Inverse Interval Matrix: A New Approach

T. Nirmala
Department of Mathematics, Faculty of Engineering and Technology
SRM University, Kattankulathur, Chennai - 603203, India.
nirmalaselvan_20@rediffmail.com

D. Datta
Health Physics Division, Bhabha Atomic Research Centre,
Trombay, Mumbai-400085, India
ddatta@barc.gov.in, dbbtr_datta@yahoo.com

H.S. Kushwaha
Director, Health Safety and Environment Group,
Bhabha Atomic Research Centre, Trombay, Mumbai-400085, India
kushwaha@barc.gov.in

K. Ganesan
Department of Mathematics, Faculty of Engineering and Technology
SRM University, Kattankulathur, Chennai - 603203, India.
ganesank@ktr.srmuniv.ac.in,hod.ma@ktr.srmuniv.ac.in

Abstract
We propose a new method to compute the inverse of an interval matrix based on the modified interval arithmetic. We introduce the notions of determinant, regularity and the inverse matrix of an interval matrix.

Mathematics Subject Classification: 15A09, 65F05, 65G30
Keywords: Interval arithmetic, Interval matrix, Inverse matrix.
1 Introduction

It is well known, that matrices play major role in various areas such as science, engineering and technology, social sciences and many others. In real life, due to the inevitable measurement inaccuracy, we do not know the exact values of the measured quantities; we know, at best, the intervals of possible values. Consequently, we can not successfully use traditional classical matrices and hence the use of interval matrices is more appropriate.

Hansen and Smith [6] started the use of Interval arithmetic in matrix computations. After this motivation and inspiration, several authors such as Alefeld and Herzberger [1], Hansen et al [6, 7], Jaulin et al [15], Neumaier [18], Rohn [23] and Ganesan and Veeramani [3, 4] etc have studied Interval matrices.

Interval matrices, product of interval matrices, inverse interval matrices and powers of interval matrices can be used in describing system dynamics, interval errors [10, 17], time invariant fuzzy systems [5] and systems analysis [20] etc.

In the existing literature there is no method available for finding the exact solution for the system of linear interval equations. But there are methods available for computing the smallest box \(\tilde{x}\) containing the exact solution of the system. In contrast to the problem of solving system of linear interval equations, the problem of inverting interval matrices has been given less attention. In this paper we propose a method for finding the inverse of an interval matrix which in turn helps us to solve system of linear interval equations in a better way.

Let \(\mathbb{IR} = \{\tilde{a} = [a_1, a_2] : a_1 \leq a_2 \text{ and } a_1, a_2 \in \mathbb{R}\}\) be the set of all proper intervals and \(\mathbb{IR} = \{\tilde{a} = [a_1, a_2] : a_1 > a_2 \text{ and } a_1, a_2 \in \mathbb{R}\}\) be the set of all improper intervals on the real line \(\mathbb{R}\). If \(a_1 = a_2 = a\), then \(\tilde{a} = [a, a]\) is a real number (or a degenerate interval). We shall use the terms “interval” and “interval number” interchangeably.
The mid-point and width (or half-width) of an interval number $\tilde{a} = [a_1, a_2]$ are defined as $m(\tilde{a}) = \left(\frac{a_1 + a_2}{2}\right)$ and $w(\tilde{a}) = \left(\frac{a_2 - a_1}{2}\right)$.

Algebraic properties of classical interval arithmetic defined on $\mathbb{IR}$ (see [17]) are often insufficient if we want to deal with real world problems involving interval parameters. Because, intervals with nonzero width do not have inverses in $\mathbb{IR}$ with respect to the classical interval arithmetical operations. This "incompleteness" stimulated attempts to create a more convenient interval arithmetic extending that based on $\mathbb{IR}$. In this direction, several extensions of the classical interval arithmetic have been proposed. Kaucher [13] proposed a new method, in which the set of proper intervals is extended by improper intervals and the interval arithmetic operations and functions are extended correspondingly. We denote the set of generalized intervals (proper and improper) by $\mathbb{D} = \mathbb{IR} \cup \overline{\mathbb{IR}} = \{[a_1, a_2] : a_1, a_2 \in \mathbb{R}\}$. The set of generalized intervals $\mathbb{D}$ is a group with respect to addition and multiplication operations of zero free intervals, while maintaining the inclusion monotonicity.

The algebraic properties of the generalized interval arithmetic create a suitable environment for solving algebraic problems involving interval numbers. However, the efficient solution of some interval algebraic problems is hampered by the lack of well studied distributive relations between generalized intervals. Ganesan and Veeramani [3] proposed a new interval arithmetic which satisfies the distributive relations between intervals.

The "dual" is an important monadic operator that reverses the end-points of the intervals and expresses an element-to-element symmetry between proper and improper intervals in $\mathbb{D}$. For $\tilde{a} = [a_1, a_2] \in \mathbb{D}$, its dual is defined by $\text{dual}(\tilde{a}) = \text{dual}[a_1, a_2] = [a_2, a_1]$. The opposite of an interval $\tilde{a} = [a_1, a_2]$ is $\text{opp}\{[a_1, a_2]\} = [-a_1, -a_2]$ which is the additive inverse of $[a_1, a_2]$ and $\left[\frac{1}{a_1}, \frac{1}{a_2}\right]$ is the multiplicative inverse of $[a_1, a_2]$, provided $0 \not\in [a_1, a_2]$.

