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Basic Riemann-Liouville and Caputo Impulsive Fractional Calculus

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Abstract: This paper gives formulas for Riemann- Liouville impulsive fractional integral calculus and for Riemann- Liouville and Caputo impulsive fractional derivatives.

Keywords: Rimann- Liouville fractional calculus, Caputo fractional derivative, Dirac delta, Distributional derivatives, High- order distributional derivatives.

1. Introduction

Fractional calculus has been used in a set of applications, mainly, to deal with modelling errors in differential equations and dynamic systems. There are also applications in Signal Processing and sampling and hold algorithms, [1-3]. Fractional integrals and derivatives can be of non-integer orders and even of complex order. This facilitates the description of some problems which are not easily descxribed by ordinary calculus due to modelling errors, [1-5]. There are several approaches for the integral fractional calculus, the most popular ones being the Riemann-Liouville fractional integral. There is also a fractional Riemann- Liouville derivative. However, the well-known Caputo fractional derivative are less involved since the associated integral operator manipulates the derivatives of the primitive function under the integral symbol. This paper extends the basic fractional differ-integral calculus to impulsive functions described through the use of Dirac distributions and Dirac distributional derivatives, [5], of real fractional orders. In the general case, it is admitted a presence of infinitely many impulsive terms at certain isolated point of the relevant function domains.

2. Extended Riemann- Liouville fractional integral

Let us denote the set of positive real numbers by $\boldsymbol{R}_{+}=\{ r\in\boldsymbol{R} : r>0\}$ and left-sided and right-sided Lebesgue integrals, respectively, as:

$\int_0^x g(\tau)d\tau := \lim_{t\to x\equiv x^-} \int_0^t g(\tau)d\tau$ (the identification $x\equiv x^-$ is used for all x in order to simplify the notation), and

$\int_0^{x^+} g(\tau)d\tau := \lim_{t\to x^+} \int_0^t g(\tau)d\tau$

Now, consider real functions $f, \bar f : \boldsymbol{R}_+ \to \boldsymbol{R}$,such that $\int_0^x (x-t)^{\mu-1} \bar f(t)dt$ exists, $\forall x\in\boldsymbol{R}_+$, fulfilling:

$f(x)=\bar f(x)+\sum_{x_i\in IMP} K_i\,\delta(x-x_i)=\bar f(x)+\sum_{i\in I(\infty)} K_i\,\delta(x-x_i)$

$\delta(x)$ denotes the Dirac delta distribution, $K_i\delta(0)=f(x_i^+)-f(x_i)$ with $K_i\in\mathbf{R}$; $\forall i\in I(\infty)\subset\boldsymbol{Z}_+$, [5], and $IMP := \bigcup_{x\in\boldsymbol{R}_+} IMP(x)=\bigcup_{x\in\boldsymbol{R}_+} IMP(x^+)$ of indexing set $I(\infty)$ is the whole impulsive set defined via empty or non-empty) partial impulsive strictly ordered denumerable sets:

$$IMP(x):=\left\{x_i\in\mathbf{R}_+ : f\left(x_i^+\right)-f(x_i)=K_i\delta(0),\, x_i<x\right\} \tag{1}$$

of indexing set $I(x):=\{i\in\boldsymbol{Z}_{0+} : x_i\in IMP(x)\}\subset I\left(x^+\right)\subset\boldsymbol{Z}_+$, for each $x\in\boldsymbol{R}_+$; and

$$IMP(x)\subset IMP\left(x^+\right):=\left\{x_i\in\mathbf{R}_+ : f\left(x_i^+\right)-f(x_i)=K_i\delta(0),\, x_i\le x^+\right\}\subset\mathbf{R}_+ \tag{2}$$

of indexing set $I(x)\subset I\left(x^+\right):=\left\{i\in\boldsymbol{Z}_{0+} : x_i\in IMP\left(x^+\right)\right\}\subset\boldsymbol{Z}_+$, for each $x\in\boldsymbol{R}_+$.

with the indexing set of IMP being $I(\infty)=\bigcup_{x\in IMP(x)} I(x)=\bigcup_{x\in IMP(x^+)} I\left(x^+\right)$. If we are interested in studying the fractional derivative of the impulsive function $f:\boldsymbol{R}_+\to\boldsymbol{R}$ then $\bar f:\boldsymbol{R}_+\to\boldsymbol{R}$ is non- uniquely defined as $\bar f(x)=f(x)$ for $x\in\boldsymbol{R}_+\backslash IMP$, and $f(x_i)=\bar f(x_i)$, $f\left(x_i^+\right)=f(x_i)+K_i\,\delta(0)=\bar f(x_i)+K_i\,\delta(0)$, for $x_i\in IMP$ with $\bar f\left(x^+\right)\in\boldsymbol{R}$ (non-uniquely) defined being bounded arbitrary (for instance, being zero or $\bar f\left(x^+\right)=f(x)$) if $x\in IMP$. Note that IMP and $I(\infty)$ have infinite cardinals if there are infinitely many impulsive values of the function f(t).

