The Application of Homotopy Analysis Method for Solving the Prey and Predator Problem

D. Rostamy
rostamy@khayam.ut.ac.ir

F. Zabihi
zabihi@ikiu.ac.ir

K. Karimi
kobra.karimi@yahoo.com

Abstract
Here, the homotopy analysis method (HAM), which is powerful tool for nonlinear problems, applied to solve the prey and predator problem. The homotopy analysis method contains the auxiliary parameter $\bar{h}$, which provides us with a simple way to adjust and control convergence region of solution series.

Mathematics Subject Classification: 35A24, 35A35

Keywords: Prey and Predator problem, Homotopy analysis method, System of nonlinear differential equations

1 Introduction

In this paper, we have the following system of nonlinear equation

\[
\begin{aligned}
\frac{\partial x}{\partial t} &= ax(t) - bx(t)y(t), \\
\frac{\partial y}{\partial t} &= -cy(t) + dx(t)y(t).
\end{aligned}
\]

With initial conditions

\[
x(0) = x_0, \quad y(0) = y_0.
\]

This system called Prey and predator problem because it designed for the growth and the death rates of the preys and predators. An environment,
foxes eat rabbits and rabbits eat clover. Apparently, there will be an increase or decrease in the number of both rabbits and foxes. In this system, \( x(t) \) and \( y(t) \) indicate the populations of rabbits and foxes at time \( t \) and \( x(t), y(t) \) indicate the amount of two organisms encountering each other. In (1.1), \( a, b, c \) and \( d \) are the growth rate of the prey, the efficiency of the predator’s ability to capture the prey, the death rate of the predator and the growth rate of predator respectively. Biazar et al used the adomian decomposition method (ADM) and the power series method for solve system (1.1) \([3, 4]\). Rafei et al used the variational iteration method (VIM) for (1.1) \([20]\). Also, Yusufoglu and Erbas applied VIM for solve this system in different cases\([22]\). Here, we applied homotopy analysis method (HAM) for solve the prey and predator problem.

HAM, firstly was developed by S.J. Liao in 1992 \([15, 16, 17, 18, 19]\) and used by many authors for engineering and physical applications. The HAM contains a certain auxiliary parameter \( \tilde{h} \) which provide us with a simple way to adjust and control the convergence region and rate of convergence of the series solution. Moreover, by means of the so-called \( \tilde{h} \)-curve, it is easy to determine the valid regions of \( \tilde{h} \) to gain a convergent series solution.

Therefore, this paper is organized as follows: In section 2, we introduce HAM and we use HAM for the prey and predator problem in section 3. We obtain numerical results by HAM for two cases of this problem in section 4 and section 5 is devoted to the conclusion remarks.

## 2 Basic idea of HAM

In this Letter, we apply the homotopy analysis method to the discussed problem. To show the basic idea, let us consider the following differential equation

\[
\mathcal{N}[u(t)] = 0,
\]

where \( \mathcal{N} \) is a nonlinear operator, \( t \) denote independent variable, \( u(t) \) is an known function, respectively. For simplicity, we ignore all boundary or initial conditions, which can be treated in the similar way. By means of generalizing the traditional homotopy method, Liao contrusts the so-called zero-order deformation equation

\[
(1 - p)\mathcal{L}[\phi(t; p) - u_0(t)] = p\tilde{h}\mathcal{H}(t)\mathcal{N}[\phi(t; p)],
\]

where \( p \in [0, 1] \) is the embedding parameter, \( \tilde{h} \) is a nonzero auxiliary, parameter \( \mathcal{H} \) is an auxiliary function, \( \mathcal{L} \) is an auxiliary linear operator, \( u_0(t) \) is an initial guess of \( u(t) \), \( \phi(t; p) \) is a unknown function,respectively. It is important that one has great freedom to chose auxiliary things in HAM. Obviously, when
The application of HAM for solving the prey and predator problem

$p = 0$ and $p = 1$, it holds

$$
\phi(t; 0) = u_0(t), \quad \phi(t, 1) = u(t),
$$

(3)

respectively. Thus as $p$ increases from 0 to 1, the solution $\phi(t; p)$ varies from the initial guess $u_0(t)$ to the solution $u(t)$. Expanding $\phi(t; p)$ in Taylor series with respect to $p$, one has

$$
\phi(t; p) = u_0(t) + \sum_{m=1}^{+\infty} u_m(t)p^m,
$$

(4)

where

$$
u_m(t) = \frac{1}{m!} \frac{\partial^m \phi(t; p)}{\partial p^m} \bigg|_{p=0}.
$$