That is $\tilde{a} + (-\text{dual} \tilde{a}) = \tilde{a} - \text{dual}(\tilde{a}) = [a_1, a_2] - \text{dual}(a_1, a_2)$

$= [a_1, a_2] - [a_2, a_1] = [a_1 - a_1, a_2 - a_2] = [0, 0]$
and \( \tilde{a} \times \left( \frac{1}{\text{dual } \tilde{a}} \right) = [a_1, a_2] \times \left( \frac{1}{\text{dual}([a_1, a_2])} \right) = [a_1, a_2] \times \frac{1}{[a_2, a_1]} \)

\( = [a_1, a_2] \times \left[ \frac{1}{a_1}, \frac{1}{a_2} \right] = \left[ \frac{a_1}{a_1}, \frac{a_2}{a_2} \right] = [1, 1] \)

In this paper we incorporate the concept of dual in the interval arithmetic given in [3] so that the general interval arithmetic possesses group properties with respect to addition and multiplication operations and satisfying the distributive relations between intervals, while maintaining the inclusion monotonicity.

The rest of this paper is organized as follows: In section 2, we extend the Sengupta and Pal’s [2] method of comparison of interval numbers to the set of all generalized intervals \( \mathbb{D} \). Ganesan and Veeramani [3] proposed a new interval arithmetic on \( \mathbb{IR} \). We extend this arithmetic operations to the set of generalized interval numbers \( \mathbb{D} \) by incorporating the concept of dual [13]. In section 3, we introduce interval matrices, determinants, regularity and inverse matrix of an interval matrix and a method for the solution of system of interval linear equations. Numerical examples are also given to illustrate the theory developed in this paper.

2 Preliminary Notes

The aim of this section is to present some notations, notions and results which are of useful in our further considerations.

2.1 Comparing Interval Numbers

Sengupta and Pal [2] proposed a simple and efficient index for comparing any two intervals on \( \mathbb{IR} \) through decision maker’s satisfaction. We extend this concept to the set of all generalized intervals on \( \mathbb{D} \).
Definition 2.1. Let $\preceq$ be an extended order relation between the interval numbers $\tilde{a} = [a_1, a_2]$ and $\tilde{b} = [b_1, b_2]$ in $\mathbb{D}$, then for $m(\tilde{a}) < m(\tilde{b})$, we construct a premise $(\tilde{a} \preceq \tilde{b})$ which implies that $\tilde{a}$ is inferior to $\tilde{b}$ (or $\tilde{b}$ is superior to $\tilde{a}$). Here, the term 'inferior to' ('superior to') is analogous to 'less than' ('greater than').

Definition 2.2. An acceptability function $A_{\preceq} : \mathbb{D} \times \mathbb{D} \rightarrow [0, \infty)$ is defined as:

$$A_{\preceq}(\tilde{a}, \tilde{b}) = A(\tilde{a} \preceq \tilde{b}) = \left(\frac{m(\tilde{b}) - m(\tilde{a})}{w(\tilde{b}) + w(\tilde{a})}\right)$$

where $w(\tilde{b}) + w(\tilde{a}) \neq 0$.

$A_{\preceq}$ may be interpreted as the grade of acceptability of the 'first interval number $\tilde{a}$ to be inferior to the second interval number $\tilde{b}$'.

For any two interval numbers $\tilde{a}$ and $\tilde{b}$ in $\mathbb{D}$, either $A(\tilde{a} \preceq \tilde{b}) > 0$ or $A(\tilde{b} \preceq \tilde{a}) > 0$ or $A(\tilde{a} \preceq \tilde{b}) = A(\tilde{b} \preceq \tilde{a}) = 0$ and $A(\tilde{a} \preceq \tilde{b}) + A(\tilde{b} \preceq \tilde{a}) = 0$. Also the proposed $A$-index is transitive; for any three interval numbers $\tilde{a}, \tilde{b}$ and $\tilde{c}$ in $\mathbb{D}$, if $A(\tilde{a} \preceq \tilde{b}) \geq 0$ and $A(\tilde{b} \preceq \tilde{c}) \geq 0$, then $A(\tilde{a} \preceq \tilde{c}) \geq 0$. But it does not mean that $A(\tilde{a} \preceq \tilde{c}) \geq \max\{A(\tilde{a} \preceq \tilde{b}), A(\tilde{b} \preceq \tilde{c})\}$.

If $A(\tilde{a} \preceq \tilde{b}) = 0$, then we say that the interval numbers $\tilde{a}$ and $\tilde{b}$ are equivalent (or non-inferior to each other) and we denote it by $\tilde{a} \approx \tilde{b}$. In particular, whenever $A(\tilde{a} \preceq \tilde{b}) = 0$ and $w(\tilde{a}) = w(\tilde{b})$, then $\tilde{a} = \tilde{b}$. Also if $A(\tilde{a} \preceq \tilde{b}) \geq 0$, then we say that $\tilde{a} \preceq \tilde{b}$ and if $A(\tilde{b} \preceq \tilde{a}) \geq 0$, then we say that $\tilde{b} \preceq \tilde{a}$.

Remark 2.1. For any two interval numbers $\tilde{a}, \tilde{b} \in \mathbb{D}$, we have $A(\tilde{a} \preceq \tilde{b}) + A(\tilde{b} \preceq \tilde{a}) = 0$.