Note that the existence of $\int_0^x (x-t)^{\mu-1}\bar f(t)dt$ implies that of $\int_0^x (x-t)^{\mu-1} f(t)dt=\int_0^x (x-t)^{\mu-1}\bar f(t)dt$ if $x\notin IMP(x)$, since $\int_0^x (x-t)^{\mu-1}\bar f(t)dt$ exists, and that of

$$\int_0^{x_i^+} (x-t)^{\mu-1} f(t)dt=\int_0^{x_i} (x-t)^{\mu-1}\bar f(t)dt+(x-x_i)^{\mu-1}\left(f\left(x_i^+\right)-f(x_i)\right) \text{if } x_i\in IMP\left(x^+\right) \tag{3}$$

Theorem 2.1. The extended fractional Riemann- Liouville integrals by considering impulsive functions are defined for any fixed order $\mu\in\boldsymbol{R}_+$ and all $x\in\boldsymbol{R}_+$ by

$$\left(J^{\mu} f\right)(x) := \frac{1}{\Gamma(\mu)} \int_0^x (x-t)^{\mu-1} f(t) dt$$
$$= \frac{1}{\Gamma(\mu)} \left(\int_0^x (x-t)^{\mu-1} \bar{f}(t) dt + \sum_{i \in I(x)} (x-x_i)^{\mu-1} \left(f\left(x_i^+\right) - f(x_i) \right) \right)$$
$$= \frac{1}{\Gamma(\mu)} \left(\sum_{i \in I(x) \cup \{0\}} \int_{x_i^+}^{x_{i+1}} (x-t)^{\mu-1} f(t) dt + \int_{x_{n(x)}^+}^{x} (x-t)^{\mu-1} f(t) dt + \sum_{i \in I(x)} (x-x_i)^{\mu-1} \left(f\left(x_i^+\right) - f(x_i) \right) \right) \tag{4}$$

$$\left(J^{\mu} f\right)\left(x^+\right) := \frac{1}{\Gamma(\mu)} \int_0^{x^+} (x-t)^{\mu-1} f(t) dt$$
$$= \frac{1}{\Gamma(\mu)} \left(\int_0^x (x-t)^{\mu-1} \bar{f}(t) dt + \sum_{i \in I(x^+)} (x-x_i)^{\mu-1} \left(f\left(x_i^+\right) - f(x_i) \right) \right)$$
$$= \frac{1}{\Gamma(\mu)} \left(\sum_{i \in I(x^+) \cup \{0\}} \int_{x_i^+}^{x_{i+1}} (x-t)^{\mu-1} f(t) dt + \sum_{i \in I(x^+)} (x-x_i)^{\mu-1} \left(f\left(x_i^+\right) - f(x_i) \right) \right) \tag{5}$$

$$\left(J^{0} f\right)\left(x^+\right) \equiv \left(J^{0} f\right)(x) := f(x)$$

where $\Gamma : \boldsymbol{R}_{0+} \to \boldsymbol{R}_+$ is the Γ- function , [1-5] and $n : IMP \to \boldsymbol{Z}_+$ is defined by $n(x) = card\, I(x) = card\, IMP(x)$.
□

Note that if $x \in IMP$ then

$$\left(J^{\mu} f\right)\left(x^+\right) = \frac{1}{\Gamma(\mu)} \left(\sum_{i \in I(x^+) \cup \{0\}} \int_{x_i^+}^{x_{i+1}} (x-t)^{\mu-1} f(t) dt + \sum_{i \in I(x^+)} (x-x_i)^{\mu-1} \left(f\left(x_i^+\right) - f(x_i) \right) \right)$$
$$= \left(J^{\mu} f\right)(x) + \left(x - x_{n(x)}\right)^{\mu-1} \left(f\left(x_{n(x)}^+\right) - f\left(x_{n(x)}\right) \right)$$
$$\neq \left(J^{\mu} f\right)(x) = \frac{1}{\Gamma(\mu)} \left(\sum_{i \in I(x) \cup \{0\}} \int_{x_i^+}^{x_{i+1}} (x-t)^{\mu-1} f(t) dt + \sum_{i \in I(x)} (x-x_i)^{\mu-1} \left(f\left(x_i^+\right) - f(x_i) \right) \right) \tag{6}$$

and if $x \notin IMP$, since $I(x^+) = I(x)$, then $\left(J^{\mu} f\right)\left(x^+\right) = \left(J^{\mu} f\right)(x)$.

3. Extended Riemann- Liouville fractional derivative

Assume that $f \in C^{m-1}(\boldsymbol{R}_+, \boldsymbol{R})$ and its $m-th$ derivative exists everywhere in $\boldsymbol{R}_+$. Then, the Caputo fractional derivative of order $\mu \geq 0$ with $m-1 \leq \mu (\in \mathbf{R}_+) \leq m$, $m \in \boldsymbol{Z}_+$ is for any $x \in \boldsymbol{R}_+$:

$$\left(D^{\mu} f\right)(x) := \left(\frac{d}{dx}\right)^{m}\left(J^{m-\mu} f\right)(x) = \frac{1}{\Gamma(m-\mu)}\left(\frac{d}{dx}\right)^{m}\left(\int_0^x (x-t)^{m-\mu-1} f(t)dt\right) \qquad (7)$$

The following particular cases follow from this formula for $\mu = m-1$:

(a) $\mu=-1;\ m=0$ yields $\left(D^{-1} f\right)(x) = \int_0^x f(t)dt$ which is the standard integral of the function f . This case does not verifies the "derivative constraint " $0 \le m-1 \le \mu(\in \boldsymbol{R}_+) < m$ leading to an integral result.

