Generalized Exact Solution for a Spherical Symmetric Perfect Fluid Sphere Model of Embedding Class Two

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Abstract
Some spherical symmetric perfect fluid model of embedding class two have been investigated by constructing a fluid sphere of variable pressure and density with property that the pressure and density vanishes at the boundary beyond which Schwarzschild exterior solution prevails. New general solutions have been obtained by introducing the mathematical form of $e^{-\lambda}$ as a polynomial of degree two, the solutions so obtained are physically acceptable and the non dimensional parameter $m/R$ appears in the solution, where $m$ is the relativistic mass of the fluid model and $R$ is the radius have been depicted in the form of graphs. In addition to that an energy conditions have been obtained and analyzed.

Keywords: GRG, Perfect fluid model, Applied Differential equation
1. Introduction

In general relativity, an exact solution is a Lorentzian manifold equipped with certain tensor fields which are taken to model states of ordinary matter, such as fluid or non-gravitational fields (electromagnetic field). These tensors obey any relevant physical law following by a standard recipe which is widely used in mathematical physics [3].

Perfect fluid distributions in general can be divided in two categories, in the 1st category we may include these class one perfect fluid conformally and non-conformally while in the 2nd category are included those perfect fluid which are characterized by non-vanishing Wyle conformal curvature tensor. Barnes, 1974 has obtained many interesting solutions of both these categories [2]. Pandey and Sharma, 1980 has studied in greater details Taub’s plane symmetric space-time in search of perfect fluid distributions of class one and of class two [5]. A class of McVittie's new non-quadratic solutions has been obtained by Henning Knutsen (1986). Delgaty, M S R and Kayll Lake, 1998 have disucssed the physical acceptability of isolated, static, spherically, symmetric perfect fluid solutions of Einstein's field equations and the solutions were subjected to the following elementary tests:

(i) isotropy of the pressure
(ii) regularity at the origin
(iii) positive definiteness of the energy density and pressure at the origin
(iv) vanishing of the pressure at some finite radius
(v) monotonic decreases of the energy density and pressure with increasing radius and
(vi) Subluminal sound speed.

Sharif, M and Iqbal, T (2002), have investigate solutions of the Einstein field equations for the case of non-static spherically symmetric perfect fluid using different equations of state. Chaisi, M and Maharaj, S D, 2006, have establish a new algorithm that generate a new solution to the Einstein field equations with an anisotropic matter distribution, from a seed isotropic solutions.

In this article some spherical symmetric perfect fluid models of embedding class two have been investigated by referring to Karmarkar condition 1948. Our main interest is constructing a spherical model of finite size with vanishing pressure and density at the interface beyond which Schwarzschild exterior solution prevails. We are guided by the form of Schwarzschild interior metric and we have obtained some perfect fluid models of class two. A general solution has been obtained by power series method; the coefficients appearing in the polynomial have been determined by matching the interior metric tensors and its derivatives of various orders across the interface with the exterior Schwarzschild metric. The non dimensional parameter $m/R$ appears in all these solutions, where m is the relativistic mass of the fluid model.
and R is the radius. In addition to that, some of these models are analyzes numerically and the results so obtained are new, physically acceptable and depicted in the form of graphs.

2. Mathematical formulation for spherical symmetric model

2.1 Einstein field equations

The static spherical symmetric line element is

\[ ds^2 = -e^{\lambda(r)} \, dr^2 - r^2 (d\theta^2 + \sin^2 \theta \, d\phi^2) + e^{\mu(r)} \, dt^2, \]

where, the functions \( \lambda(r) \) and \( \mu(r) \) are gravitational potentials gives rise to the following expressions for the energy momentum tensor and flow-vector \( v^i \) \( (i=1,2,3,4) \) respectively

\[ T_{ij} = (P + \rho) v^i v^j - P \delta_{ij}, \]

\[ g_{ij} \, v^i v^j = 1 \]

For a perfect fluid distribution is substituted in the Einstein’s field equations can be expressed as follows:

\[ 8\pi P = e^{-\lambda} \left( \frac{\mu'}{r} + \frac{1}{r^2} \right) - \frac{1}{r^2}, \]

\[ 8\pi P = e^{-\lambda} \left( \frac{\mu''}{2} - \frac{\lambda' - \mu'}{4} + \frac{\mu'^2}{4} + \frac{\mu' - \lambda'}{2r} \right), \]

\[ 8\pi P = e^{-\lambda} \left( \frac{\lambda'}{r} - \frac{1}{r^2} \right) + \frac{1}{r^2}, \]

\[ v^i = (0, 0, 0, e^{-\mu/2}) , \]

overhead dash denotes ordinary differentiation with respect to \( r \).