Remark 2.2. If $m(\tilde{x}) = 0$ then we say that $\tilde{x}$ is a zero interval number. In particular, if $m(\tilde{x}) = 0$ and $w(\tilde{x}) = 0$, then $\tilde{x} = [0, 0]$. Also, if $m(\tilde{x}) = 0$ and $w(\tilde{x}) \neq 0$, then $\tilde{x} \approx 0$. It is to be noted that if $\tilde{x} = [0, 0] = 0$, then $\tilde{x} \approx 0$, but the converse need not be true. If $\tilde{x} \neq 0$ (i.e. $\tilde{x}$ is not equivalent to 0), then $\tilde{x}$ is said to be a non-zero interval number. It is to be noted that if $\tilde{x} \neq 0$, then
\( \hat{x} \neq \hat{0} \), but the converse need not be true. If \( A(\hat{x} \succeq \hat{0}) \geq 0 \) and \( A(\hat{x} \succeq \hat{0}) \neq 0 \), that is, if \( \hat{x} \succeq \hat{0} \) and \( \hat{x} \neq \hat{0} \), then \( \hat{x} \) is said to be a positive interval number and is denoted by \( \hat{x} \succ \hat{0} \).

### 2.2 A New Interval Arithmetic

Ganesan and Veeramani [3] proposed a new interval arithmetic on \( \mathbb{I}\mathbb{R} \). We extend this arithmetic operations to the set of generalized interval numbers \( \mathbb{D} \) and incorporating the concept of dual. For \( \hat{x} = [x_1, x_2], \hat{y} = [y_1, y_2] \in \mathbb{D} \) and for \( * \in \{+, -, \cdot, \div\} \), we define \( \hat{x} * \hat{y} = [m(\hat{x}) * m(\hat{y}) - k, m(\hat{x}) * m(\hat{y}) + k] \), where \( k = \min \{(m(\hat{x}) * m(\hat{y})) - \alpha, \beta - (m(\hat{x}) * m(\hat{y}))\} \), \( \alpha \) and \( \beta \) are the end points of the interval \( \hat{x} \odot \hat{y} \) under the existing interval arithmetic. In particular

(i) Addition:
\[
\hat{x} + \hat{y} = [x_1, x_2] + [y_1, y_2] = [(m(\hat{x}) + m(\hat{y})) - k, (m(\hat{x}) + m(\hat{y})) + k],
\]
where \( k = \frac{1}{2} \left( \frac{y_2 + x_2 - (y_1 + x_1)}{2} \right) \).

(ii) Subtraction:
\[
\hat{x} - \hat{y} = [x_1, x_2] - [y_1, y_2] = [(m(\hat{x}) - m(\hat{y})) - k, (m(\hat{x}) - m(\hat{y})) + k],
\]
where \( k = \frac{1}{2} \left( \frac{y_2 + x_2 - (y_1 + x_1)}{2} \right) \).

Also if \( \hat{x} = \hat{y} \), i.e. if \( [x_1, x_2] = [y_1, y_2] \), then
\[
\hat{x} - \hat{y} = \hat{x} - \text{dual}(\hat{x}) = [x_1, x_2] - [x_2, x_1] = [x_1 - x_1, x_2 - x_2] = [0, 0]
\]

(iii) Multiplication:
\[
\hat{x} \cdot \hat{y} = \hat{x}\hat{y} = [x_1, x_2][y_1, y_2] = [m(\hat{x})m(\hat{y}) - k, m(\hat{x})m(\hat{y}) + k],
\]
where \( k = \min \{(m(\hat{x})m(\hat{y})) - \alpha, \beta - (m(\hat{x})m(\hat{y}))\} \),
\[
\alpha = \min(x_1y_1, x_1y_2, x_2y_1, x_2y_2) \text{ and } \beta = \max(x_1y_1, x_1y_2, x_2y_1, x_2y_2).
\]

(iv) Division:
\[
1 \div \hat{x} = \frac{1}{\hat{x}} = \frac{1}{[x_1, x_2]} = \left[ \frac{1}{m(\hat{x})} - k, \frac{1}{m(\hat{x})} + k \right], \text{ where}
\]
Inverse interval matrix: A new approach

\[ k = \min \left\{ \frac{1}{x_2} \left( \frac{x_2 - x_1}{x_1 + x_2} \right), \frac{1}{x_1} \left( \frac{x_2 - x_1}{x_1 + x_2} \right) \right\} \text{ and } 0 \not\in [x_1, x_2]. \]

Also if \( \tilde{x} = \tilde{y} \) i.e. \([x_1, x_2] = [y_1, y_2]\), then

\[
\frac{\tilde{x}}{\tilde{y}} = \frac{\tilde{x}}{\tilde{x}} = \frac{\tilde{x}}{\text{dual}(\tilde{x})} = [x_1, x_2]. \left[ \frac{1}{x_1}, \frac{1}{x_2} \right] = \left[ \frac{x_1}{x_2}, \frac{x_2}{x_1} \right] = [1, 1]
\]

From (iii), it is clear that \( \lambda \tilde{x} = \begin{cases} [\lambda x_1, \lambda x_2], & \text{for } \lambda \geq 0 \\ [\lambda x_2, \lambda x_1], & \text{for } \lambda < 0 \end{cases} \)

It is to be noted that we use \( \odot \) to denote the existing interval arithmetic and \( \ast \) to denote the modified interval arithmetic. But wherever there is no confusion we use the same notation for both the cases.

