The Homotopy Perturbation Method (HPM) for Nonlinear Parabolic Equation with Nonlocal Boundary Conditions

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Abstract

In this paper, we have presented Homotopy Perturbation Method (HPM) to solve nonlinear parabolic equation with nonlocal boundary conditions. This method provides an analytical solution by utilizing only the initial condition. The HPM allows for the solution of the nonlinear parabolic equations to be calculated in the form of a series with easily computable components. The obtained results are presented and only four terms are required to obtain an approximate solution that is accurate and efficient.

Keywords: Homotopy perturbation method, Parabolic equation, Nonlocal boundary condition

1 Introduction

Many physical problems can be described by mathematical models that involve partial differential equations. Mathematical modeling involves physical observation, selection of the relevant physical variables, formulation of the equations, analysis of the equations, simulation, and finally, validation of the model. The means of mathematical models is an equation, or set of equations, whose solution describe the physical behavior of a related physical system. In
other words, a mathematical model is a simplified description of physical reality expressed in mathematical terms. The behavior of each model is governed by the input data for the particular problem: the boundary and initial conditions, the coefficient functions of the partial differential equation, and the forcing function. This input data cause the solution of the model problem to possess highly localized properties in space, in time, or in both. Thus, the investigation of the exact or approximate solution helps us to understand the means of these mathematical models. In most case, it is difficult, or infeasible, to find the analytical solution or good numerical solution of the problems. Numerical solutions or approximate analytical solutions become necessary. The main goal of this paper is to apply the Homotopy Perturbation Method (HPM) to obtain the approximate solution of the nonlinear parabolic partial differential equations with nonlocal boundary conditions. Parabolic equations in one dimension that involve nonlocal boundary conditions have been studied by several authors [2, 7, 8, 11, 4]. The general form is as follows:

\[
\begin{align*}
  u_t &= F(x, t, u, u_x, u_{xx}) \\
  u(a, t) &= \int_a^b \phi_1(x, t)u(x, t)dx + g_1(t), \\
  u(b, t) &= \int_a^b \phi_2(x, t)u(x, t)dx + g_2(t), \\
  u(x, 0) &= f(x),
\end{align*}
\]

where \( f, \phi_1, \phi_2, g_1 \) and \( g_2 \) are known functions and \( F \) is a continues function. The question of existence and uniqueness of the solutions and theoretical discussion of the one-dimensional parabolic equation with nonlocal boundary conditions have been addressed by many researchers [11, 13, 20]. Finite difference methods have been frequently used to solve such equations [2, 3, 8, 9, 10, 12].

Approximate analytical schemes such as Adomian Decomposition Method (ADM), Variational Iteration Method (VIM), Homotopy Perturbation Method (HPM) and Homotopy Analysis Method (HAM) have been widely used to solve operator equations. These schemes generate an infinite series of solutions and do not have the problem of rounding error. The solution obtained by using these methods shows the applicability, accuracy and efficiency in solving a large class of nonlinear physics, engineering and various branches of mathematics. The approximate analytical methods also unlike Finite Difference Method (FDM) do not provide any linear or nonlinear system of equations.

The Homotopy Perturbation Method (HPM), was first proposed by He in 1998, was developed and improved by He [15, 16, 17]. The HPM is a novel and effective method, and can solve various nonlinear equations of various branches in mathematics and physics. This method has been successfully applied to solve many types of problems. Examples can be found in [6, 14, 18, 19].
The rest of our paper is organized as follows:

In section 2 we present an analysis of the HPM applied to nonlinear parabolic partial differential equations. In section 3 we have used the HPM to solve some nonlinear parabolic equation with nonlocal boundary conditions and compare the solution of them with exact solution. Finally in section 4, the conclusion of this method is provided.