2.2 Karmarker’s Condition

The class one condition for (1), in case of a perfect fluid distribution, is expressed by Karmarker (1948) condition [1]:

\[ 3F^2 - 8\pi F (\rho + 3P) + 64\pi^2 P \rho = 0 \]

where

\[ F = (1/r^2)(1 - e^{-\lambda}) \]

The condition (8) implies that \( e^{-\lambda} \) takes one of the following forms:

\[ e^{-\lambda} = 1 - \frac{8\pi}{3} \rho \, r^2 \]

\[ e^{-\lambda} = 1 - 8\pi P \, r^2 \]
Karmarker, 1948 has shown that the only class one spherically symmetric static perfect fluid distribution, which is physically meaningful, is the well-known Schwarzschild interior solution and follows from (10i).

3. Exact solutions for some models of class two

In this section, we have constructed some spherical sphere models filled with a perfect fluid. These models are of class two and may be guided by the mathematical form of \( \text{Exp}(-\lambda) \) for Schwarzschild interior solutions.

In view of (4) and (5), the functions \( \lambda(r) \) and \( \mu(r) \) are related by the equation:

\[
2r^2 \mu'' + r^2 \mu' - \mu' (2r + r^2 \lambda') - 2r \lambda' - 4(1 - e^\lambda) = 0 \tag{11}
\]

If we set

\[
x = r^2 / R^2 \quad \text{and} \quad y^2 = e^\mu, \tag{12}
\]

where \( R \) is a constant, then the equation (11) reduces to the form:

\[
\frac{d^2 y}{dx^2} - \frac{1}{2} \frac{d \lambda}{dx} \frac{dy}{dx} - \frac{1}{4x} \left[ \frac{d \lambda}{dx} + \frac{1 - e^\lambda}{x} \right] y = 0 \tag{13}
\]

Hence any solution of (13), which does not satisfy (10), will be of class two. This is because of a spherical symmetric space-time is in general of class two. It is evident that equation (13) is under consideration. Consequently, we need to specify one of the variables \( \lambda(r) \) or \( \mu(r) \) in advance, so that a solution can be obtained. Hence, according to Karmarker’s condition above we assume that

\[
e^{-\lambda} = \sum_{i=0}^{N} \alpha_i x^i \tag{14}
\]

If \( \alpha_0 = 1 \) for \( i \geq 1 \) then (14) becomes

\[
e^{-\lambda} = 1 + \sum_{i=1}^{N} \alpha_i x^i \tag{15}
\]

This form ensures that

\[
e^{-\lambda} = 1 + O(r^2) \tag{16}
\]

near \( r = 0 \) for suitable choice of \( N \). In fact this is a sufficient condition for a static perfect fluid sphere to be regular at the center as pointed out by Maartens and Maharaj [5].
The simple expressions for $e^{-\lambda} = \alpha_0$, $e^{-\lambda} = \alpha_0 + \alpha_1 x$, where $\alpha_i$ is a positive constant have been considered by Jasim, M K (2008). The solutions so obtained have been discussed and analyzed with concerned to physical acceptability conditions.

Now, we assume the expression $e^{-\lambda} = \alpha_0 + \alpha_1 x + \alpha_2 x^2$ where $\alpha_i$ are constants.