(b)) $\mu=0;\ m=1$ yields $\left(D^{0} f\right)(x) = f(x)$ which so that $D^0 f$ is the identity operator

(c) $\mu=1;\ m=2$ yields $\left(D^{1} f\right)(x) = f^{(1)}(x)$

(d) $\mu=2;\ m=3$ yields $\left(D^{2} f\right)(x) = f^{(2)}(x)$ which is the standard first- derivative of the function f.

Compared to the parallel cases with the Caputo fractional derivative, note that the Riemann- Liouville fractional derivative, compared to the Caputo corresponding one, does not depend on the conditions at zero of the function and its derivatives. Define the Kronecker delta $\delta(a,b)$ of any pair of real numbers (a,b) as $\delta(a,b)=1$ if $a=b$ and $\delta(a,b)=0$ if $a \ne b$ and then evaluate recursively the Riemann – Liouville fractional derivative of order $\mu \ge 0$ from the above formula by using Leibniz´s differentiation rule by noting that , since $\mu \ne m-j;\forall j(\in \boldsymbol{Z}_+) > 1$, only the differential part corresponding to the differentiation of the integrand is non zero for $j > m-\mu$. This yields the following:

Theorem 3.1. Assume that $f \in C^{m-2}(\boldsymbol{R}_+, \boldsymbol{R})$ and $f^{(m-1)}$ exists everywhere in $\boldsymbol{R}_+$ and that $f(t)$ is integrable on $\mathbf{R}_+$, then:

$$\begin{aligned}
\left(D^{\mu} f\right)(x) &= \frac{1}{\Gamma(m-\mu)}\left(\frac{d}{dx}\right)^{m}\left(\int_0^x (x-t)^{m-\mu-1} f(t)dt\right) \\
&= \frac{1}{\Gamma(m-\mu)}\left(\frac{d}{dx}\right)^{m-1}\left[\int_0^x (m-\mu-1)(x-t)^{m-\mu-2} f(t)dt + f(x)\delta(\mu, m-1)\right] \\
&= \frac{1}{\Gamma(m-\mu)}\left[f^{(m-1)}(x)\delta(\mu, m-1) + \left(\frac{d}{dx}\right)^{m-1}\left(\int_0^x (m-\mu-1)(x-t)^{m-\mu-2} f(t)dt\right)\right] \\
&= \frac{1}{\Gamma(m-\mu)}\left[f^{(m-1)}(x)\delta(\mu, m-1) + \left(\sum_{i=1}^{m-2}\prod_{j=i+1}^{m-1}[j-\mu]\right) f^{(i)}(x)\delta(\mu, m-i) + \left[\prod_{j=0}^{m-1}[j-\mu]\right]\left(\int_0^x (x-t)^{-(\mu+1)} f(t)dt\right)\right]
\end{aligned} \qquad (8)$$

If $f \in PC^{k}(\boldsymbol{R}_+, \boldsymbol{R})$ with $f^{(k)}(x)$ being discontinuous of first class then $f^{m-1}(x) = \delta^{(j(x))}(x)$ with $j(x) = m-1-k(x)$, one uses to obtain the right value of (8) the perhaps high-order distributional derivatives formula:

$$\left| f^{(m-1)}\left(x^+\right) - f^{(m-1)}(x)\right| = \frac{(-1)^k k!}{x^k}\left| f^{(m-1-k)}\left(x^+\right) - f^{(m-1-k)}(x)\right| \delta(0) = \infty \qquad (9)$$

to yield

$$\left(D^{\mu} f\right)\left(x^{+}\right)=\frac{1}{\Gamma(m-\mu)}\left[\frac{(-1)^{k(x)} k(x)!}{x^{k(x)}}\left|f^{(m-1-k(x))}\left(x^{+}\right)-f^{(m-1-k(x))}(x)\right| \delta(0) \delta(\mu, m-1)\right.$$
$$\left.+\left(\sum_{i=1}^{m-2} \prod_{j=i+1}^{m-1}[j-\mu]\right) f^{(i)}(x) \delta(\mu, m-i)+\left[\prod_{j=0}^{m-1}[j-\mu]\right]\left(\int_0^x (x-t)^{-(\mu+1)} f(t) d t\right)\right] \tag{10}$$

If $\mu=m-1$ then

$$\left(D^{m-1} f\right)(x)=f^{(m-1)}(x)+\left[\prod_{j=0}^{m-1}[j-\mu]\right]\left(\int_0^x (x-t)^{-m} f(t) d t\right) \tag{11}$$

provided that $\left(\int_0^x (x-t)^{-(\mu+1)} f(t) d t\right)$ exists for $x \in \mathbf{R}_{+}$(which is guaranteed if $f(t)$ is Lebesgue-integrable on $\mathbf{R}_{+}$), $f \in C^{m-2}\left(\boldsymbol{R}_{+}, \boldsymbol{R}\right)$and f^{m-1} exists everywhere in $\mathbf{R}_{+}$. The correction (10) applies when the derivative does not exist. □

