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Abstract

In this paper, two well-known numerical methods, the Adomian decomposition method and the homotopy perturbation method, are applied to solve the Abel integral equation of the second kind. The results show that the homotopy perturbation method with a specific convex homotopy is equivalent to the Adomian decomposition method for solving the Abel integral equation.

Keywords: Adomian’s decomposition method; Homotopy perturbation method; Abel’s integral equation

1 Introduction

The mathematical model of many physical problems takes the form of the well-known Abel integral equation. This equation was applied by Niels Abel in 1823 to describe a sliding point mass in a vertical plane on a unknown curve under gravitational force. The point mass starts its motion without initial velocity from a point which has a vertical distance \( x \) from the lowest point of the curve [1]. Using the work-energy theorem, the equation of the unknown curve that obtained is the well-known Abel integral equation

\[
f(x) = \int_0^x \frac{\varphi(t)}{\sqrt{2g(x - t)}} \, dt,
\]
in which g is the acceleration due to gravity.

Recently, the homotopy perturbation method (HPM) [2, 3], the modified homotopy perturbation method (MHPM), the Adomian decomposition method (ADM) [4, 5], and the modified Adomian decomposition method (MADM) have been applied to solve the Volterra integral equation with the Abel kernel [6].

S. Abbasbandy [7] used the ADM and the HPM to solve integral equations of the second kind and showed that these methods gave the same solutions. In this paper, we use a specific convex homotopy in the HPM to solve the Abel integral equation and compare these results with the solution by the ADM.

2 The Methods

2.1 Adomian decomposition method

Consider the functional equation

\[ y - Ny = f, \]

where \( N \) is a nonlinear operator from a Hilbert space \( H \) into \( H \) and \( f \) is a given function in \( H \). Now, we are looking for \( y \in H \) satisfying (2).

At the beginning of the 1980s, Adomian developed a very powerful method to solve Equation (2) in which the solution \( y \) was considered as the sum of a decomposition series

\[ y = \sum_{i=0}^{\infty} y_i, \]

and \( Ny \) as the sum of the decomposition series

\[ Ny = \sum_{n=0}^{\infty} A_n. \]

The method consists of the following recursion scheme

\[ y_0 = f, \quad y_{n+1} = A_n(y_0, y_1, \ldots, y_n), \quad n = 0, 1, \ldots \]

where the \( A_n \)'s are polynomials depending on \( y_0, y_1, \ldots, y_n \) and are called the Adomian polynomials [8, 9]; these are defined as

\[ n!A_n = \frac{d^n}{d\lambda^n} N(\sum \lambda^i y_i)]_{\lambda=0}, \quad n = 0, 1, \ldots \]
2.2 Homotopy perturbation method

To illustrate the basic concept of the HPM, we consider a general equation of type

\[ L(u) = 0, \tag{7} \]

where \( L \) is an integral or differential operator. We define a convex homotopy \( H(u, p) \) by

\[ H(u, p) = (1 - p)F(u) + pL(u), \tag{8} \]

where \( F(u) \) is a functional operator with the known solution \( v_0 \), which can be easily obtained. We observe that if

\[ H(u, p) = 0, \tag{9} \]

then we have

\[ H(u, 0) = F(u) \quad \text{and} \quad H(u, 1) = L(u). \]

This shows that \( H(u, p) \) continuously traces an implicitly defined curve from a starting point \( H(v_0, 0) \) to a solution function \( H(u, 1) \). The embedding parameter \( p \) monotonously increases from zero to unity as the trivial problem \( F(u) = 0 \) continuously deforms to the original problem \( L(u) = 0 \). The embedding parameter \( p \in [0, 1] \) can be considered as an expanding parameter [2] to obtain:

\[ u(x) = \sum_{n=0}^{\infty} p^n u_i = u_0 + pu_1 + p^2 u_2 + \ldots \tag{10} \]

When \( p \to 1 \), (9) corresponds to (7) and becomes the approximate solution of (7), i.e.

\[ f = \lim_{p \to 1} u = \sum_{i=0}^{\infty} u_i(x) \tag{11} \]

It is well known that the series (11) is convergent for most cases and its rate of convergence depends on \( L(u) \) [2].