(5)

If the auxiliary linear operator, the initial guess, the auxiliary parameter $\bar{h}$ and the auxiliary function are so properly chosen, the series (2.4) converges at $p = 1$, one has

$$u(t) = u_0(t) + \sum_{m=1}^{+\infty} u_m(t),
$$

(6)

which must be one of solutions of original nonlinear equation, as proved by Liao [15]. As $\bar{h} = -1$ and $\mathcal{H}(t) = 1$, Eq. (2.2) becomes

$$(1 - p)\mathcal{L}[\phi(t; p) - u_0(t)] + p\mathcal{N}[\phi(t; p)] = 0,
$$

(7)

which is used mostly in the homotopy perturbation method, whereas the solution obtained directly, without using Taylor series [13, 14]. The comparison between HAM and HPM can be found in [12, 17]. According to the definition (2.5), the governing equation can be deduced from the zero-order deformation equation (2.2). Define the vector

$$
\vec{u}_k = \{u_0, u_1, \ldots, u_k\}
$$

Differentiating Eq. (2,2), $m$ times with respect to embedding parameter $p$ and then setting $p = 0$ and finally dividing them by $m!$, we have the so-called $m$th-order deformation equation

$$
\mathcal{L}[u_m(t) - \chi_m u_{m-1}(t)] = \bar{h}\mathcal{H}(t)R_m(\vec{u}_{m-1}),
$$

(8)

where

$$
R_m(\vec{u}_{m-1}) = \frac{1}{(m-1)!} \frac{\partial^{m-1} \mathcal{N}[\phi(t; p)]}{\partial p^{m-1}} \bigg|_{p=0},
$$

(9)

and

$$
\chi_m = \begin{cases} 
0, & m \leq 1, \\
1, & m > 1.
\end{cases}
$$


It should be emphasized that $u_m(t)$ for $m = 1, 2, 3, \ldots$ is governed by the linear equation (2.8) with the linear boundary conditions that come from original problem, which can be easily solved by symbolic computation software such as Maple and Mathematica.

3 applications

Consider the system of nonlinear differential (1.1) with initial conditions (1.2). We assume that the solution of system (1.1), $x(t)$ and $y(t)$ can be expressed by following set of base functions

$$\{t^m|m = 0, 1, 2, 3, \ldots\},$$

in the form

$$x(t) = \sum_{m=1}^{+\infty} a_m t^m, \quad y(t) = \sum_{m=1}^{+\infty} b_m(t) t^m,$$

where $a_m, b_m$ are coefficient to be determined. This provides us with the so-called rule of solution expression, i.e., the solution of (1.1) must be expressed in the same form as (3.1) and the other expressions must be avoided. According to (1.1) and (3.1), we chose the linear operator

$$\mathcal{L}(t;p) = \frac{\partial \phi(t;p)}{\partial t},$$

with the property

$$\mathcal{L}[c_i] = 0 \quad i = 1, 2,$$

where $c_i$ is constant. From (1.1), we define nonlinear operator

$$\mathcal{N}_1[\phi(t;p)] = \frac{\partial \phi_1(t;p)}{\partial t} - a\phi(t;p) + b\phi_1(t;p)\phi_2(t;p),$$

$$\mathcal{N}_2[\phi(t;p)] = \frac{\partial \phi_2(t;p)}{\partial t} + c\phi_2(t;p) - d\phi_1(t;p)\phi_2(t;p),$$

According to (1.1) and the rule of solution expression (3.1), it is straightforward that the initial approximation should be in the form $x_0(t) = x_0$ and $y_0(t) = y_0$ and the initial condition of the zero-order deformation equation (1.2) are $\phi_1(0;p) = x_0$ and $\phi_2(0;p) = y_0$. From Eqs.(2.9), (3.4) and (3.5), we have

$$R_m(\overrightarrow{x}_{m-1}) = x'_{m-1}(t) - ax_{m-1}(t) + b \sum_{n=0}^{m-1} x_n(t)y_{m-1-n}(t),$$
$$R_m(\overline{y}_{m-1}) = y'_{m-1}(t) + cy_{m-1}(t) - d \sum_{n=0}^{m-1} x_n(t) y_{m-1-n}(t), \quad (8)$$