If $\mu \neq m-1$ with $m-1 \leq \mu\left(\in \mathbf{R}_{+}\right) \leq m$ then after defining the impulsive sets, its associated indexing sets and the function $\bar{f}: \mathbf{R}_{+} \rightarrow \mathbf{R}$ as for the extended Riemann- Liouville fractional integral, one gets:

$$\begin{aligned}\left(D^{\mu} f\right)(x) &=\frac{1}{\Gamma(m-\mu)}\left[\prod_{j=0}^{m-1}[j-\mu]\right]\left(\int_0^x (x-t)^{-(\mu+1)} f(t) d t\right) \\ &=\frac{1}{\Gamma(m-\mu)}\left[\prod_{j=0}^{m-1}[j-\mu]\right]\end{aligned}$$
$$\times\left(\sum_{i \in I(x) \cup\{0\}} \int_{x_i^{+}}^{x_{i+1}} (x-t)^{-(\mu+1)} f(t) d t+\int_{x_{n(x)}^{+}}^{x} (x-t)^{-(\mu+1)} f(t) d t+\sum_{i \in I(x)} \left(x-x_i\right)^{-(\mu+1)}\left(f\left(x_i^{+}\right)-f\left(x_i\right)\right)\right) \tag{12}$$

$$\left(D^{\mu} f\right)\left(x^{+}\right)=\frac{1}{\Gamma(m-\mu)}\left[\prod_{j=0}^{m-1}[j-\mu]\right]$$
$$\times\left(\int_0^x (x-t)^{-(\mu+1)} \bar{f}(t) d t+\sum_{i \in I(x)} \left(x-x_i\right)^{-(\mu+1)}\left(f\left(x_i^{+}\right)-f\left(x_i\right)\right)\right) \tag{13}$$

4. Extended Caputo fractional derivative

Assume that $f \in C^{m-1}\left(\boldsymbol{R}_{+}, \boldsymbol{R}\right)$ and its $m-th$ derivative exists everywhere in $\boldsymbol{R}_{+}$. Then, the Caputo fractional derivative of order $\mu \geq 0$ with $m-1 \leq \mu\left(\in \boldsymbol{R}_{+}\right)<m$, $m \in \boldsymbol{Z}_{+}$ is for any $x \in \boldsymbol{R}_{+}$:

$$\left(D_*^{\mu} f\right)(x):=\left(J^{m-\mu} f^{(m)}\right)(x)=\frac{1}{\Gamma(m-\mu)} \int_0^x (x-t)^{m-\mu-1} f^{(m)}(t) d t \tag{14}$$

$; \quad m-1 \leq \mu<m, \quad m \in \boldsymbol{Z}_{+},$

$x \in \boldsymbol{R}_{+}$

The following particular cases occur with $\mu = m-1$ leading to

$$\left(D_*^{m-1} f\right)(x) = \int_0^x f^{(m)}(t)dt = f^{(m-1)}(x) - f^{(m-1)}\left(0^+\right) \tag{15}$$

(a) $\mu=-1; m=0$ yields $\left(D_*^{-1} f\right)(x) = f^{(-1)}(x) - f^{(-1)}\left(0^+\right)$ which is an integral result f . Note that this case does not verifies the "derivative constraint" $0 \le \mu(\in \boldsymbol{R}_+) < m$ leading to an integral result.
(b)) $\mu=0; m=1$ yields $\left(D_*^{0} f\right)(x) = f^{(0)}(x) - f^{(0)}\left(0^+\right) = f(x) - f\left(0^+\right)$
(c) $\mu=1; m=2$ yields $\left(D_*^{1} f\right)(x) = f^{(1)}(x) - f^{(1)}\left(0^+\right)$
(d) $\mu=2; m=3$ yields $\left(D_*^{2} f\right)(x) = f^{(2)}(x) - f^{(2)}\left(0^+\right)$

We can extend the above formula to real functions with impulsive $m-th$ derivative as follows. Assume that $f \in C^{m-2}(\mathbf{R}_+, \mathbf{R})$ with bounded piecewise $(m-1)-th$ derivative existing everywhere in $\boldsymbol{R}_+$ and $f^{(m)}(x) \equiv \dfrac{d^m f(x)}{dx^m}$ being impulsive with $f^{(m)}(x_i) = K_i \delta(0) = \left(f^{(m-1)}\left(x_i^+\right) - f^{(m-1)}(x_i)\right)\delta(0);\ \forall x_i \in IMP$, equivalently , $\forall i \in I(\infty)$, at the eventual discontinuity points $x_i > 0$ at the impulsive set $IMP := \bigcup_{x \in \boldsymbol{R}_+} IMP(x)$, where the partial impulsive sets are re-defined as follows:

$$IMP(x) := \left\{x_i \in \mathbf{R}_+ : f^{(m-1)}\left(x_i^+\right) - f^{(m-1)}(x_i) = K_i,\ x_i < x\right\} \subset IMP\left(x^+\right) \tag{16}$$

$$IMP\left(x^+\right) := \left\{x_i \in \mathbf{R}_+ : f^{(m-1)}\left(x_i^+\right) - f^{(m-1)}(x_i) = K_i,\ x_i \le x^+\right\} \subset IMP\left(x^+\right) \tag{17}$$

