$P_9$ - Factorization of Complete Bipartite Graph

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Abstract

$P_k$ -factorization of a complete bipartite graph for $k$, an even integer was studied by H. Wang [1]. Further, Beiling Du [2] extended the work of H.Wang, and studied the $P_{2k}$-factorization of complete bipartite multigraph. For odd value of $k$ the work on factorization was done by a number of researchers[3,4,5]. $P_3$-factorization of complete bipartite graph was studied by K.Ushio [3]. $P_7$-factorization of complete bipartite graph was studied by J.Wang etal [4]. Further, $P_7$-factorization was studied by J.Wang [5] and he gave necessary and sufficient conditions for its existence. In the present paper we study the $P_9$-factorization of complete bipartite graph $K_{m,n}$ and show that the necessary and sufficient conditions for its existence are:

1. $5m \geq 4n$,
2. $5n \geq 4m$,
3. $m + n \equiv 0(\text{mod } 9)$ and
4. $9mn/8(m + n)$ is an integer.

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1 Introduction

Let $K_{m,n}$ be a complete bipartite graph with two partite sets $X$ and $Y$. The $P_k$ -factorization of $K_{m,n}$ is a partition of the set of $P_k$ -factors of $K_{m,n}$. Ushio[6] gave the following necessary and sufficient conditions for the existence of $P_k$ -factorization of $K_{m,n}$, when $k$ is odd and $m$ and $n$ are positive integers :

1. $(k + 1)m \geq (k - 1)n$,
2. $(k + 1)n \geq (k - 1)m$,
3. $m + n \equiv 0(\text{mod } k)$ and
4. $kmn/[(k - 1)(m + n)]$is an integer.
The generalization was proved by J. Wang et al. [4] for $k = 5$ and J. Wang [5] for $k = 7$. In the present paper we will prove that this generalization is also true for $k = 9$. To verify this result the necessary lemmas and theorem are developed.

## 2 Mathematical Analysis

Theorem 1. Let $m$ and $n$ be positive integers. Then $K_{m,n}$ has $P_9$-factorization if and only if the following are satisfied:

1. $5m \geq 4n$;
2. $5n \geq 4m$;
3. $m + n \equiv 0 \pmod{9}$ and
4. $9mn/(8(m + n))$ is an integer.

To prove this theorem the following well established number theoretic result is used.

**Lemma 1.** Let $g$, $h$, $p$ and $q$ be positive integers. If $\gcd(p, q) = 1$, then $\gcd(pq, p + gq) = \gcd(p, q)$. Similarly, if $\gcd(gp, hq) = 1$ then $\gcd(gp + hq, pq) = 1$.

The following theorem will also be used in the proof.

**Theorem 2.** If $K_{m,n}$ has $P_9$-factorization then $K_{sm, sn}$ has also $P_9$-factorization for every positive integer $s$.

**Proof.** Let $K_{s, s}$ be 1-factorable [7], and $H_1, H_2, \ldots, H_s$ be a 1-factorization of it. For each $i$ with $1 \leq i \leq s$, replace every edge of $H_i$ with a $K_{m,n}$ to get spanning sub graphs $G_i$ of $K_{sm, sn}$ such that the $G_i$’s $1 \leq i \leq s$ are pair wise edge disjoint and there sum is $K_{sm, sn}$. Since $K_{m,n}$ is $P_9$-factorable, therefore it is obvious that each $G_i$ is also $P_9$-factorable, and hence, $K_{sm, sn}$ is also $P_9$-factorable.