It is also to be noted that \( \tilde{x} \ast \tilde{y} \subseteq \tilde{x} \odot \tilde{y} = \{ x \odot y / x \in \tilde{x}, y \in \tilde{y} \} \), where \( \odot \in \{ \oplus, \odot, \otimes, \oslash \} \) is the existing interval arithmetic.

For example if \( \tilde{x} = [-1, 2] \) and \( \tilde{y} = [3, 5] \), then

\[
\tilde{x} \odot \tilde{y} = [-1, 2] \odot [3, 5] = [\min(-3, -5, 6, 10), \max(-3, -5, 6, 10)] = [-5, 10]
\]

so that \( \tilde{x} \ast \tilde{y} = \tilde{x} \odot \tilde{y} = [-1, 2][3, 5] = [-5, 9] \)

It is also important to note that by using this modified interval arithmetic we are able to prove the distributive law for interval numbers and hence many other important results.

3 Main Results

An interval matrix \( \tilde{A} \) is a matrix whose elements are interval numbers. An interval matrix \( \tilde{A} \) will be written as: \( \tilde{A} = \begin{pmatrix} \tilde{a}_{11} & \cdots & \tilde{a}_{1n} \\ \vdots & \ddots & \vdots \\ \tilde{a}_{m1} & \cdots & \tilde{a}_{mn} \end{pmatrix} = (\tilde{a}_{ij})_{1 \leq i \leq m, \ 1 \leq j \leq n} \),

where each \( \tilde{a}_{ij} = [a_{ij}, \bar{a}_{ij}] \) (or) \( \tilde{A} = [\underline{A}, \overline{A}] \) for some \( \underline{A}, \overline{A} \) satisfying \( \underline{A} \leq \overline{A} \).

We use \( \mathbb{D}^{m \times n} \) to denote the set of all \( (m \times n) \) interval matrices. The midpoint (center) of an interval matrix \( \tilde{A} \) is the matrix of midpoints of its interval elements defined as \( m(\tilde{A}) = \begin{pmatrix} m(\tilde{a}_{11}) & \cdots & m(\tilde{a}_{1n}) \\ \vdots & \ddots & \vdots \\ m(\tilde{a}_{m1}) & \cdots & m(\tilde{a}_{mn}) \end{pmatrix} \). The width
of an interval matrix \( \tilde{A} \) is the matrix of widths of its interval elements defined as
\[
w(\tilde{A}) = \begin{pmatrix} w(\tilde{a}_{11}) & \cdots & w(\tilde{a}_{1n}) \\ \vdots & \ddots & \vdots \\ w(\tilde{a}_{m1}) & \cdots & w(\tilde{a}_{mn}) \end{pmatrix}
\]
which is always nonnegative. We use \( O \) to denote the null matrix
\[
\begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix}
\]
and \( \tilde{O} \) to denote the null interval matrix
\[
\begin{pmatrix} \tilde{0} & \cdots & \tilde{0} \\ \vdots & \ddots & \vdots \\ \tilde{0} & \cdots & \tilde{0} \end{pmatrix}
\]. Also we use \( I \) to denote the identity matrix
\[
\begin{pmatrix} 1 & \cdots & 0 \\ \vdots & 1 & \cdots \\ 0 & \cdots & 1 \end{pmatrix}
\]
and \( \tilde{I} \) to denote the identity interval matrix
\[
\begin{pmatrix} \tilde{1} & \cdots & \tilde{0} \\ \cdots & \tilde{1} & \cdots \\ \tilde{0} & \cdots & \tilde{1} \end{pmatrix}
\].

If \( m(\tilde{A}) = m(\tilde{B}) \), then the interval matrices \( \tilde{A} \) and \( \tilde{B} \) are said to be equivalent and is denoted by \( \tilde{A} \approx \tilde{B} \). In particular if \( m(\tilde{A}) = m(\tilde{B}) \) and \( w(\tilde{A}) = w(\tilde{B}) \), then \( \tilde{A} = \tilde{B} \).

If \( m(\tilde{A}) = O \), then we say that \( \tilde{A} \) is a zero interval matrix. In particular if \( m(\tilde{A}) = O \) and \( w(\tilde{A}) = O \), then \( \tilde{A} = \begin{pmatrix} [0,0] & \cdots & [0,0] \\ \vdots & \ddots & \vdots \\ [0,0] & \cdots & [0,0] \end{pmatrix} \). Also, if \( m(\tilde{A}) = O \) and \( w(\tilde{A}) \neq O \), then \( \tilde{A} = \begin{pmatrix} \tilde{0} & \cdots & \tilde{0} \\ \vdots & \ddots & \vdots \\ \tilde{0} & \cdots & \tilde{0} \end{pmatrix} \approx \tilde{O} \). If \( \tilde{A} \neq \tilde{O} \) (i.e. \( \tilde{A} \) is not equivalent to \( \tilde{O} \)), then \( \tilde{A} \) is said to be a non-zero interval matrix.