Now, consider $f \in C^{m-1}(0, \infty)$ with $f^{(m)}(x) \equiv \dfrac{d^m f(x)}{dx^m}$ being almost everywhere piecewise continuous in $\boldsymbol{R}_+$ except possibly on a non-empty discrete impulsive set IMP .Define a non-impulsive real function $\bar{f} : \boldsymbol{R}_+ \to \boldsymbol{R}$ defined as $\bar{f}^{(m)}(x) = f^{(m)}(x)$ for $x \in \boldsymbol{R}_+ \backslash IMP$, and $f^{(m)}(x_i) = \bar{f}^{(m)}(x_i)$, $f^{(m)}(x_i) = \bar{f}^{(m)}(x_i) + K_i \delta(0)$ for $x_i \in IMP$ with $\bar{f}^{(m)}\left(x^+\right) = f^{(m)}(x)$; $x \in IMP$ (defined being bounded arbitrary (for instance, zero) if $x \in IMP$. Through a similar reasoning as that used for Riemann- Liouville fractional integral by replacing the function $f : \boldsymbol{R}_+ \to \boldsymbol{R}$ by its m-th derivative, one obtains the following result:

Theorem 4.1. The Caputo fractional derivative of order $\mu \in \boldsymbol{R}_+$ satisfying $m-1 < \mu \le m$; $m \in \boldsymbol{Z}_+$ and all $x \in \boldsymbol{R}_+$ is given below:

$$\left(D_*^{\mu} f\right)(x) := \frac{1}{\Gamma(m-\mu)} \int_0^x (x-t)^{m-\mu-1} f^{(m)}(t)dt$$

$$=$$

$$\frac{1}{\Gamma(m-\mu)}\left(\int_0^x (x-t)^{m-\mu-1} \bar{f}^{(m)}(t)\,dt + \sum_{i \in I(x)} (x-x_i)^{m-\mu-1}\left(f^{(m-1)}\left(x_i^+\right) - f^{(m-1)}(x_i)\right)\delta(x-x_i) \right)$$

$$= \frac{1}{\Gamma(m-\mu)}$$

$$\times\left(\sum_{i\in I(x)\cup\{0\}}\int_{x_i^+}^{x_{i+1}}(x-t)^{m-\mu-1}\bar{f}^{(m)}(t)dt+\int_{x_{n(x)}^+}^{x}(x-t)^{m-\mu-1}\bar{f}^{(m)}(t)dt+\sum_{i\in I(x)}(x-x_i)^{m-\mu-1}\left(f^{(m-1)}\left(x_i^+\right)-f^{(m-1)}(x_i)\right)\delta(x-x_i)\right)$$

(18)

$$\left(D_*^{\mu} f\right)\left(x^+\right):=\frac{1}{\Gamma(m-\mu)}\int_0^{x^+}(x-t)^{m-\mu-1}f^{(m)}(t)dt$$

$$=\frac{1}{\Gamma(m-\mu)}\left(\int_0^{x}(x-t)^{m-\mu-1}\bar{f}^{(m)}(t)dt+\sum_{i\in I(x^+)}(x-x_i)^{m-\mu-1}\left(f^{(m-1)}\left(x_i^+\right)-f^{(m-1)}(x_i)\right)\delta(x-x_i)\right)$$

$$=\frac{1}{\Gamma(m-\mu)}\left(\sum_{i\in I(x^+)\cup\{0\}}\int_{x_i^+}^{x_{i+1}}(x-t)^{m-\mu-1}\bar{f}^{(m)}(t)dt+\sum_{i\in I(x^+)}(x-x_i)^{m-\mu-1}\left(f^{(m-1)}\left(x_i^+\right)-f^{(m-1)}(x_i)\right)\delta(x-x_i)\right)$$

(19)

where $n: IMP\to \boldsymbol{Z}_+$ is a discrete function defined by $n(x)=card\, I(x)=card\, IMP(x)$. □

Note that if $x\in IMP$ then

$$\left(D_*^{\mu} f\right)\left(x^+\right)=\frac{1}{\Gamma(m-\mu)}$$

$$\times\left(\sum_{i\in I(x^+)\cup\{0\}}\int_{x_i^+}^{x_{i+1}}(x-t)^{m-\mu-1}\bar{f}^{(m)}(t)dt+\sum_{i\in I(x^+)}(x-x_i)^{m-\mu-1}\left(f^{(m-1)}\left(x_i^+\right)-f^{(m-1)}(x_i)\right)\delta(x-x_i)\right)$$

$$=\left(D_*^{\mu} f\right)(x)+\left(x-x_{n(x)}\right)^{m-\mu-1}\left(f^{(m-1)}\left(x_{n(x)}^+\right)-f^{(m-1)}\left(x_{n(x)}\right)\right)\delta(0)$$