Now to prove the theorem (1), consider three cases:-

**Case I ($5m = 4n$):** In this case $K_{m,n}$ has a $P_9$-factorization, since $K_{8,10}$ (trivial case) has $P_9$-factorization

\[ y_1x_1y_2x_2y_3x_3y_4x_4y_5, y_6x_5y_7x_7y_8x_8y_9x_9y_10, \]
\[ y_1x_1y_2x_2y_3x_3y_4x_4y_5, y_6x_5y_7x_7y_8x_8y_9x_9y_10, \]
\[ y_1x_3y_3x_3y_4x_4y_5, y_6x_5y_7x_7y_8x_8y_9x_9y_10, \]
\[ y_1x_4y_2x_5y_3x_6y_4x_7y_7x_8y_8x_9y_10, y_6x_5y_7x_7y_8x_8y_9x_9y_10, \]
\[ y_1x_5y_2x_6y_3x_7y_4x_8y_5, y_6x_5y_7x_7y_8x_8y_9x_9y_10. \]

**Case II ($5n = 4m$), swapping the values of $m, n$):** Obviously $K_{m,n}$ has $P_9$-factorization.

**Case III ($5m > 4n$ and $5n > 4m$):** Let $a = \frac{5m-4n}{9}$, $b = \frac{5n-4m}{9}$, $t = \frac{m+n}{9}$ and $r = \frac{9mn}{8(m+n)}$.

Where $a, b, t$ and $r$ must be integers, (for $m, n$ satisfying theorem1). We have $m = 4a + 5b, n = 5a + 4b, r = \frac{5(a+b)}{2} + z$, where $z = \frac{ab}{8(a+b)}$. 


Case (2): If \( \gcd(p, a) = 10 \) and \( \gcd(p, b) = 20 \), there exist a positive integer \( s \) such that
\[
\begin{aligned}
& \text{a positive integer } \alpha \text{ exists a positive integer } s \text{ exist a positive integer } m, n, a, b \text{ such that} \\
& \text{m, n, a, b exist a positive integer } s \text{ exist a positive integer } m, n, a, b \text{ such that} \\
& \text{m, n, a, b exist a positive integer } s \text{ exist a positive integer } m, n, a, b \text{ such that}
\end{aligned}
\]

Let \( \gcd(4a, 5b) = d \) and therefore \( 4a = dp, 5b = dq \) for some \( p, q \); where \( \gcd(p, q) = 1 \). Consequently \( z = \frac{dpq}{8(5p+4q)} \).

These equalities imply the following equalities:
\[
\begin{aligned}
& d = \frac{8(5p+4q)}{pq}, m = \frac{8(p+q)(5p+4q)}{pq}, n = \frac{2(25p+16q)(5p+4q)}{pq}, r = \frac{(p+q)(25p+16q)}{pq}, a = \frac{2p(5p+4q)}{pq} \text{ and } b = \frac{8q(5p+4q)}{pq}.
\end{aligned}
\]

Now to compute the values of \( m, n, a, b \) and \( r \), we established the following lemma:

**Lemma 2.**

Case (1): If \( \gcd(p, 16) = 1 \) and \( \gcd(q, 25) = 1 \), then there exist a positive integer \( s \) such that,
\[
\begin{aligned}
& m = 40(p + q)(5p + 4q), n = 2(25p + 16q)(5p + 4q), s, \\
& a = 10p(5p + 4q), b = 8q(5p + 4q) \text{ and } r = 5(p + q)(25p + 16q).
\end{aligned}
\]

Case (2): If \( \gcd(p, 16) = 1 \) and \( \gcd(q, 25) = 5 \), let \( q = 5q_1 \). Then there exist a positive integer \( s \) such that
\[
\begin{aligned}
& m = 40(p + 5q_1)(p + 4q_1), n = 10(5p + 16q_1)(p + 4q_1), s, \\
& a = 10p(p + q_1), b = 40q_1(p + 4q_1) \text{ and } r = 5(p + 5q_1)(5p + 16q_1).
\end{aligned}
\]

Case (3): If \( \gcd(p, 16) = 1 \) and \( \gcd(q, 25) = 25 \), let \( q = 25q_2 \). Then there exist a positive integer \( s \) such that
\[
\begin{aligned}
& m = 8(p + 25q_2)(p + 20q_2), n = 10(p + 16q_2)(p + 20q_2), s, \\
& a = 2p(p + 20q_2), b = 40q_2(p + 20q_2) \text{ and } r = 5(p + 25q_2)(p + 16q_2).
\end{aligned}
\]