By substituting in equation (13) we have:

$$x^2(\alpha_0 + \alpha_1 x + \alpha_2 x^2)y'' + 0.5x^2(\alpha_1 + 2\alpha_2 x)y' + 0.25\alpha_2 x^2 y - 0.25(\alpha_0 - 1)y = 0 \quad (17)$$

To solve this equation we have two sub cases:

**Sub case (I):**

For singularity free model, we choose $\alpha_0 = 1$, and then equation (13) will be reduced to:

$$(1 + \alpha_1 x + \alpha_2 x^2)y'' + \frac{1}{2}(\alpha_1 + 2\alpha_2 x)y' + \frac{1}{4}\alpha_2 y = 0 \quad (18)$$

We can solve this equation by power series method, we get:

$$a_0 \neq 0, \quad a_1 \neq 0, \quad a_2 = -\frac{(2a_1 + a_0 a_2)}{8}, \quad a_3 = -\frac{(12a_2 a_1 + 5a_1 a_2)}{24}$$

... $a_n = \frac{[(4n^2 - 16n + 17)a_{n-2} a_2 + (4n^2 - 10n + 6)a_{n-1} a_1]}{4n(n-1)}$

.$\therefore \quad y = a_0 + a_1 x - \frac{(2a_1 a_0 + a_1 a_2)}{8}x^2 - \frac{(12a_2 a_1 + 5a_1 a_2)}{24}x^3 - \cdots$

$$\frac{[(4n^2 - 16n + 17)a_{n-2} a_2 + (4n^2 - 10n + 6)a_{n-1} a_1]}{4n(n-1)} x^n + \cdots$$

If $\alpha_1 = 0$, then equation (18) becomes: $(1 + \alpha_2 x^2)y'' + \alpha_2 xy' + 0.25\alpha_2 y = 0$, and

$$a_0 \neq 0, \quad a_2 = -\frac{\alpha_2}{4} \frac{1/4}{2 \times 1}, \quad a_4 = -\frac{\alpha_2}{4} \frac{4 + (1/4)}{4 \times 3} a_2$$

... $a_{2k} = -\frac{\alpha_2}{4} \left(\frac{(2k - 2)^2 + 1/4}{(2k)(2k - 1)}\right) a_{2k-2}$

$$a_1 \neq 0, \quad a_3 = -\frac{\alpha_2}{4} \frac{1 + 1/4}{3 \times 2} a_1, \quad a_5 = -\frac{\alpha_2}{4} \frac{9 + 1/4}{5 \times 4} a_3$$

... $a_{2k+1} = -\frac{\alpha_2}{4} \left(\frac{(2k - 1)^2 + 1/4}{(2k + 1)(2k)}\right) a_{2k-1}$

Then,
By the ratio test we can show that the two series (19) above are convergences and then

\[ y = (a_0 - \left(\frac{\alpha_2}{4}\right) \frac{1}{2} \prod_{i=1}^{k} \frac{1}{(2i-2)^2 + 1/4} x^2 + \cdots + \left(\frac{\alpha_2}{4}\right)^k \frac{1}{(2k)!} x^{2k}) a_0 \]

\[ + (a_1 x - \left(\frac{\alpha_2}{4}\right) \frac{1}{3} x^3 \prod_{i=1}^{k} \frac{1}{(2i-1)^2 + 1/4} x^{2k+1}) a_1 \]

**Sub case (II):**

If \( \alpha_0 \neq 1 \), then equation (13) can be solved by Frobeniu’s method, we get:

\[(\alpha_0 m(m-1) - \frac{(\alpha_0 - 1)}{4}) a_0 = 0 \]

\[ a_0 \neq 0 \]

\[ a_1 = -\frac{(1m(m-1) + \frac{1}{2} m) \alpha_1 a_0}{4\alpha_0 m(m+1) - (\alpha_0 - 1) a_0} \]

\[ a_n = -\frac{[(m + n - 1)(m + n - \frac{3}{2}) \alpha_1 a_{n-1} + ((m + n - 2)^2 + \frac{1}{4}) \alpha_2 a_{n-2}]}{4\alpha_0 (m + n - 1)(m + n) - (\alpha_0 - 1)} \]

Now: we have two cases:

1. If \( m_1 - m_2 = 0 \), then \( m_1 = m_2 = \frac{1}{2} \), and by replace in equations (20) we get:

\[ a_0 \neq 0 \]

\[ a_1 = -\frac{\alpha_1}{2(2\alpha_0 + 1)} a_0 \]

\[ a_2 = -\frac{(3\alpha_1 a_1 + \alpha_2 a_0)}{2(14\alpha_0 + 1)} \]

\[ a_3 = -\frac{(10\alpha_1 a_2 + 5\alpha_2 a_1)}{2(34\alpha_0 + 1)} \]

\[ \ldots a_n = -\frac{[(n - 1)(n - \frac{1}{2}) \alpha_1 a_{n-1} + ((n - \frac{3}{2})^2 + \frac{1}{4}) \alpha_2 a_{n-2}]}{4\alpha_0 n^2 - 2\alpha_0 + 1} \]

\[ \therefore y_1 = \sum_{n=0}^{\infty} a_n x^{n+\frac{1}{2}} \text{ and } y_2 = y'_1 \]

\[ \therefore y_G = a_0 Ax^m + ma_0 Bx^{m-1} \]
Hence, \[ ds^2 = -(\alpha_0 + \alpha_1 x + \alpha_2 x^2)^{-1} dr^2 - r^2 (d\theta^2 + \sin^2(\theta) d\varphi^2) + (Ay_1 + By_2)^2 dt^2 \]
where \( A \) and \( B \) are arbitrary constants.

For the above result if \( \alpha_i = 0 \) then \( a_1 = a_2 = \cdots = a_n = 0 \),
and \( y_1 = a_n x^{\frac{1}{2}} \) and \( y_2 = y_1' = 0.5a_0 x^{-\frac{1}{2}} \), thus \( y_G = a_0 Ax^{\frac{1}{2}} + 0.5a_0 Bx^{-\frac{1}{2}} \)
where \( A \) and \( B \) are arbitrary constants.

(2) If \( m_1 - m_2 \neq 0 \)

(a) If \( m_1 - m_2 = a \), where \( a \) is not integer number

\[
y_1(x) = a_0 x^{m_1} - \frac{[m_1(m_1 - 1) + \frac{1}{2} m_1] \alpha_1}{4\alpha_0 m_1(m_1 + 1) - (\alpha_0 - 1)} a_0 x^{m_1 + 1} - \frac{[(m_1 + 1)(m_1 + 2)a_1 \alpha_1 + (m_1^2 + \frac{1}{4})a_0 \alpha_2]}{4\alpha_0 m_1(m_1 + 1)} x^{m_1 + 2} - \cdots \]

\[
y_2(x) = a_0 x^{m_2} - \frac{[m_2(m_2 - 1) + \frac{1}{2} m_2] \alpha_1}{4\alpha_0 m_2(m_2 + 1) - (\alpha_0 - 1)} a_0 x^{m_2 + 1} - \frac{[(m_2 + 1)(m_2 + 2)a_1 \alpha_1 + (m_2^2 + \frac{1}{4})a_0 \alpha_2]}{4\alpha_0 m_2(m_2 + 1)} x^{m_2 + 2} - \cdots \]

\[ \therefore y_G = A y_1(x) + B y_2(x) \], where \( A, B \) are arbitrary constants.

(b) If \( m_1 - m_2 = b \), where \( b \) is an integer number.

\[
y_1(x) = a_0 x^{m_1} - \frac{[m_1(m_1 - 1) + \frac{1}{2} m_1] \alpha_1}{4\alpha_0 m_1(m_1 + 1) - (\alpha_0 - 1)} a_0 x^{m_1 + 1} - \frac{[(m_1 + 1)(m_1 + 2)a_1 \alpha_1 + (m_1^2 + \frac{1}{4})a_0 \alpha_2]}{4\alpha_0 m_1(m_1 + 1)} x^{m_1 + 2} - \cdots \]

\[
y_2(x) = \left( \frac{\partial y_1}{\partial m_1} \right)_m \], thus \( y_G = A y_1(x) + B y_2(x) \), where \( A, B \) are arbitrary constants.

4. Physical Analysis of the Solutions

The physical acceptability static spherical symmetric perfect fluid solutions of Einstein field equations must comply with the following conditions [6]:

(i) The matter density \( \rho \) and the fluid pressure \( P \) should be vanished at the boundary and equal \( \infty \) at \( r = R \).
(ii) The matter density $\rho$ and the fluid pressure $P$ should be positive throughout the distribution.

(iii) The gradients $\frac{d\rho}{dr}, \frac{dP}{dr}$ should be negative with increasing radius.

