
This equation is similar to the classical Black-Scholes PDE, so one can easily transform it to a parabolic equation.
We have $x = \frac{t A}{S}$, so $x = 0$ at $t = 0$ otherwise $x > 0$
Firstly, assume $x > 0$ and value the option at $t > 0$ as we have done above.
We can solve the equation at $x = 0$ using Millen transform as following
\[
\frac{\partial \hat{f}}{\partial \tau} = \left(\frac{\sigma^2}{2} \left(z^2 + z\right) - (\sigma^2 - \mu) z - r\right) \hat{f} \quad (48)
\]
\[
\hat{f} = ce^{\left(\frac{\sigma^2}{2} \left(z^2 + z\right) - (\sigma^2 - \mu) z - r\right) \tau} \quad (49)
\]
\[
\hat{f}(0, z, A) = \psi(z, A)
\]
\[
\hat{f} = \psi(z, A) e^{\left(\frac{\sigma^2}{2} \left(z^2 + z\right) - (\sigma^2 - \mu) z - r\right) \tau} \quad (50)
\]
Where $\psi(z, A)$ is depends on the kind of the option.

3 Conclusion

Pricing continuous arithmetic Asian option is the difficult issue in finance for several decades since there is no analytical solution for the PDE of arithmetic Asian option. In this paper we solved the problem with stochastic and partial differential equation approach. We have derived the modified arithmetic Asian option equation using means of stochastic differential equation, and we have shown that the modified PDE can be transformed to the classical Black-Scholes equation and then to the parabolic equation which one can easily solve it analytically.
References


J. E. Zhang, Theory of continuously-sampled Asian option pricing, City University of Hong Kong (2000).

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