Case (4): If \( \gcd(p, 16) = 2 \) and \( \gcd(q, 25) = 1 \), let \( p = 2p_1 \). Then there exist a positive integer \( s \) such that
\[
\begin{aligned}
& m = 40(2p_1 + q)(5p_1 + 2q), n = 4(25p_1 + 8q)(5p_1 + 2q), s, \\
& a = 20p_1(5p_1 + 2q), b = 8q(5p_1 + 2q) \text{ and } r = 5(2p_1 + q)(25p_1 + 8q).
\end{aligned}
\]

Case (5): If \( \gcd(p, 16) = 2 \) and \( \gcd(q, 25) = 5 \), let \( p = 2p_1, q = 5q_1 \). Then there exist a positive integer \( s \) such that
\[
\begin{aligned}
& m = 40(2p_1 + 5q_1)(p_1 + 2q_1), n = 20(5p_1 + 8q_1)(p_1 + 2q_1), s, \\
& a = 20p_1(p_1 + 2q_1), b = 40q_1(p_1 + 2q_1) \text{ and } r = (2p_1 + 5q_1)(5p_1 + 8q_1).
\end{aligned}
\]

Case (6): If \( \gcd(p, 16) = 2 \) and \( \gcd(q, 25) = 25 \), let \( p = 2p_1, q = 25q_2 \). Then there exist a positive integer \( s \) such that
\[
\begin{aligned}
& m = 40(2p_1 + 25q_2)(p_1 + 10q_2), n = 20(p_1 + 8q_2)(p_1 + 10q_2), s, \\
& a = 40p_1(p_1 + 10q_2), b = 40q_2(p_1 + 10q_2) \text{ and } r = 5(2p_1 + 25q_2)(p_1 + 8q_2).
\end{aligned}
\]

Case (7): If \( \gcd(p, 16) = 4 \) and \( \gcd(q, 25) = 1 \), let \( p = 4p_2 \). Then there exist a positive integer \( s \) such that
Case (1): If $\gcd(p, 16) = 4$, $\gcd(q, 25) = 5$, let $p = 4p_2, q = 5q_1$. Then there exist a positive integer $s$ such that

$m = 40(4p_2 + q)(5p_2 + q)s, n = 8(25p_2 + 4q)(5p_2 + q)s,$
$a = 40p_2(5p_2 + q)s, b = 8q(5p_2 + q)s$ and $r = 5(25p_2 + 4q)(4p_2 + q)s.$

Case (8): If $\gcd(p, 16) = 8$, $\gcd(q, 25) = 8$ let $p = 4p_2, q = 25q_2$. Then there exist a positive integer $s$ such that

$m = 8(4p_2 + 25q_2)(p_2 + 5q_2)s, n = 40(p_2 + 4q)(p_2 + 5q_2)s, a = 8p_2(p_2 + 5q_2)s,$
$b = 40q_2(p_2 + 5q_2)s$ and $r = 5(4p_2 + 25q_2)(p_2 + 4q_2)s.$

Case (9): If $\gcd(p, 16) = 5, \gcd(q, 25) = 25$ let $p = 8p_3, q = 25q_2$. Then there exist a positive integer $s$ such that

$m = 20(8p_3 + q)(10p_3 + q)s, n = 8(25p_3 + 2q)(10p_3 + q)s,$
$a = 40p_3(10p_3 + q)s, b = 4q(10p_3 + q)s$ and $r = 5(8p_3 + q)(25p_3 + 2q)s.$

Case (11): If $\gcd(p, 16) = 8, \gcd(q, 25) = 5$, let $p = 8p_3, q = 5q_1$. Then there exist a positive integer $s$ such that

$m = 20(8p_3 + 5q_1)(2p_3 + q_1)s, n = 40(5p_3 + 2q_1)(2p_3 + q_1)s,$
$a = 40p_3(2p_3 + q_1)s, b = 20q_1(2p_3 + q_1)s$ and $r = 5(8p_3 + 5q_1)(5p_3 + 2q_1)s.$

Case (12): If $\gcd(p, 16) = 8, \gcd(q, 25) = 25$, let $p = 8p_3, q = 25q_2$. Then there exist a positive integer $s$ such that