2 Homotopy Perturbation Method (HPM)

To illustrate the basic idea of the homotopy technique [15] on the nonlinear parabolic equation (1), we can construct a homotopy \( v(r, p) : \Omega \times [0, 1] \rightarrow \mathbb{R} \) which satisfies

\[
H(v, p) = (1 - p)(\frac{\partial v}{\partial t} - \frac{\partial u_0}{\partial t}) + p \left[ \frac{\partial v}{\partial t} - F \left( x, t, v, \frac{\partial v}{\partial x}, \frac{\partial^2 v}{\partial x^2} \right) \right] = 0 \quad p \in [0, 1]
\]

or

\[
H(v, p) = \frac{\partial v}{\partial t} - \frac{\partial u_0}{\partial t} + p \frac{\partial u_0}{\partial t} - pF \left( x, t, v, \frac{\partial v}{\partial x}, \frac{\partial^2 v}{\partial x^2} \right) = 0 \quad p \in [0, 1]
\]

where \( p \in [0, 1] \) is an embedding parameter, and \( u_0 \) is the initial approximation of Equation (1) which satisfies the boundary conditions. Obviously, from Eqs. (2) and (3) we will have

\[
H(v, 0) = \frac{\partial v}{\partial p} - \frac{\partial u_0}{\partial t} = 0 \quad (4)
\]

\[
H(v, 1) = \frac{\partial v}{\partial t} - F(x, t, v, v_x, v_{xx}) = 0 \quad (5)
\]

According to the (HPM), we can first use the embedding parameter \( p \) as a small parameter, and assume that the solutions of Eqs. (2) and (3) can be written as a power series in \( p \):

\[
v = \sum_{i=0}^{\infty} v_ip^i
\]

by substituting (2.6) into (2.3) we have

\[
\sum_{i=0}^{\infty} p^i \frac{\partial v_i}{\partial t} - \frac{\partial u_0}{\partial t} = p \left( -\frac{\partial u_0}{\partial t} + F \left( x, t, \sum_{i=0}^{\infty} v_ip^i, \sum_{i=0}^{\infty} p^i \frac{\partial v_i}{\partial x}, \sum_{i=0}^{\infty} p^i \frac{\partial^2 v_i}{\partial x^2} \right) \right)
\]

setting \( p = 1 \), results in the approximate solution of Eq. (1.1)

\[
u = \lim_{p \to 1} v = v_0 + v_1 + v_2 + ...
\]
According to [1], a combination of the perturbation method and the homotopy method is called the homotopy perturbation method (HPM), which has eliminated the limitations of traditional perturbation methods. On the other hand, the HPM has eliminated limitations of traditional perturbation methods whilst retaining the full advantages. The convergence of the series in (3.6) has been studied and discussed in [1, 15].

3 Illustrative examples

In this section, the HPM will be demonstrated on two examples of nonlinear parabolic equation with nonlocal boundary conditions. For our numerical computation, let the expression

$$\psi_m(x, t) = \sum_{k=0}^{m-1} u_k(x, t)$$

(9)

denote the $m$-term approximation to $u(x, t)$. We compare the approximate analytical solution obtained for our nonlinear parabolic equations with known exact solution.

We define $E_m(x, t)$ to be the absolute error between the exact solution and $m$-term approximate solution $\psi_m(x, t)$ as follows

$$E_m(x, t) = |u(x, t) - \psi_m(x, t)|$$

(10)

Example 1:

We first study the following nonlinear parabolic equation in a bounded domain

$$u_t = u_{xx}^2 - 4 \tanh^2(2t - x)u_x^2 - 2u_x$$

$10 < x < 20 \quad t > 0$

(11)

with initial conditions

$$u(x, 0) = \tanh x$$

and nonlocal boundary conditions

$$u(10, t) = \int_{10}^{20} \coth(x - 2t)u(x, t)dx + (\tanh(10 - 2t) - 10), \quad t > 0$$

(12)

$$u(20, t) = \int_{10}^{20} u(x, t)dx + \left(\tanh(20 - 2t) - \ln \frac{\cosh(20 - 2t)}{\cosh(10 - 2t)}\right), \quad t > 0$$

(13)

It is easy to check that the exact solution is

$$u(x, t) = \tanh(x - 2t)$$
To solve Eq. (11) with initial condition \( u(x, 0) = \tanh x \), according to the homotopy perturbation technique, we construct the following homotopy:

\[
\frac{\partial v}{\partial t} - \frac{\partial u_0}{\partial t} = p \left( \frac{\partial u_0}{\partial t} + \left( \frac{\partial^2 v}{\partial x^2} \right)^2 - 4 \tanh^2(2t - x) \left( \frac{\partial v}{\partial x} \right)^2 - 2 \frac{\partial v}{\partial x} \right) \tag{14}
\]

Suppose the solution of Eq. (14) has the form

\[
v = \sum_{i=0}^{\infty} v_i p^i \tag{15}
\]

