About the General KdV6 and its Exact Solutions

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Abstract

We consider some conditions over the coefficients of the sixth-order KdV equation (KdV6) under which this equation has exact solutions. An algebraic condition for the existence of exact solutions to KdV6 is obtained. A new ansatz is considered to obtain analytic solutions for several forms of it.

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1 Introduction

It is well know that the general form of the sixth-order KdV equation (KdV6) is given by

\[ u_{xxxxxx} + au_xu_{xxx} + bu_{xx}u_{xxx} + cu_x^2u_{xx} + du_{tt} + eu_{xxx} + fu_xu_{xt} + gu_tu_{xx} = 0, \]

(1)

where \( a, b, c, d, e, f, g \) are arbitrary parameters, and \( u(x, t) \) is a differentiable function. Several forms can be constructed from it by changing the values of the parameters. For instance,

\[
\begin{align*}
\left\{ \begin{array}{l}
a = 20, b = 40, c = 120, d = 0, e = 1, f = 8, g = 4 : \\
u_{xxxxxx} + 20u_xu_{xxx} + 40u_{xx}u_{xxx} + 120u_x^2u_{xx} + u_{xxx} + 8u_xu_{xt} + 4u_tu_{xx} = 0.
\end{array} \right.
\end{align*}
\]

(2)
\[
\begin{align*}
\left\{ \begin{array}{l}
a = -9, \quad b = -18, \quad c = 18, \quad d = -\frac{1}{2}, \quad e = \frac{1}{2}, \quad f = 0, \quad g = 0 : \\
u_{xxxxxx} - 9u_xu_{xxxx} - 18u_{xx}u_{xxx} + 18u_x^2u_{xx} - \frac{1}{2}u_{tt} + \frac{1}{2}u_{xxxx} = 0.
\end{array} \right.
\end{align*}
\] (3)

\[
\begin{align*}
\left\{ \begin{array}{l}
a = -15, \quad b = -15, \quad c = 45, \quad d = -5, \quad e = -5, \quad f = 15, \quad g = 15 : \\
u_{xxxxxx} - 15u_xu_{xxxxxx} - 15u_{xx}u_{xxxx} + 45u_x^2u_{xx} - 5u_{tt} - 5u_{xxxxxx} + 15u_xu_{xt} + 15u_tu_{xx} = 0.
\end{array} \right.
\end{align*}
\] (4)

and

\[
\begin{align*}
\left\{ \begin{array}{l}
a = -15, \quad b = -\frac{75}{2}, \quad c = 45, \quad d = -5, \quad e = -5, \quad f = 15, \quad g = 15 : \\
u_{xxxxxx} - 15u_xu_{xxxxxx} - \frac{75}{2}u_{xx}u_{xxxx} + 45u_x^2u_{xx} - 5u_{tt} - 5u_{xxxxxx} + 15u_xu_{xt} + 15u_tu_{xx} = 0.
\end{array} \right.
\end{align*}
\] (5)

respectively. It has been proved that (2), (3), (4) and (5) are particular integrable cases of (1). More exactly, the five authors of [1] have found the Lax Pair, an auto-Bäcklund transformation, traveling wave solutions and third-order generalized symmetries for (2). More recently, Kupershmidt [2] showed that (2) is integrable in the usual sense. The two authors of [3] found a Bäcklund self-transformation for (3), and multisoliton solutions for it were studied by the authors of [4]. On the other hand, (4) and (5) have been obtained from equations

\[
5\partial_x^{-1}v_{tt} + 5v_{xxxxxx} - 15vv_t - 15v_x\partial_x^{-1}v_t - 45v_x^2v_x + 15v_xv_{xx} + 15vv_{xxx} - v_{xxxxxx} = 0.
\] (6)

and

\[
5\partial_x^{-1}v_{tt} + 5v_{xxxxxx} - 15vv_t - 15v_x\partial_x^{-1}v_t - 45v_x^2v_x + \frac{45}{2}v_xv_{xx} + 15vv_{xxx} - v_{xxxxxx} = 0.
\] (7)

respectively, after the use of the potential transformation

\[
v(x, t) = u_x(x, t).
\] (8)

Equations (6) and (7) are fifth-order nonlinear equations which govern wave propagation in two opposite directions. More exactly, (6) is related to Sawada–Kotera-Caudrey–Dodd–Gibbon equation [5], [6], and (7) may be considered a bidirectional version of the Kaup–Kupershmidt equation [7] (see [1][8]). The two authors of [8] have been constructed Lax pair for (6) and (7).