$m = 4(8p_3 + 25q_2)(2p_3 + 5q_2)s, n = 40(p_3 + 2q_2)(2p_3 + 5q_2)s,$
$a = 8p_3(2p_3 + 5q_2)s, b = 20q_2(2p_3 + 5q_2)s$ and $r = 5(8p_3 + 25q_2)(p_3 + 2q_2)s.$

Case (13): If $\gcd(p, 16) = 16, \gcd(q, 25) = 1$, let $p = 16p_4$. Then there exist a positive integer $s$ such that

$m = 10(16p_4 + q)(20p_4 + q)s, n = 8(25p_4 + q)(20p_4 + q)s,$
$a = 40p_4(20p_4 + q)s, b = 2q(10p_4 + q)s$ and $r = 5(16p_4 + q)(25p_4 + q)s.$

Case (14): If $\gcd(p, 16) = 16, \gcd(q, 25) = 5$, let $p = 16p_4, q = 5q_1$. Then there exist a positive integer $s$ such that

$m = 10(16p_4 + 5q_1)(4p_4 + q_1)s, n = 40(5p_4 + q_1)(4p_4 + q_1)s,$
$a = 8p_4(4p_4 + q_1)s, b = 10q_1(4p_4 + q_1)s$ and $r = 5(16p_4 + 5q_1)(5p_4 + q_1)s.$

Case (15): If $\gcd(p, 16) = 16, \gcd(q, 25) = 25$, let $p = 16p_4, q = 25q_2$. Then there exist a positive integer $s$ such that

$m = 2(16p_4 + 25q_2)(4p_4 + 5q_2)s, n = 40(p_4 + q_2)(4p_4 + 5q_2)s,$
$a = 8p_4(4p_4 + 5q_2)s, b = 10q_2(4p_4 + 5q_2)s$ and $r = 5(16p_4 + 25q_2)(p_4 + q_2)s.$

Proof: We are giving the proof of case (1). Let $\gcd(p, q) = 1$, $\gcd(p, 16) = 1$and $\gcd(q, 25) = 1$, then $\gcd(25p + 16q, 5) = 1 = \gcd(5p + 4q, 5)$ and $\gcd(25p, 16q) = \gcd(5p, 4q) = 1.$

Hence, $\gcd(25p + 16q, pq) = \gcd(5p + 4q, pq) = 1$ (lemma 1). Since, $n = \frac{2p + 16q}{5p_2}$ is an integer, hence we observe that $\frac{z}{5pq}$ (call it $s$) will be an integer. Then the equalities in (1) hold.

The proofs of other equalities in different cases are similar to (1). Now for each case of lemma (2) we will establish the values of $m$ and $n$ for $P_3$-factorization.
We observe that cases (1) and (15), (2) and (14), (3) and (13), (4) and (12), (5) and (11), (6) and (10) and (7) and (9) are symmetrical. Therefore we give the direct construction of some cases the rest will be obvious.

Lemma 3. For any positive integer $p$ and $q$ let $m = 40(p + q)(5p + 4q)$, and $n = 2(25p + 16q)(5p + 4q)$. Then $K_{m,n}$ has $P_9$-factorization.

Proof: Let $a = 10p(5p + 4q)$ and $b = 8q(5p + 4q)$. Hence $t = a + b = 2(5p + 4q)^2$, and $r = r_1.r_2$, where $r_1 = 5(p + q)$ and $r_2 = (25p + 16q)$.

Let $X, Y$ be two partite sets of $K_{m,n}$ and set $X = \{x_{i,j}; 1 \leq i \leq r_1, 1 \leq j \leq m_0\}$, and $Y = \{y_{i,j}; 1 \leq i \leq r_2, 1 \leq j \leq n_0\}$, where $m_0 = m/r_1 = 8(5p + 4q)$ and $n_0 = n/r_2 = 2(5p + 4q)$.

In this case there will be $a = 10p(5p + 4q)$, type $M$, $P_9$ - factor and $b = 8q(5p + 4q)$ type $W$, $P_9$ - factor. Here type $M$ denotes the $P_9$-factor with its end points in $Y$ and type $W$ denote the $P_9$-factor with its end points in $X$.