Substituting (15) into (14), and comparing coefficients of terms with identical powers of \( p \), leads to:

\[
\begin{align*}
(v_0)_t - (u_0)_t &= 0, \\
(v_1)_t &= -(u_0)_t + (v_0)^2_{xx} - 4 \tanh^2(2t - x)(v_0)^2_x - 2(v_0)_x, \\
(v_2)_t &= 2(v_0)_{xx}(v_1)_{xx} - 8 \tanh^2(2t - x)(v_0)_x(v_1)_x - 2(v_1)_x, \\
(v_3)_t &= [(v_1)^2_{xx} + 2(v_0)_{xx}(v_2)_{xx}] - 4 \tanh^2(2t - x)[(v_1)^2_x + 2(v_0)_x(v_2)_x] - 2(v_2)_x, \\
&\vdots
\end{align*}
\]

with condition

\[
v_i(x, 0) = 0, \quad i = 1, 2, \ldots
\]

we can easily obtain the components of series (8) as

\[
\begin{align*}
(\nu_0)_t &= \tanh x, \\
(\nu_1)_t &= -2t \sec h^2 x - 2(2t - x) \sec h^4 x + 2 \sec h^4 x \tanh(2t - x) + 4t \sec h^4 x \tanh^2 x, \\
(\nu_2)_t &= -4t \sec h^4 x - 8(2t - x) \sec h^6 x + 2 \sec h^4 x \tanh(2t - x) + 8t \sec h^6 x \times \\
&\quad \tanh^2(2t - x) + \frac{8}{3} \sec h^6 x \tanh^2(2t - x) - 16t^2 \sec h^4 x \tanh x + \\
&\quad - 4t^2 \sec h^2 x \tanh x + \cdots \\
&\vdots
\end{align*}
\]

Thus, we have the solution given by (8)

\[
u = \sum_{k=0}^{\infty} \nu_k = \nu_0 + \nu_1 + \nu_2 + \cdots \tag{16}
\]

In table 1, we have shown the absolute error between solution obtained using HPM with four terms and the exact solution for various \( x \in (10, 20) \) and \( t \in (0, 1) \). It is to be note that four terms only were used in evaluating
the approximate solutions. We achieved a very good approximation with the actual solution of the equations by using four terms only of the decomposition derived above. Furthermore, as the HPM does not require discretization of the variables, i.e. time and space, it is not effected by computation round off errors and one is not faced with necessity of large computer memory and time. It is evident that the overall errors can be made smaller by adding new terms of the HPM series (15).

Table 1: Absolute error \( E_4 = |u(x, t) - \psi_4(x, t)| \) for various values of \( x \) and \( t \).

<table>
<thead>
<tr>
<th>t/x</th>
<th>12</th>
<th>14</th>
<th>16</th>
<th>18</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>( 8.74856 \times 10^{-14} )</td>
<td>( 1.66533 \times 10^{-15} )</td>
<td>0.</td>
<td>0.</td>
</tr>
<tr>
<td>0.3</td>
<td>( 8.46534 \times 10^{-12} )</td>
<td>( 1.54987 \times 10^{-13} )</td>
<td>( 2.77556 \times 10^{-15} )</td>
<td>( 1.11022 \times 10^{-16} )</td>
</tr>
<tr>
<td>0.5</td>
<td>( 7.97100 \times 10^{-11} )</td>
<td>( 1.45983 \times 10^{-12} )</td>
<td>( 2.68674 \times 10^{-14} )</td>
<td>( 5.55112 \times 10^{-16} )</td>
</tr>
<tr>
<td>0.7</td>
<td>( 3.82495 \times 10^{-10} )</td>
<td>( 7.00562 \times 10^{-12} )</td>
<td>( 1.28342 \times 10^{-13} )</td>
<td>( 2.33147 \times 10^{-15} )</td>
</tr>
<tr>
<td>0.9</td>
<td>( 1.33959 \times 10^{-9} )</td>
<td>( 2.45354 \times 10^{-11} )</td>
<td>( 4.49418 \times 10^{-13} )</td>
<td>( 8.21564 \times 10^{-15} )</td>
</tr>
</tbody>
</table>

The absolute error between solution obtained using HPM with four terms and the exact solution for various \( x \) and \( t \) have been displayed in figures 1 and 2.