If \( m(\tilde{A}) = I \), then we say that \( \tilde{A} \) is a identity interval matrix. In particular if \( m(\tilde{A}) = I \) and \( w(\tilde{A}) = O \), then \( \tilde{A} = \begin{pmatrix} [1,1] & \cdots & [0,0] \\ \vdots & \ddots & \vdots \\ [0,0] & \cdots & [1,1] \end{pmatrix} \). Also, if \( m(\tilde{A}) = I \) and \( w(\tilde{A}) \neq O \), then \( \tilde{A} = \begin{pmatrix} \tilde{1} & \cdots & \tilde{0} \\ \cdots & \tilde{1} & \cdots \\ \tilde{0} & \cdots & \tilde{1} \end{pmatrix} \approx \tilde{I} \). If \( \tilde{A} \neq \tilde{I} \) (i.e. \( \tilde{A} \) is not equivalent to \( \tilde{I} \)), then \( \tilde{A} \) is said to be a non-identity interval matrix.
\[ m(\hat{A}) = I \] and \( w(\hat{A}) \neq O \), then \( \hat{A} = \begin{pmatrix} \hat{1} & \cdots & \hat{0} \\ \cdots & \hat{1} & \cdots \\ \hat{0} & \cdots & \hat{1} \end{pmatrix} \approx \hat{I} \).

We introduce the following arithmetic operations on interval matrices. As with interval numbers, we define the arithmetic operations on interval matrices as follows.

If \( \hat{A}, \hat{B} \in \mathbb{D}^{m \times n}, \hat{x} \in \mathbb{D}^n \) and \( \hat{\alpha} \in \mathbb{D} \), then

(i). \( \hat{\alpha}\hat{A} = (\hat{\alpha}\hat{a}_{ij})_{1 \leq i \leq m, 1 \leq j \leq n} \)

(ii). \( \hat{A} + \hat{B} = (\hat{a}_{ij} + \hat{b}_{ij})_{1 \leq i \leq m, 1 \leq j \leq n} \)

(iii). \( \hat{A} - \hat{B} = \begin{cases} (\hat{a}_{ij} - \hat{b}_{ij})_{1 \leq i \leq m, 1 \leq j \leq n}, & \text{if } \hat{A} \neq \hat{B} \\ \hat{A} - \text{dual}(\hat{A}) = \hat{O} = O, & \text{if } \hat{A} = \hat{B} \end{cases} \)

(iv). \( \hat{A}\hat{B} = (\sum_{k=1}^{n} \hat{a}_{ik}\hat{b}_{kj})_{1 \leq i \leq m, 1 \leq j \leq n} \)

(v). \( \hat{A}\hat{x} = (\sum_{j=1}^{n} \hat{a}_{ij}\hat{x}_j)_{1 \leq i \leq m} \)

### 3.1 Inverse of an Interval Matrix

We define the determinant of a square interval matrix as in the case of real square matrix except that the determinant of an interval matrix is an interval number.

That is \( \text{det} \hat{A} = | \hat{A} | = \sum \hat{a}_{ij}\hat{A}_{ij} \), where \( \hat{A}_{ij} \) is the cofactor of \( \hat{a}_{ij} \) with usual meaning.

It is easy to see that most of the properties of determinants of classical matrices are hold good (up to equivalent) for the determinants of interval matrices under the modified interval arithmetic.

**Definition 3.1.** A square interval matrix \( \hat{A} \) is said to be non singular or regular if \( | \hat{A} | \) is invertible (i.e. \( 0 \notin | \hat{A} | \)). Alternatively, a square interval matrix \( \hat{A} \) is said to be invertible if \( | \hat{A} | \) is invertible (i.e. \( 0 \notin | \hat{A} | \)).
Example 3.1. Let \( \tilde{A} = [\underline{A}, \overline{A}] = \begin{pmatrix} [1, 2] & [3, 4] \\ [-9, 1] & [8, 10] \end{pmatrix} \).

Then \( |\tilde{A}| = \begin{vmatrix} [1, 2] & [3, 4] \\ [-9, 1] & [8, 10] \end{vmatrix} = [1, 2][8, 10] - [-9, 1][3, 4] = [8, 19] - [-32, 4] = [4, 51] \neq [0, 0] \).

We see that \( 0 \not\in |\tilde{A}| = [4, 51] \) and hence \( |\tilde{A}| \) is invertible. So that \( \tilde{A} \) is regular.

Definition 3.2. Let \( \tilde{A} \) be a square interval matrix. The adjoint matrix \( \tilde{A}^* \) of \( \tilde{A} \) is the transpose of the matrix of cofactors of the elements of \( \tilde{A} \). That is \( \tilde{A}^* = \text{adj}(\tilde{A}) = (\tilde{b}_{ij}) \), where \( \tilde{b}_{ij} = |A_{ji}| \), for all \( i, j = 1, 2, 3, \ldots, n \).

Definition 3.3. For any \( \tilde{A} \in \mathbb{IR}^{n \times n} \), if \( |\tilde{A}| \) is invertible, then the common solution of equations \( \tilde{A}\tilde{X} = \tilde{I} \) and \( \tilde{X}\tilde{A} = \tilde{I} \) is called the inverse of \( \tilde{A} \) and is denoted by \( \tilde{A}^{-1} = \frac{\text{adj}(\tilde{A})}{|\tilde{A}|} = \frac{\tilde{A}^*}{\text{det}(\tilde{A})} \). It is to be noted that, if \( \tilde{A} \) is invertible, then \( m(\tilde{A}^{-1}) = [m(\tilde{A})]^{-1} \).