3 ADM and HPM for Abel integral equations

In this section, we will consider the Abel integral equation of the second kind. The Abel integral equation has the structure of the Volterra integral equation with a kernel of the form \( \frac{1}{\sqrt{x-t}} \) as follow

\[ y(x) = f(x) + \int_0^x \frac{y(t)}{\sqrt{x-t}} dt, \quad 0 \leq x \leq 1. \tag{12} \]
In the following, we show that the HPM with a specific convex hopotopy is equivalent to the ADM for solution of the Abel integral equation. This fact can be shown in following theorem.

**Theorem.** The homotopy perturbation method with the convex homotopy

\[ L(u) = u(x) - f(x) - p \int_0^x \frac{u(t)}{\sqrt{x-t}} dt = 0, \quad (13) \]

is the Adomian decomposition method for the Abel integral Equation (12).

**Proof.** To apply the HPM to (12), we define the convex homotopy \( H(u, p) \) by

\[ H(u, p) = (1 - p)F(u) + pL(u), \quad (14) \]

where

\[ F(u) = u(x) - f(x). \quad (15) \]

Substituting (13) and (15) into (14), we have

\[ (1 - p)[u(x) - y_0(x)] + p[u(x) - f(x) - \int_0^x \frac{u(t)}{\sqrt{x-t}} dt] = 0, \quad (16) \]

The initial approximate solution \( u_0(x) = y_0(x) \) is taken to be \( f(x) \), therefore \( u_0(x) = y_0(x) = f(x) \). Substituting (10) into (16) and equating the coefficients of \( p \) with the same power, one gets

\[ p^0 : \quad u_0(x) = y_0(x) = f(x), \quad (17) \]

\[ p^1 : \quad u_1(x) - \int_0^x \frac{u_0(t)}{\sqrt{x-t}} dt = 0 \implies u_1(x) = \int_0^x \frac{u_0(t)}{\sqrt{x-t}} dt, \quad (18) \]

\[ p^2 : \quad u_2(x) - \int_0^x \frac{u_1(t)}{\sqrt{x-t}} dt = 0 \implies u_2(x) = \int_0^x \frac{u_1(t)}{\sqrt{x-t}} dt, \quad (19) \]

\[ p^3 : \quad u_3(x) - \int_0^x \frac{u_2(t)}{\sqrt{x-t}} dt = 0 \implies u_3(x) = \int_0^x \frac{u_2(t)}{\sqrt{x-t}} dt, \ldots \quad (20) \]

dependence, the approximate solution by the HPM is given by following recursive relations

\[ u_0(x) = f(x), \quad (21) \]
\begin{equation}
u_{i+1}(x) = \int_0^x \frac{u_i(t)}{\sqrt{x-t}} dt.
\end{equation}

Note that, the solution of equation (12) by the HPM is given by

\begin{equation}
y_{HPM}(x) = \lim_{p \to 1} u(x) = \sum_{i=0}^\infty u_i(x),
\end{equation}

On the other hand, the ADM for equation (12) consists of replacing \( y(x) \) by the decomposition series

\begin{equation}
y(x) = \sum_{i=0}^\infty y_i(x),
\end{equation}

now by substituting (24) in (12), we will have

\begin{equation}
\sum_{i=0}^\infty y_i(x) = f(x) + \int_0^x \frac{\sum_{i=0}^\infty y_i(t)}{\sqrt{x-t}} dt,
\end{equation}

so, by the ADM we obtain the recursive relations

\begin{equation}
y_0(x) = f(x),
\end{equation}

\begin{equation}
y_{i+1}(x) = \int_0^x \frac{y_i(t)}{\sqrt{x-t}} dt,
\end{equation}

Hence, the solution of equation (12) by the ADM is given by

\begin{equation}
y_{ADM}(x) = y(x) = \sum_{i=0}^\infty y_i(x).
\end{equation}

Comparing the relations (21) and (22) with (26) and (27) shows that \( y_{HPM}(x) = y_{ADM}(x) \).

4 Conclusion

In this work, we applied the ADM and the HPM to the Abel integral equation of the second kind. We showed that the ADM and the HPM for solving the Abel integral equation of the second kind are equivalent.

References


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