where the prime denotes differentiation with respect to similarity variable $t$. Now, the solution of the $m$th-order deformation Eq.(2.8) for $m = 1, 2, 3, \ldots$ become

$$x_m(t) = \chi_m x_{m-1} + \bar{h} \int_0^t \mathcal{H}(\eta) R_m(\overline{y}_{m-1}) d\eta + c_1, \quad (9)$$

$$y_m(t) = \chi_m y_{m-1} + \bar{h} \int_0^t \mathcal{H}(\eta) R_m(\overline{y}_{m-1}) d\eta + c_2, \quad (10)$$

where the integral constants $c_i$ is determined by initial condition of (1.2). According to the rule of solution expression denoted by (3.1) and from Eq.(2.8), the auxiliary function $\mathcal{H}(t)$ should be in the form $\mathcal{H}(t) = t^k$, where $k$ is in integer. It is found that, when $k \leq -1$, the solution of high-order deformation Eq.(2.8) contains the terms $\ln(t)$ and $1/t^s$ ($s \geq 1$), which incidentally disobey the rule of solution expression (3.1). When $k = 1, 2, 3, \ldots$, the base $t$ always disappears in the solution expression of the high-order deformation Eq.(2.8), so that the coefficient of the term $t$ cannot be modified even if the order of approximation tends to infinity. This rule called rule of coefficient ergodicity by Liao [16]. Thus, we had to set $k = 0$, which uniquely determines the corresponding auxiliary function $\mathcal{H}(t) = 1$. Therefore, we have

$$x_1(t) = (bx_0y_0 - ax_0)t\bar{h},$$

$$x_2(t) = (\bar{h}bx_0y_0 - \bar{h}ax_0 + \bar{h}^2bx_0y_0 - \bar{h}^2ax_0)t + (-a\bar{h}^2bx_0y_0 + 0.5\bar{h}^2a^2x_0$$

$$-0.5\bar{h}^2bdx_0^2y_0 + 0.5\bar{h}^2bcx_0y_0 + 0.5\bar{h}^2b^2x_0y_0^2)t^2,$$

$$\vdots$$

$$y_1(t) = (-dx_0y_0 + cy_0)t\bar{h},$$

$$y_2(t) = (-\bar{h}dx_0y_0 + \bar{h}cy_0 - \bar{h}^2dx_0y_0 + \bar{h}^2cy_0)t + (-c\bar{h}^2dx_0y_0 + 0.5\bar{h}^2c^2y_0$$

$$+0.5\bar{h}^2d^2x_0^2y_0 - 0.5d\bar{h}^2bx_0y_0^2 + 0.5dy_0\bar{h}^2ax_0)t^2,$$

$$\vdots$$

Hence, the $m$th-order approximation of $x(t)$, $y(t)$ can be generally expressed by

$$x(t) \approx \sum_{n=0}^{m} x_n(t) = \sum_{n=0}^{m} \eta_{m,n}(\bar{h})t^n, \quad x(t) \approx \sum_{n=0}^{m} y_n(t) = \sum_{n=0}^{m} \phi_{m,n}(\bar{h})t^n, \quad (11)$$

where $\eta_{m,n}, \phi_{m,n}$ are coefficient dependent of $\bar{h}$.
4 Results

For comparison with the results done by Biazar [3] and Rafei [20] and Yusufoğlu [22] by ADM and VIM respectively, we consider the following values for parameters.

Table 1: Two different sets of the values of the constant $a, b, c, d, x_0, y_0$.

<table>
<thead>
<tr>
<th>Case</th>
<th>$a$</th>
<th>$b$</th>
<th>$c$</th>
<th>$d$</th>
<th>$x_0$</th>
<th>$y_0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>3</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>0.1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>14</td>
<td>18</td>
</tr>
</tbody>
</table>

4.1 Case. 1

In system (1.1), we assume $a = b = c = d = 1$ with initial conditions $x_0 = 3$ and $y_0 = 2$. As pointed by Liao [15], the auxiliary parameter $\tilde{h}$ can be employed to adjust the convergence region of the homotopy analysis solution. To investigate the influence of $\tilde{h}$ on the solution series, we plot the so-called $\tilde{h}$-curve of $x'(0), x''(0), y'(0), y''(0)$ obtained from the 5th-order HAM approximation solution as show in Fig.1,2. According to this $\tilde{h}$-curve, it is easy to discover the valid region of $\tilde{h}$ which corresponds to the line segment nearly parallel to the horizontal axis. It is clear that the series of solutions for this case is convergent when $-1.2 < \tilde{h} < -0.8$. 