Theorem 3.4. Let \( \tilde{A}^* \) be the adjoint matrix of \( \tilde{A} \). Then \( \tilde{A}\tilde{A}^* = \tilde{A}^*\tilde{A} = |\tilde{A}|\tilde{I} \).

Proof. Let \( \tilde{A} = (\tilde{a}_{ij}) \), \( \tilde{A}^* = (\tilde{b}_{ij}) \) so that \( \tilde{b}_{ij} = |A_{ji}| \). Then for \( i, j = 1, 2, 3, \ldots, n \), we have

\[
(\tilde{A}\tilde{A}^*)_{ij} = \sum_{k=1}^{n} \tilde{a}_{ik}\tilde{b}_{kj} = \sum_{k=1}^{n} \tilde{a}_{ik}\tilde{A}_{jk} = |\tilde{A}|\tilde{\delta}_{ij} = |\tilde{A}|\tilde{I} \quad (3.1)
\]

and

\[
(\tilde{A}^*\tilde{A})_{ij} = \sum_{k=1}^{n} \tilde{b}_{ik}\tilde{a}_{kj} = \sum_{k=1}^{n} \tilde{a}_{kj}\tilde{A}_{ki} = |\tilde{A}|\tilde{\delta}_{ji} = |\tilde{A}|\tilde{I}, \quad (3.2)
\]

where \( \tilde{\delta}_{ji} = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases} \).

From equations (3.1) and (3.2), we see that \( \tilde{A}\tilde{A}^* = \tilde{A}^*\tilde{A} = |\tilde{A}|\tilde{I} \). \(\square\)

Example 3.2. Let \( \tilde{A} = [\underline{A}, \overline{A}] = \begin{pmatrix} [0, 2] & [1, 3] \\ [3, 5] & [5, 7] \end{pmatrix} \).
Then $|\tilde{A}| = \begin{bmatrix} 0, 2 \\ 3, 5 \end{bmatrix} - \begin{bmatrix} 1, 3 \\ 5, 7 \end{bmatrix} = [0, 2] [5, 7] - [1, 3] [3, 5] = [−13, 9] ≠ [0, 0] = 0.$

Now the adjoint matrix $\tilde{A}^*$ of $\tilde{A}$ is $\tilde{A}^* = \begin{pmatrix} [5, 7] & [1, 3] \\ −[3, 5] & [0, 2] \end{pmatrix}$ so that $\tilde{A}\tilde{A}^* = \begin{pmatrix} [−13, 9] & [0, 0] \\ [0, 0] & [−13, 9] \end{pmatrix} = [−13, 9] \begin{pmatrix} [1, 1] & [0, 0] \\ [0, 0] & [1, 1] \end{pmatrix} = |\tilde{A}| I \Rightarrow \tilde{A}\tilde{A}^* = |\tilde{A}| I$.

Also it is easy to prove $\tilde{A}^*\tilde{A} = |\tilde{A}| I$ and hence $\tilde{A}\tilde{A}^* = \tilde{A}^*\tilde{A} = |\tilde{A}| I$.

**Theorem 3.5.** (i) If $\tilde{A}$ is invertible, then the matrix equations $\tilde{A}\tilde{X} = \tilde{I}$ and $\tilde{X}\tilde{A} = \tilde{I}$ both possesses a common solution $\tilde{X} = \frac{1}{|\tilde{A}|} \tilde{A}^*$.

(ii) If atleast one of the equations is solvable for $\tilde{X}$, then $|\tilde{A}|$ is invertible and so both equations are solvable and possesses a common solution $\tilde{X} = \frac{1}{|\tilde{A}|} \tilde{A}^*$.

**Proof.** (i) If $|\tilde{A}|$ is invertible, then by theorem (3.4) the matrix $\tilde{X} = \frac{1}{|\tilde{A}|} \tilde{A}^*$ satisfies $\tilde{A}\tilde{X} = \tilde{X}\tilde{A} = \tilde{I}$ and is therefore a solution to both $\tilde{A}\tilde{X} = \tilde{I}$ and $\tilde{X}\tilde{A} = \tilde{I}$.

(ii) Suppose that $\tilde{A}\tilde{X} = \tilde{I}$ is solvable. Then $|\tilde{A}| |\tilde{X}| = \tilde{I}$ and so $|\tilde{A}|$ is invertible. \(\square\)

**Proposition 3.1.** [14] Let $\hat{A} = [A, \overline{A}] \in IR^{n×n}$. If $A$ and $\overline{A}$ are regular and $A^{-1} ≥ 0$, $\overline{A}^{-1} ≥ 0$, then $\hat{A}$ is regular and $\hat{A}^{-1} = [\overline{A}^{-1}, A^{-1}] ≥ 0$.

**Example 3.3.** Let $\hat{A} = [A, \overline{A}] = \begin{pmatrix} [1, 2] & [−1, 0] \\ [−2, −1] & [3, 4] \end{pmatrix} ⇒ A = \begin{pmatrix} 1 & −1 \\ −2 & 3 \end{pmatrix}$ and $\overline{A} = \begin{pmatrix} 2 & 0 \\ −1 & 4 \end{pmatrix}$. Hence $A^{-1} = \begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix}$ and $\overline{A}^{-1} = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$. Here $A^{-1} ≥ 0$, $\overline{A}^{-1} ≥ 0$. Then by proposition (3.1), we get

$\hat{A}^{-1} = [\overline{A}^{-1}, A^{-1}] = \begin{pmatrix} [1, 2] & [0, 1] \\ [1, 3] & [1, 4] \end{pmatrix} ≥ 0$. 