$$\neq\left(D_*^{\mu} f\right)(x)=\frac{1}{\Gamma(m-\mu)}$$

$$\times\left(\sum_{i\in I(x)\cup\{0\}}\int_{x_i^+}^{x_{i+1}}(x-t)^{m-\mu-1}\bar{f}^{(m)}(t)dt+\sum_{i\in I(x)}(x-x_i)^{m-\mu-1}\left(f^{(m-1)}\left(x_i^+\right)-f^{(m-1)}(x_i)\right)\delta(x-x_i)\right)$$

and if $x\notin IMP$, since $I(x^+)=I(x)$, then $\left(D_*^{\mu} f\right)\left(x^+\right)=\left(D_*^{\mu} f\right)(x)$. The above formalism applies when $f^{(m-1)}:\boldsymbol{R}_+\to\boldsymbol{R}$ is piecewise continuous with isolated first- class discontinuity points, that is $f\in PC^{m-1}(\boldsymbol{R}_+,\boldsymbol{R})$ implying that $f\in C^{m-2}(\boldsymbol{R}_+,\boldsymbol{R})$. A more general situation arises when the discontinuities can point-wise arise for points of the function itself of for any successive derivative up- till order m. This would lead to a more general description than that given as follows. Define partial sets of positive integers as $\bar{k}:=\{1,2,....,k\}$

Assume that $f\in PC^{j}(\boldsymbol{R}_+,\boldsymbol{R})$ and x is a discontinuity point of first class of $f^{(j)}(x)$ for some $j\in\overline{m-1}\cup\{0\}$. Then , $f^{(j+\ell)}(x)$ are impulsive for $\ell\in\overline{m-j}$ of high order being increasing with ℓ. Define the (j+1) – th impulsive sets of the function *f* on $(0,x)\subset\boldsymbol{R}$ as:

$$IMP_{j+1}(x):=\left\{z\in\boldsymbol{R}_+: z<x,\ 0<\left|f^{(j)}\left(z^+\right)-f^{(j)}(z)\right|<\infty\right\};\ j\in\overline{m-1}\cup\{0\},\ x\in\boldsymbol{R}_+ \quad (20)$$

This leads directly the definition of the following impulsive sets:

$$IMP_{j+1} := \left\{ x \in \boldsymbol{R}_+ :\ 0 < \left| f^{(j)}\left(x^+\right) - f^{(j)}(x) \right| < \infty \right\} \equiv \bigcup_{x \in \boldsymbol{R}_+} IMP_{j+1}(x) \tag{21}$$

$$IMP := \left\{ x \in \boldsymbol{R}_+ :\ 0 < \left| f^{(j)}\left(x^+\right) - f^{(j)}(x) \right| < \infty \text{, some } j \in \overline{m-1} \cup \{0\} \right\} \equiv \bigcup_{x \in \boldsymbol{R}_+} \left(\bigcup_{j \in \overline{m-1} \cup \{0\}} IMP_{j+1}(x) \right) \tag{22}$$

which can be empty . Thus , if $z \in IMP_{j+1}$ then $f^{(j-1)}\left(x^+\right) = f^{(j-1)}(x)$ exists with identical left and right limits, $f^{(j)}\left(x^+\right) - f^{(j)}(x) = K = K(x) \neq 0$ and $f^{(j)}(x) = K\delta(0)$ with successive higher-order derivatives represented by higher- order Dirac distributional derivatives

The above definitions yield directly the following simple results:

Assertion 5.2. $x \in IMP \Rightarrow x \in IMP_j$ for a unique $j = j(x) \in \overline{m}$.

Proof: Proceed by contradiction. Assume that $x \in \left(IMP_{i+1} \cap IMP_{j+1} \right)$ for $i, j (\neq i) \in \overline{m-1} \cup \{0\}$.Then:

$0 < \left| f^{(i)}\left(x^+\right) - f^{(i)}(x) \right| < \infty$; $0 < \left| f^{(j)}\left(x^+\right) - f^{(j)}(x) \right| < \infty$

Assume with no loss of generality that $j = i + k > i$ for some $k (\leq m - i - 1) \in \boldsymbol{Z}_+$. Then,

$$\left| f^{(j)}\left(x^+\right) - f^{(j)}(x) \right| = \left| f^{(i+k)}\left(x^+\right) - f^{(i+k)}(x) \right| = \frac{(-1)^k k!}{x^k} \left| f^{(i)}\left(x^+\right) - f^{(i)}(x) \right| \delta(0) = \infty$$

with $x \in \boldsymbol{R}_+$. If $\left| f^{(i)}\left(x^+\right) - f^{(i)}(x) \right| \neq 0$ which contradicts $0 < \left| f^{(i)}\left(x^+\right) - f^{(i)}(x) \right| < \infty$ so that $i = j$.□

Assertion 5.3. $x \in IMP \Rightarrow \left(x \in IMP_j \Leftrightarrow \exists \text{ a unique } j = j(x) = \max_{i \in \overline{m}} \left| f^{(i-1)}\left(x^+\right) - f^{(i-1)}(x) \right| < \infty \right)$.