Now for each $1 \leq i \leq 5p$, let $1 \leq j \leq 2(5p + 4q)$, $0 \leq u \leq 1$, $0 \leq v, w \leq 1$, $0 \leq s, t \leq 1$

$\chi_q = \{x_{i,j} + (5p+4q)u + 3(5p+4q)v+3(5p+4q)w y_{5(i-1)+s+t+u+v+1,j+2(i-1)+s+t}\}$

and for each $1 \leq i \leq q$, let $1 \leq j \leq 2(5p + 4q)$, $1 \leq u \leq 3, 0 \leq v, w, t \leq 1$,

$E_{5q+i} = \{x_{5q+5(i-1)+2(u-1)+v,j+2(5p+4q)+3(5p+4q)w+(5p+4q)t}$

$y_{25p+16(i-1)+2u+v+1,j+10p+8(i-1)+2u+v+w}\}$.

Let $F = U_{1 \leq i \leq 5p+q}E_i$. Obviously $F$ contains $t = a + b = 2(5p + 4q)^2 = (5p + 4q)r_0$ number of vertex disjoint and edge disjoint $P_9$ component and span $K_{m,n}$. Then the graph $F$ is a $P_9$- factor of $K_{m,n}$. Define a bijection $\sigma : X \cup Y \rightarrow X \cup Y$ such that $\sigma(x_{i,j}) = x_{i+1,j}$ and $\sigma(y_{i,j}) = y_{i+1,j}$ for each $i \in \{1, 2, ..., r_1\}$ and each $j \in \{1, 2, ..., r_2\}$.

Let $F_{\xi,\eta} = z_1 \{\sigma^2(x)\sigma^\eta(y) : x \in X, y \in Y, xy \in F\}$. It is shown that the graph, $F_{\xi,\eta} \{1 \leq \xi \leq r_1, 1 \leq \eta \leq r_2\}$ are edge disjoint $P_9$-factor of $K_{m,n}$ and its union is also $K_{m,n}$. Thus $\{F_{\xi,\eta} : 1 \leq \xi \leq r_1, 1 \leq \eta \leq r_2\}$ is a $P_9$-factorization of $K_{m,n}$.

The proofs of following lemmas (4.7) are similar to lemma 3, with only changes in the values as given below.

Lemma 4. For any positive integer $p$ and $q$, let $m = 40(p + 5q)(p + 4q)$ and $n = 10(5p + 16q)(p + 4q)$ then $K_{m,n}$ has $P_9$-factorization.

Proof: Let $a = 10p(p + 4q)$, $b = 40q(p + 4q)$ and $r = r_1.r_2$ where $r_1 = 5(p + 5q)$, $r_2 = (5p + 16q)$.

Let $X = \{x_{i,j} : 1 \leq i \leq r_1, 1 \leq j \leq 8(p + 4q)\}$,

$Y = \{y_{i,j} : 1 \leq i \leq r_2, 1 \leq j \leq 10(p + 4q)\}$.

For each $1 \leq i \leq 5p$, let

$E_i = \{x_{i,j} + 3(5p+4q)u y_{i,j} + 2(i-1) + (u-1) + v : 1 \leq j \leq 2(p + 4q), 0 \leq u \leq 2, 0 \leq v \leq 1\}$.

Now for each $1 \leq i \leq 5q$, let

$E_{5q+i} = \{x_{5q+5(i-1)+2u+v,j+3(p+4q)u y_{5q+3(i-1)+q+u+v,j+10p+8(i-1)+4u+v-1}$

$+ 1 \leq j \leq 2(p + 4q), 0 \leq u \leq 2, 0 \leq v \leq 1\}$.