But we are not able to get $\tilde{A}^{-1} \tilde{A} = \tilde{A}\tilde{A}^{-1} = \tilde{I}$ or $\tilde{A}^{-1} \tilde{A} \approx \tilde{A}\tilde{A}^{-1} \approx \tilde{I}$ either by applying the existing interval arithmetic or by applying the modified interval arithmetic. On the other hand, if we apply the method introduced here for finding the inverse of an interval matrix, we get $\tilde{|A|} = \begin{bmatrix} 3 & 15 \\ 2 & 2 \end{bmatrix}$, so that $\tilde{A}^{-1} = \frac{1}{\tilde{|A|}} \begin{bmatrix} 3 & 15 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 15 \\ 2 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 3 & 15 \\ 2 & 2 \end{bmatrix} = \frac{1}{\tilde{|A|}} \begin{bmatrix} 1, 1 \\ 0, 0 \end{bmatrix} = \tilde{I}$.

Theorem 3.6. Let $\tilde{A}\tilde{x} = \tilde{b}$ be a system of linear equations involving interval numbers. If the $(n \times n)$ interval matrix $\tilde{A}$ is invertible, then it is possible to find a smallest box $\tilde{x} = (\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \ldots, \tilde{x}_n)$, where each $\tilde{x}_i = \frac{|\tilde{A}^{(i)}|}{|A|}$, $\tilde{A}^{(i)}$ is the interval matrix obtained when the $i$th column of $\tilde{A}$ is replaced by the vector $\tilde{b} = (\tilde{b}_1, \tilde{b}_2, \tilde{b}_3, \ldots, \tilde{b}_n)$.

Proof. Given that $\tilde{A}$ is invertible. Multiply both sides of the system $\tilde{A}\tilde{x} = \tilde{b}$ by $\tilde{A}^{-1}$, we have $\tilde{A}^{-1}\tilde{A}\tilde{x} = \tilde{A}^{-1}\tilde{b} = \tilde{I}\tilde{x} = \tilde{A}^{-1}\tilde{b} = \tilde{x} = \tilde{A}^{-1}\tilde{b} = \frac{1}{|A|} \tilde{A}^{*}\tilde{b}$. That is

$$
\begin{pmatrix}
\tilde{x}_1 \\
\tilde{x}_2 \\
\vdots \\
\tilde{x}_n
\end{pmatrix} = \frac{1}{|A|} \begin{pmatrix}
\tilde{A}_{11} & \tilde{A}_{21} & \cdots & \tilde{A}_{n1} \\
\tilde{A}_{12} & \tilde{A}_{22} & \cdots & \tilde{A}_{n2} \\
\vdots & \vdots & \ddots & \vdots \\
\tilde{A}_{1n} & \tilde{A}_{2n} & \cdots & \tilde{A}_{nn} \\
\end{pmatrix} \begin{pmatrix}
\tilde{b}_1 \\
\tilde{b}_2 \\
\vdots \\
\tilde{b}_n
\end{pmatrix} = \frac{1}{|A|} \begin{pmatrix}
\tilde{A}_{11}\tilde{b}_1 + \tilde{A}_{21}\tilde{b}_2 + \cdots + \tilde{A}_{n1}\tilde{b}_n \\
\tilde{A}_{12}\tilde{b}_1 + \tilde{A}_{22}\tilde{b}_2 + \cdots + \tilde{A}_{n2}\tilde{b}_n \\
\vdots \\
\tilde{A}_{1n}\tilde{b}_1 + \tilde{A}_{2n}\tilde{b}_2 + \cdots + \tilde{A}_{nn}\tilde{b}_n
\end{pmatrix}
$$
which implies 
\[ \hat{x}_1 = \frac{1}{|\hat{A}|}(\hat{A}_{11}\hat{b}_1 + \hat{A}_{21}\hat{b}_2 + \cdots + \hat{A}_{n1}\hat{b}_n), \]
\[ \hat{x}_2 = \frac{1}{|\hat{A}|}(\hat{A}_{12}\hat{b}_1 + \hat{A}_{22}\hat{b}_2 + \cdots + \hat{A}_{n2}\hat{b}_n), \]
\[ \vdots \]
\[ \hat{x}_n = \frac{1}{|\hat{A}|}(\hat{A}_{1n}\hat{b}_1 + \hat{A}_{2n}\hat{b}_2 + \cdots + \hat{A}_{nn}\hat{b}_n). \]

\[ \square \]

**Example 3.4.** Let us consider an example given in Ning et al [19]. The system of interval equations \( \hat{A}\hat{x} = \hat{b} \) be given with
\[
\hat{A} = \begin{pmatrix}
[3.7, 4.3] & [-1.5, -0.5] & [0, 0] \\
[-1.5, -0.5] & [3.7, 4.3] & [-1.5, -0.5] \\
[0, 0] & [-1.5, -0.5] & [3.7, 4.3]
\end{pmatrix}, \quad \hat{b} = \begin{pmatrix}
[-14, 14] \\
[-9, 9] \\
[-3, 3]
\end{pmatrix}.
\]