Furthermore, such a unique *j=j(x)* satisfies $\left| f^{(j-1)}\left(x^+\right) - f^{(j-1)}(x) \right| > 0$.

Proof: The existence is direct by contradiction. If $\neg \exists j = j(x) \in \overline{m-1} \cup \{0\}$ such that $\left| f^{(j)}\left(x^+\right) - f^{(j)}(x) \right| < \infty$ then $x \notin IMP$. Now, assume there exist two nonnegative integers $i = i(x) = \left| f^{(i-1)}\left(x^+\right) - f^{(i-1)}(x) \right| < \infty$ and $j = j(x) = i + k = \left| f^{(i+k-1)}\left(x^+\right) - f^{(i+k-1)}(x) \right| < \infty$; for some $k \in \overline{m-i}$. But for $x > 0$,

$$\infty = \frac{(-1)^k k!}{x^k} \left| f^{(i-1)}\left(x^+\right) - f^{(i-1)}(x) \right| \delta(0) = \left| f^{(i+k-1)}\left(x^+\right) - f^{(i+k-1)}(x) \right| < \infty$$

which is a contradiction. Then,

$x \in IMP_j \Rightarrow \exists\ j = j(x) = \max_{i \in \overline{m}} \left| f^{(i-1)}\left(x^+\right) - f^{(i-1)}(x) \right| < \infty$ which is unique. Also, from the definition of the impulsive sets $IMP_i(x)$, $\left| f^{(j-1)}\left(x^+\right) - f^{(j-1)}(x) \right| < \infty \Rightarrow x \in \bigcup_{i \in \overline{j} \cup \{0\}} IMP_i(x)$.

Now , assume that $x \in \bigcup_{i \in \overline{j-1} \cup \{0\}} IMP_i(x)$.

Thus, $0<\left|f^{(j-1)}\left(x^{+}\right)-f^{(j-1)}(x)\right|<\infty \Rightarrow\left|f^{(j)}\left(x^{+}\right)-f^{(j)}(x)\right|=\infty$ from the definition of the impulsive sets. Then, $x\in IMP_{j}(x)$. The opposite logic implication $j=j(x)=\max_{i\in\overline{m}}\left|f^{(i-1)}\left(x^{+}\right)-f^{(i-1)}(x)\right|<\infty\Rightarrow x\in IMP_j$ is proved. Then, it has been fully proved that

$$x\in IMP\Rightarrow\left(x\in IMP_j\Leftrightarrow\exists \text{ a unique } j=j(x)=\max_{i\in\overline{m}}\left|f^{(i-1)}\left(x^{+}\right)-f^{(i-1)}(x)\right|<\infty\right).$$

Now, establish again a contradiction by assuming that

$$j=j(x)=\left|f^{(k-1)}\left(x^{+}\right)-f^{(k-1)}(x)\right|=\max\left|f^{(i-1)}\left(x^{+}\right)-f^{(i-1)}(x)\right|=0<\infty;\ \forall k\in\overline{m}$$

what contradicts $x\in IMP$. This proves that the unique *j=j(x)* implying and being implied by $x\in IMP_j$ satisfies $\left|f^{(j-1)}\left(x^{+}\right)-f^{(j-1)}(x)\right|>0$.

□

Using the necessary – high order distributional derivatives, one gets that

$$x\in IMP\Rightarrow f^{(m)}(x)=\frac{(-1)^{m-j}(m-j)!}{x^{m-j}}\left(f^{(j)}\left(x^{+}\right)-f^{(j)}(x)\right)\delta(0);\quad \text{with}\quad j\in\overline{m-1}\cup\{0\}\quad\text{being}$$

uniquely defined so that $0<\left|f^{(j)}\left(x^{+}\right)-f^{(j)}(x)\right|<\infty$. Thus, the m-th distributional derivative of $f:\boldsymbol{R}_{+}\to\boldsymbol{R}$ can be represented as:

$$f^{(m)}(x)=\bar{f}^{(m)}(x)+\sum_{x_i\in IMP_{j_i+1}}\frac{(-1)^{j_i}(m-j_i)!}{x_i^{m-j_i}}\left(f^{(j_i)}\left(x_i^{+}\right)-f^{(j_i)}\left(x_i\right)\right)\delta\left(x-x_i\right),\ x\in\boldsymbol{R}_{+}$$

with $j_i=j_i(x_i)$ being uniquely defied for each $x_i\in IMP$ so that $x_i\in IMP_{j_i}$, where $\bar{f}\in C^{m-1}(\boldsymbol{R}_{+},\boldsymbol{R})$ with everywhere continuous first-derivative defined as $\bar{f}^{(j)}(x)=f^{(j)}(x);\ x\in\boldsymbol{R}_{+},\ \bar{f}(0)=f(0)$. The above formula is applicable if $f\notin PC^{m}(\boldsymbol{R}_{+},\boldsymbol{R})$ but it is also applicable if $f\in PC^{m}(\boldsymbol{R}_{+},\boldsymbol{R})$ yielding:

$$f^{(m)}\left(x^{+}\right)=f^{(m)}(x)=\bar{f}^{(m)}(x)\ \text{ if } x\notin IMP$$