The main objective of this work is to obtain an algebraic equation involving the coefficients of the KdV6 (1) that allows to obtain an exact solution in certain cases.
2 Search of exact solutions to general KdV6

We seek solutions to equation (1) in the form

$$u(\xi) = \frac{p}{1 + k \exp(\xi)} + C_0, \quad k \neq 0, \quad p \neq 0$$

and

$$\xi = \mu(x + \lambda t + \xi_0), \quad \lambda \neq 0, \quad \mu \neq 0.$$  \hspace{1cm} (9)

Inserting ansatz (9) into equation (1) we obtain a polynomial equation in the variable $\zeta = \exp(\xi)$. Equating the coefficients of $\zeta^i$ ($i = 0, 1, 2, \ldots$) to zero gives the following algebraic system in the variables $a, b, c, d, e, f, g, p, \lambda$ and $\mu$:

$$d \lambda^2 + e \lambda \mu^2 + \mu^4 = 0. \hspace{1cm} (10)$$

$$-ap \mu^3 - b p \mu^3 + 8d \lambda^2 - 4e \lambda \mu^2 - 2gp \lambda \mu - 52 \mu^4 = 0. \hspace{1cm} (11)$$

$$9ap \mu^3 + 3bp \mu^3 + cp \mu^2 + 18d \lambda^2 - 18e \lambda \mu^2 - 6gp \lambda \mu + 198 \mu^4 = 0. \hspace{1cm} (12)$$

Eliminating $p, \lambda$ and $\mu$ from system of equations (10)-(11)-(12) gives equation

$$4(25d - 4e^2)^2 c^2 + Qc + R = 0, \hspace{1cm} (13)$$

where

$$Q = Q(a, b, d, e, f, g) = -2(250a^2d^2 - 70a^2e^2 + 3a^2e^4 + 250abd^2 - 75abde^2 + 2abe^4 - 200adeg + 68ae^3g - 5b^2de^2 - b^2e^4 - 50bdeg + 44be^3g + 1000dg^2 - 340e^2g^2). \hspace{1cm} (14)$$

and

$$R = R(a, b, d, e, f, g) = ((a + b)^2d - 2eg(a + b) + 4g^2) ((25d - 6e^2) a^2 (45)$$

$$+ (10g - be)ae + (10g - be)^2).$$

We shall call equation (13) the discriminant equation for the KdV6 (1). It is remarkable the fact that equations (2),(3),(4) and (5) satisfy their discriminant equation. This may be verified by direct substitution of the respective values of $a, b, c, d, e, f$ and $g$ into equation (13). Condition (13) is necessary for the existence of solutions to (1) in the form (9). On the other hand, if a given KdV6 (1) satisfies its discriminant equation (13), it admits exact solutions in the form (9) under certain additional restrictions over its coefficients. Indeed, suppose that condition (13) holds. Then, from equations (10)-(11)-
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(12)-(13) a solution is given by

\[
\begin{align*}
    c &= \frac{-Q + 2((a + b)e - 5(f + g))((3a - b)e^2 - 15(f + g)e + 25bd) \sqrt{e^2 - 4d}}{8(25d - 4e^2)^2}, \\
    \lambda &= -\frac{(e \mp \sqrt{e^2 - 4d}) \mu^2}{2d}, \\
    p &= -\frac{6e((a + b)(10d - e^2) - 3(f + g)e \pm ((a + b)e - 5(f + g))\sqrt{e^2 - 4d})}{(a^2 + b^2)d - b(f + g)e + (f + g)^2 + a(2bd - (f + g)e)}, \\
    u(x, t) &= \frac{p}{1 + k \exp\left(\mu \left(\frac{x - \mu^2}{2d} \left(e + \sqrt{e^2 - 4d}\right) t + \xi_0\right)\right)} + C_0.
\end{align*}
\]