Using interval Gaussian elimination with existing interval arithmetic and using their technique, Ning et al [19] obtained the same solution set (box)
\[
\begin{pmatrix}
[-6.38, 6.38] \\
[-6.40, 6.40] \\
[-3.40, 3.40]
\end{pmatrix}
\]

If we apply the above theorem with modified interval arithmetic, we have
\[
|\hat{A}| = \begin{vmatrix}
[3.7, 4.3] & [-1.5, -0.5] & [0, 0] \\
[-1.5, -0.5] & [3.7, 4.3] & [-1.5, -0.5] \\
[0, 0] & [-1.5, -0.5] & [3.7, 4.3]
\end{vmatrix}
= [44.178, 75.822] + [-7.075, -0.925]
= [37.103, 74.897] \text{ and } 0 \not\in |\hat{A}|.
\]

\[
|\hat{A}^{(1)}| = \begin{vmatrix}
[-14, 14] & [-1.5, -0.5] & [0, 0] \\
[-9, 9] & [3.7, 4.3] & [-1.5, -0.5] \\
[-3, 3] & [-1.5, -0.5] & [3.7, 4.3]
\end{vmatrix}
= [-317.64, 317.64].
\]
\[ |\tilde{A}(2)| = \left| \begin{array}{ccc} [3.7, 4.3] & [-14, 14] & [0, 0] \\ [-1.5, -0.5] & [-9, -9] & [-1.5, -0.5] \\ [0, 0] & [-3, -3] & [3.7, 4.3] \end{array} \right| = [-271.86, 271.86]. \]

Also \[ |\tilde{A}(3)| = \left| \begin{array}{ccc} [3.7, 4.3] & [-1.5, -0.5] & [-14, 14] \\ [-1.5, -0.5] & [3.7, 4.3] & [-9, -9] \\ [0, 0] & [-1.5, -0.5] & [-3, -3] \end{array} \right| = [-144.77, 144.77]. \]

Then by the above theorem we see that
\[ \tilde{x}_1 = \frac{|\tilde{A}^{(1)}|}{|A|} = \frac{|-317.64, 317.64|}{[37.103, 74.897]} = [-317.64, 317.64][0.014, 0.022] \]
\[ = [-6.988, 6.988], \]
\[ \tilde{x}_2 = \frac{|\tilde{A}^{(2)}|}{|A|} = \frac{|-271.86, 271.86|}{[37.103, 74.897]} = [-271.86, 271.86][0.014, 0.022] \]
\[ = [-5.981, 5.981] \]
and \[ \tilde{x}_3 = \frac{|\tilde{A}^{(3)}|}{|A|} = \frac{|-144.77, 144.77|}{[37.103, 74.897]} = [-144.77, 144.77][0.014, 0.022] \]
\[ = [-3.185, 3.185]. \]

In this case, we obtain the solution set (box) \( \begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{x}_3 \end{pmatrix} = \begin{pmatrix} [-6.988, 6.988] \\ [-5.981, 5.981] \\ [-3.185, 3.185] \end{pmatrix}. \)

It is to be noted that the solution set obtained by our method is sharper than the solution set obtained by other techniques.

**Example 3.5.** Let us consider another example given in Ning et al [19]. The system of interval equations \( \tilde{A}\tilde{x} = \tilde{b} \) be given with
\[ \tilde{A} = \begin{pmatrix} [3.7, 4.3] & [-1.5, -0.5] & [0, 0] \\ [-1.5, -0.5] & [3.7, 4.3] & [-1.5, -0.5] \\ [0, 0] & [-1.5, -0.5] & [3.7, 4.3] \end{pmatrix}, \quad \tilde{b} = \begin{pmatrix} [-14, 0] \\ [-9, 0] \\ [-3, 0] \end{pmatrix}. \]
By our method, we obtain the solution set (box) \[
\begin{bmatrix}
\tilde{x}_1 \\
\tilde{x}_2 \\
\tilde{x}_3
\end{bmatrix} = \begin{bmatrix}
[-4.482, 0] \\
[-3.816, 0] \\
[-1.776, 0.066]
\end{bmatrix}.
\]

Using interval Gaussian elimination with existing interval arithmetic, Ning et al. [19] obtained the solution set (box)
\[
\begin{bmatrix}
[-6.38, 0] \\
[-6.40, 0] \\
[-3.40, 0]
\end{bmatrix}.
\]

Using Hansen’s technique [9] or Rohn’s reformulation of [22], Ning et al. [19] obtained the solution set (wider box)
\[
\begin{bmatrix}
[-6.38, 1.12] \\
[-6.40, 1.54] \\
[-3.40, 1.40]
\end{bmatrix}.
\]

Using their technique, Ning et al. [19] obtained the solution set (much wider box)
\[
\begin{bmatrix}
[-6.38, 1.67] \\
[-6.40, 2.77] \\
[-3.40, 2.40]
\end{bmatrix}.
\]

It is to be noted that the solution set (box) obtained by our method is sharper than the solution sets obtained by other techniques.

**Acknowledgements**

The authors gratefully acknowledge (BRNS - DAE) the Board of Research in Nuclear Sciences, Department of Atomic Energy, Government of India for its support through the funded research project grant No. 2008/36/35/BRNS/1999 for the investigation presented here. Also the authors would like to thank the anonymous reviewers for their critical comments and valuable suggestions which helped the authors to improve the presentation of this paper.
References


Received: September, 2010