(iv) The speed of the sound should not exceed the speed of light as implication of causality fulfillment.

(v) The interior metric should match continuously with the Schwarzschild exterior solution

$$ds^2 = -(1 - \frac{2M}{r})^{-1} dr^2 - r^2 (d\theta^2 + \sin^2(\theta)d\phi^2) + c^2 (1 - \frac{2M}{r}) dt^2$$

(vi) In addition to that the weak energy (WE) and strong energy (SE) conditions should be positive.

Therefore, our search for spherically symmetric model of radius across which the pressure and density both vanishes at the boundary. So, the spherically symmetric models of perfect fluid spheres have been derived and discussed according to the mathematical form of $e^{-\lambda}$ for the cases which are mentioned in a previous section.

In order to obtain a Singularity-free model we start with a polynomial expression

$$e^{-\lambda} = 1 + \alpha_1 x + \alpha_2 x^2$$

The continuity of $e^{-\lambda}$ and its first derivative with $g_{11}$ of Schwarzschild exterior metric at the boundary $r=R$, needs to determine $\alpha_0, \alpha_1, \alpha_2$ which can be furnished as follows

$$1 + \alpha_1 + \alpha_2 = 1 - \frac{2m}{R}$$

And the continuity of the first derivative of $g_{11}$ at $r=R$ gives

$$\alpha_1 + 2\alpha_2 = \frac{m}{R}$$

Solving (22) and (23), we get

$$\alpha_1 = -\frac{5m}{R}, \quad \alpha_2 = \frac{3m}{R}$$

In view of (24), (22) and (18), the density and pressure are furnished as follows

$$8\pi\rho R^2 = \frac{15m(15 - x)}{R}$$

$$\frac{dP}{dx} = \frac{1}{4x} \left[ P + \frac{15m}{R} (1 - x) \right] \left[ 1 - e^\lambda (1 + Px) \right]$$
Without an explicit knowledge of the function $y$ in the (18), it is possible to determine the solution $y$ in the form of series solution of the differential equation. Now, equation (26) can be solved numerically with initial condition that the pressure at the boundary is equal to zero i.e. $P(1) = 0$, which means that the pressure vanishes at the boundary of the model. The solution so obtained has been tabulated numerically for some restriction to $\frac{m}{R} < 0.48$ and it has shown physically acceptable.

Similarly for the case of,

$$e^{-\lambda} = \alpha_0 + \alpha_1 x + a_2 x^2$$

(27)

The continuity of $e^{-\lambda}$ and its first derivative with $g_{11}$ of Schwarzschild exterior metric at the boundary $r=R$, needs to determine $\alpha_0, \alpha_1, \alpha_2$ as follows:

$$\alpha_0 + \alpha_1 + \alpha_2 = 1 - \frac{2m}{R}$$

(28)

And the continuity of the first derivative of $g_{11}$ at $r=R$ gives

$$\alpha_1 + 2\alpha_2 = \frac{m}{R}$$

(29)

Solving (29), (28) and (27), we get:

$$\alpha_2 = -\frac{m}{R}, \quad \alpha_1 = \frac{3m}{R}, \quad \alpha_0 = 1 - \frac{4m}{R}$$

Thus, the density may calculate directly while the pressure with initial condition $P(1) = 0$ can be solved by Runge-Kutta method. The solution, so obtained is physically valid with restricted value of $0 < \frac{m}{R} < 0.39$

The above analysis may be shown more precisely in the following graphs for different values of $\frac{m}{R}$ with respect to density, pressure, weak energy conditions as well as for strong energy conditions respectively.
Figure (1): shows the behavior of the Density when $\frac{m}{R} = 0.1, 0.2, 0.3, \text{ and } 0.39$

Figure (2): Pressure when $\frac{m}{R} = 0.1, 0.2, \text{ and } 0.3$
5. Conclusion

Static spherical fluid spheres have been considered for obtaining a perfect fluid distributions with pressure and density both vanishing at the boundary beyond with the Schwarzschild exterior gravitation field. The solutions so obtained are discussed and analyzed with respect to the physical acceptability. Also, we would mention here that the velocity of the sound is not exceeding that of light in a medium.

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References


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