$$f^{(m)}(x)=\bar{f}^{(m)}(x);\ f^{(m)}\left(x^{+}\right)=f^{(m)}(x)+\frac{(-1)^{m-j}(m-j)!}{x^{m-j}}\left(f^{(j)}\left(x^{+}\right)-f^{(j)}(x)\right)\delta(0)\ \text{ if } x\in IMP$$

$$f^{(m-1)}(x)=\bar{f}^{(m-1)}(x);\ f^{(m-1)}\left(x^{+}\right)=f^{(m-1)}(x)+\frac{(-1)^{m-j}(m-1-j)!}{x^{m-1-j}}\left(f^{(j)}\left(x^{+}\right)-f^{(j)}(x)\right)\delta(0)$$

if $x\in IMP$ and $j<m-1$

$$f^{(m-1)}(x)=\bar{f}^{(m-1)}(x);\ f^{(m-1)}\left(x^{+}\right)=f^{(m-1)}(x)+\left(f^{(m-1)}\left(x^{+}\right)-f^{(m-1)}(x)\right)$$

if $x\in IMP$ and $j=m-1$

for a unique $j=j(x)\in\overline{m-1}\cup\{0\}$ from Assertion 1. Denote further sets related to impulses as follows:

$$IMP(x):=\{z\in IMP:z<x\};\ IMP\left(x^{+}\right):=\{z\in IMP:z\le x\};\ \forall x\in\boldsymbol{R}_{+}$$

Being indexed by two subsets of integers of the same corresponding cardinals defined by:

$I(x)=\bar{j}=\overline{j(x)}$ indexing the members z_i of $IMP(x)$ in increasing order
$I(x^+)$, being either $I(x)$ or $I(x)+1$, indexing the members z_i of $IMP(x^+)$ in increasing order

The following result holds:

Theorem 5.4. The Caputo fractional derivative of $f : \mathbf{R}_+ \to \mathbf{R}$ of order $\mu \in \mathbf{R}_+$ satisfying $m-1<\mu\le m\,;m\in\mathbf{Z}_+$ and all $x\in\mathbf{R}_+$ is after using distributional derivatives becomes in the most general case:

$$\left(D_*^{\mu} f\right)(x) := \frac{1}{\Gamma(m-\mu)}\int_0^x (x-t)^{m-\mu-1} f^{(m)}(t)dt$$

$$=\frac{1}{\Gamma(m-\mu)}\left(\int_0^x (x-t)^{m-\mu-1}\bar{f}^{(m)}(t)\,dt\right.$$

$$\left.+\sum_{i\in I(x)}(-1)^{m-j(x_i)-1}(x-x_i)^{m-\mu-1}\frac{(m-j(x_i)-1)!}{(x-x_i)^{m-j(x_i)-1}}\left(f^{(j(x_i))}(x_i^+)-f^{(j(x_i))}(x_i)\right)\hat{\delta}(x-x_i)\right)$$

$$=\frac{1}{\Gamma(m-\mu)}\left(\sum_{i\in I(x)\cup\{0\}}\int_{x_i^+}^{x_{i+1}}(x-t)^{m-\mu-1}\bar{f}^{(m)}(t)dt+\int_{x_{n(x)}^+}^{x}(x-t)^{m-\mu-1}\bar{f}^{(m)}(t)dt\right.$$

$$\left.+\sum_{i\in I(x)}(-1)^{m-j(x_i)-1}(x-x_i)^{m-\mu-1}\frac{(m-j(x_i)-1)!}{(x-x_i)^{m-j(x_i)-1}}\left(f^{(j(x_i))}(x_i^+)-f^{(j(x_i))}(x_i)\right)\right) \quad (23)$$

$$\left(D_*^{\mu} f\right)(x^+) := \frac{1}{\Gamma(m-\mu)}\int_0^{x^+}(x-t)^{m-\mu-1} f^{(m)}(t)dt$$

$$=\frac{1}{\Gamma(m-\mu)}\left(\int_0^x (x-t)^{m-\mu-1}\bar{f}^{(m)}(t)dt\right.$$

$$\left.+\sum_{i\in I(x^+)}(-1)^{m-j(x_i)-1}(x-x_i)^{m-\mu-1}\frac{(m-j(x_i)-1)!}{(x-x_i)^{m-j(x_i)-1}}\left(f^{(j(x_i))}(x_i^+)-f^{(j(x_i))}(x_i)\right)\right)$$

$$=\frac{1}{\Gamma(m-\mu)}\left(\sum_{i\in I(x^+)\cup\{0\}}\int_{x_i^+}^{x_{i+1}}(x-t)^{m-\mu-1}\bar{f}^{(m)}(t)dt\right.$$

$$\left.+\sum_{i\in I(x^+)}(-1)^{m-j(x_i)-1}(x-x_i)^{m-\mu-1}\frac{(m-j(x_i)-1)!}{(x-x_i)^{m-j(x_i)-1}}\left(f^{(j(x_i))}(x_i^+)-f^{(j(x_i))}(x_i)\right)\right) \quad (24)$$

□

Note that $\left|\left(D_*^{\mu} f\right)(x^+)\right|=\infty$ if $x=x_i\in IMP$, as expected.

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