Expressions in (16) are valid for

\[d \neq 0 \text{ and } g \neq -f + \frac{1}{2}(a + b) \left(e \pm \sqrt{e^2 - 4d}\right) \text{ and } \]

\[25d - 4e^2 \neq 0 \text{ and } e^2 - 4d \geq 0.\]

Thus, if the KdV6 (1) satisfies its discriminant equation and condition (17) holds, it admits an exact real-valued solution given by (9).

3 Some special cases

Suppose that he KdV6 (1) satisfies its discriminant equation and condition (17) does not meet. We may consider several cases.

3.1 First Case : \(d = 0\).

We have that if \(g \neq (a + b)e - f\) then a solution is

\[
\begin{align*}
    c &= \frac{1}{32e^2}(e(a + b) - (f + g))(3ae - be + 5(f + g)). \\
    \lambda &= -\frac{\mu^2}{e}, \quad p = \frac{48e\mu}{f + g - (a + b)e}. \\
    u(x, t) &= \frac{48e\mu}{f + g - (a + b)e} \cdot \frac{1}{1 + k \exp (\mu(x - \frac{\mu^2}{2d} t + \xi_0))} + C_0.
\end{align*}
\]

Remark 1 There is no solution in the form (9) if \(g = e(a + b) - f\).
3.2 Second Case: \( g = -f + \frac{1}{2}(a + b) \left( e \pm \sqrt{e^2 - 4d} \right) \).

A solution is given by

\[
\begin{aligned}
c &= \frac{(a + b)(ae + be - f - g)(3a^2e + 2abe + b(5(f + g) - be))}{2(4e(a + b) - 5(f + g))^2} \\
d &= \frac{(f + g)(ae + be - f - g)}{(a + b)^2}, \quad \lambda = -\frac{\mu^2(a + b)}{(a + b)e - (f + g)} \\
p &= -\frac{12\mu(4(a + b)e - 5(f + g))}{(a + b)((a + b)e - 2(f + g))} \\
u(x, t) &= \frac{p}{1 + k \exp \left( \mu(x - \frac{\mu^2(a + b)}{(a + b)e - (f + g)}t + \xi_0) \right)} + C_0.
\end{aligned}
\]

provided that

\((a + b)((a + b)e - (f + g))(4(a + b)e - 5(f + g))((a + b)e - 2(f + g)) \neq 0.\)

(20)

3.3 Third Case: \( 25d - 4e^2 = 0. \)

A solution is given by

\[
\begin{aligned}
c &= \frac{(4e(a + b) - 5(f + g))(2ae - be + 5(f + g))}{90e^2}, \quad d = \frac{4e^2}{25} \\
\lambda &= -\frac{5\mu^2}{4e}, \quad p = -\frac{180e\mu}{4(a + b)e - 5(f + g)} \\
u(x, t) &= \frac{p}{1 + k \exp \left( \mu \left( x - \frac{5\mu^2}{4e}t \right) + \xi_0 \right)} + C_0.
\end{aligned}
\]

provided that

\(4(a + b)e - 5(f + g) \neq 0.\)

(22)

It is easy to see that equations (2),(3),(4) and (5) are particular cases of the ones we have considered.

Finally, a similar approach for the study of the KdV7 may be found in [9]. Other methods we may employ to solve nonlinear pde’s exactly may be found in [10]–[31].
4 Conclusions

In this paper we have studied some relations between the coefficients of the general KdV6 to obtain an exact solution to this equation. We obtained an algebraic equation - the discriminant equation of the KdV6 that allowed us to solve the general KdV6 exactly. At the same time, our results generalize some recent results about the sixth-order KdV equation. We also think that many KdV6 equations satisfying their discriminant equation admit two and three soliton solutions and, in general, \(N\)-soliton solutions.

References


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