Lemma 5. For any positive integer $p$ and $q$, let $m = 40(2p + q)(5p + 2q)$, and $n = 4(25p + 8q)(25p + 2q)$ then $K_{m,n}$ has $P_9$-factorization.
Proof: Let \( r = r_1, r_2 \) where \( r_1 = 5(2p + q), r_2 = (25p + 8q), m_0 = 8(5p + 2q), n_0 = 4(5p + 2q) \) and
\[
X = \{x_{i,j} : 1 \leq i \leq r_1, 1 \leq j \leq m_0\},
\]
\[
Y = \{y_{i,j} : 1 \leq i \leq r_2, 1 \leq j \leq n_0\}.
\]
For each \( 1 \leq i \leq 5p, \) let
\[
E_i = \{x_{2(i-1)+u,v} + (5p+2q)u+2(5p+2q)v+v5(1-1)+u+v+4(i-1)+u+w : 1 \leq j \leq 4(5p+2q), 1 \leq u \leq 2, 0 \leq v, w \leq 1\}.
\]
Now for each \( 1 \leq i \leq q, \) let
\[
E_{5p+i} = \{x_{10p+5(i-1)+2u,v} + (5p+2q)(h-1)+2(5p+2q)w : 1 \leq j \leq 20(p+2q), 1 \leq u \leq 2, 0 \leq v \leq 1, 1 \leq h \leq 3\}.
\]
Lemma 6. For any positive integer \( p \) and \( q, \) let \( m = 40(2p+5q)(p+2q) \) and \( n = 20(5p+2q)(p+2q) \) then \( K_{m,n} \) has \( P_g \)-factorization.
Proof: Let \( r = r_1, r_2, \) where \( r_1 = (2p+5q) \) and \( r_2 = (5p+8q), m_0 = 40(p+2q), n_0 = 20(p+2q) \), let
\[
X = \{x_{i,j} : 1 \leq i \leq r_1, 1 \leq j \leq m_0\},
\]
\[
Y = \{y_{i,j} : 1 \leq i \leq r_2, 1 \leq j \leq n_0\}.
\]
Now for each \( 1 \leq i \leq p, \) let
\[
E_i = \{x_{2(i-1)+u,v,j} + (5p+2q)u+2(5p+2q)v+v5(1-1)+u+v+4(i-1)+u+v+2+w+1 : 1 \leq j \leq 20(p+2q), 1 \leq u \leq 2, 0 \leq v \leq 1, 0 \leq w, h \leq 1\}.
\]
Now for each \( 1 \leq i \leq q, \) let
\[
E_{p+i} = \{x_{2p+5(i-1)+u+v,j} + (5p+2q)u+v5(1-1)+u+v+20(i-1)+5u+5v+4w+2h+1 : 1 \leq j \leq 20(p+2q), 1 \leq u \leq 3, 1 \leq v \leq 2, 0 \leq h, 1 \leq 1\}.
\]
Lemma 7. Consider \( m = 40(4p+5q)(p+q) \) and \( n = 40(5p+4q)(p+q) \) . Then \( K_{m,n} \) has \( P_g \)-factorization.
Proof: Let \( r = r_1, r_2 \) where \( r_1 = 5(4p+5q), r_2 = (5p+4q) \) and \( m_0 = 8(p+q), n_0 = 40(p+q) \), Now consider,
\[
X = \{x_{i,j} : 1 \leq i \leq r_1, 1 \leq j \leq m_0\},
\]
\[
Y = \{y_{i,j} : 1 \leq i \leq r_2, 1 \leq j \leq n_0\}.
\]
Now for each \( 1 \leq i \leq p, \) let
\[
E_i = \{x_{20(i-1)+2h+u+v+5w,j} + 5(i-1)+u+v+4(i-1)+8h+w-1 + 1 \leq j \leq 8(p+q), 1 \leq h \leq 5, 1 \leq u \leq 3, 1 \leq v \leq 2, 0 \leq w \leq 1\}.
\]
Now \( 1 \leq i \leq q, \) let
\[
E_{p+i} = \{x_{20p+5(i-1)+h,j} + 5(i-1)+u+w+j+4(i-1)+8h+w-1 + 1 \leq j \leq 8(p+q), 1 \leq h \leq 5, 1 \leq u \leq 3, 0 \leq w \leq 1\}.
\]
Proof: By applying lemmas (2) to (7) with theorem (1) along with theorem (2), it can be seen that when the parameters \( m \) and \( n \) satisfy condition (1)-(4) in theorem (1), the graph \( K_{m,n} \) has \( P_g \)-factorization.
References


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