![Fig. 1: point curve: The $\tilde{h}$-curve, 5th-order approximation of $x'(0)$; solid curve: The $\tilde{h}$-curve, 5th-order approximation of $x''(0)$.](image)
If $\bar{h} = -1$ then, we see solution of HAM is similar to solutions of ADM and VIM and the following solutions is derived:

$$\begin{align*}
x(t) &= 3 - 3t - 4.5t^2 + 4.5t^3 + 8.875t^4 - 4.575t^5, \\
y(t) &= 2 + 4t + t^2 - 6.33334t^3 - 6.16667t^4 + 7.58334t^5. 
\end{align*}$$

(1)

4.2 Case. 2

In system (1.1), we chose $a = 0.1$, $b = c = d = 1$ and in (1.2), chose $x_0 = 14$ and $y_0 = 18$. By the above parameters and using HAM, we have

$$\begin{align*}
x_0 &= 14, \\
x_1 &= 250.6\bar{h}t, 
\end{align*}$$
\[ x_2 = 250.6\bar{h}t + 250.6\bar{h}^2t + 604.87\bar{h}^2t^2, \]
\[ x_3 = 250.6\bar{h}t + 501.2\bar{h}^2t + 1209.74\bar{h}^2t^2 + 250.6\bar{h}^3t \]
\[ + 1209.74\bar{h}^3t^2 - 19364.94233\bar{h}^3t^3, \]
\[ \vdots \]
\[ x(t) = 14 + 1253\bar{h}t + 4\bar{h}(250.6\bar{h}t + 604.87\bar{h}t^2) \]
\[ + 3\bar{h}(-19364.94233\bar{h}^2t^3 + 0.5(2419.48\bar{h}^2 + 1209.74\bar{h}^3)t^2 \]
\[ + 250.6\bar{h}^2t + 250.6\bar{h}t) + \ldots \]

and

\[ y_0 = 18, \]
\[ y_1 = -234\bar{h}t, \]
\[ y_2 = -234\bar{h}t - 234\bar{h}^2t - 734.4\bar{h}^2t^2, \]
\[ y_3 = -234\bar{h}t - 468\bar{h}^2t - 1468.8\bar{h}^2t^2 - 234\bar{h}^3t \]
\[ - 1468.8\bar{h}^3t^2 + 19099.98\bar{h}^3t^3, \]
\[ \vdots \]
\[ y(t) = 18 - 1170\bar{h}t + 4\bar{h} (-234\bar{h}t - 734.4\bar{h}t^2) \]
\[ + 3\bar{h}(19099.98\bar{h}^2t^3 + 0.5(-2937.6\bar{h}^2 - 1468.8\bar{h}^3)t^2 \]
\[ 234\bar{h}^2t - 234\bar{h}t) + \ldots \]

Fig. 4: The \( \bar{h} \)-curve, 5th-order approximation of \( x'(0) \).
The application of HAM for solving the prey and predator problem

To influence of $\bar{h}$ on the convergence of (3.9) and (3.10), we first plot the so-called $\bar{h}$-curve of $x'(0)$ and $y'(0)$ as shown in Fig. 4, Fig. 5 respectively: If we chose $\bar{h} = -1$ then, with considering before section for system of nonlinear differential equation (1.1), we will have the 5th order approximation as follow:

\[
\begin{align*}
 x(t) &= 14 - 250.6t + 604.87t^2 + 19364.94233t^3 - 1.012032419 \times 10^5 t^4 \\
 &\quad - 1.710513996 \times 10^6 t^5, \\
 y(t) &= 18 + 234t - 734.4t^2 - 19099.98t^3 + 1.064623605 \times 10^5 t^4 \\
 &\quad + 1.68719746 \times 10^6 t^5, 
\end{align*}
\]

The above solution are shown in Fig. 5.

With attention Fig. 6, we can see that solution of HAM by using $\bar{h} = -1$ is similar to solutions of ADM and VIM.
5 Conclusions

We have applied the homotopy analysis method (HAM) to the prey-predator problem to obtain approximations to solutions given in two cases in [3]. The HAM provides us with a convenient way to control the convergence of approximation series; this is a fundamental qualitative difference between the HAM and other methods for finding approximate solutions. The example in this paper gives further confirmation of the power of the HAM to solve nonlinear problems.

References


Received: August, 2010