![Figure 1: The absolute error \( E_4 \) for various \( t \in (0, 1) \) at \( x = 12 \) and \( x = 18 \).](image)

Example 2: For the second example, We consider the following nonlinear parabolic equation

\[
\begin{align*}
  u_t &= u_{xx}^3 + 8 \coth(x + \frac{t}{2}) \csc h^4(\frac{t}{2}) u^2 u_x - \frac{1}{2} u_x,
  &\quad 10 < x < 20, \quad t > 0
\end{align*}
\]
Figure 2: The absolute error $E_4$ for various $x \in (10, 20)$ at $t = 0.1$ and $t = 0.9$.

with initial condition $$u(x, 0) = \coth x$$ and nonlocal boundary conditions as follows

$$u(10, t) = \int_{10}^{20} \tanh(x + \frac{t}{2})u(x, t)dx + \coth(10 + \frac{t}{2}) - 10,$$

$$u(20, t) = \int_{10}^{20} \tanh^2(x + \frac{t}{2})u(x, t)dx + \left(\ln \frac{\cosh(20 + t/2)}{\cosh(10 + t/2)} - \coth(20 + t/2)\right).$$

It can be verified that the exact solution is

$$u(x, t) = \coth(x + \frac{t}{2})$$

Using the HPM, we can obtain

$$v_t - (u_0)_t = p \left( v_{xx}^3 + 8 \coth(x + \frac{t}{2}) \csc h^4(\frac{t}{2}) v^2 v_x \frac{1}{2} v_x - (u_0)_t \right)$$

(20)

Suppose that the solution of (17) can be represented as (6) thus Substituting (6) into (20) and comparing coefficients of terms with identical powers of $p$, leads to

$$(v_0)_t - (u_0)_t = 0,$$

$$(v_1)_t = \left( (v_0)_{xx}^3 + 8 \coth(x + \frac{t}{2}) \csc h^4(\frac{t}{2}) v^2_0(v_0)_x - \frac{1}{2}(v_0)_x - (u_0)_t \right),$$

$$(v_2)_t = \left( 3(v_0)_{xx}^2(v_1)_{xx} + 8 \coth(x + \frac{t}{2}) \csc h^4(\frac{t}{2}) [v_0^2(v_1)_x + 2v_0v_1(v_0)_x] - \frac{1}{2}(v_1)_x \right),$$
with condition of
\[ v_i(x, 0) = 0, \quad i = 1, 2, \ldots \]
We can easily obtain the components of series (8) as
\[
\begin{align*}
v_0 &= \coth x, \\
v_1 &= \frac{1}{3} \csc h^2 x + \left[1.5t - 16(-4 + \cosh 2x) \coth \frac{t}{2} \coth^3 x \csc h^2 x - 24 \coth^2 x \times \right. \\
&\quad \left. \csc h^2 x(\coth^2 \frac{t}{2} + 2 \csc h^2 x(- \log(\sin \frac{t}{2}) + \log(\sin(\frac{t}{2} + x))) + 8 \coth^3 x \times (3t \csc h^4 x + \csc h^4(\frac{t}{2}) \sinh t)) \right],
\end{align*}
\]
Thus, we have the solution given by (8)
\[
\begin{equation}
\begin{aligned}
u(x,t) &= \sum_{k=0}^{\infty} v_k = v_0 + v_1 + v_2 + \cdots.
\end{aligned}
\end{equation}
\]
The absolute error between solution obtained using HPM with four terms and the exact solution for various \( x \in (10, 20) \) and \( t \in (0, 1) \) have been shown in table 2. The errors are very small in these tables. The results provides very strong evidence that is the homotopy perturbation technique not only easy to obtain explicit solution but also easy to get approximate solution of the nonlinear equation. It is to be note that four terms only were used in evaluating the approximate solutions. It is evident that the overall errors can be made smaller by adding new terms of the HPM series (6). Moreover, the HPM do not require discretization of the variables time and space, this method is not affected by computation round-off errors and one is not faced with the necessity of large computer memory and time.

| Table 2: Absolute error \( E_4 = |u(x, t) - \psi_4(x, t)| \) for various values of \( x \) and \( t \). |
|---|---|---|---|---|
| \( t/x \) | 12 | 14 | 16 | 18 |
| 0.1 | \( 8.00545 \times 10^{-10} \) | \( 1.46626 \times 10^{-11} \) | \( 2.68452 \times 10^{-13} \) | \( 4.77396 \times 10^{-15} \) |
| 0.3 | \( 4.38667 \times 10^{-10} \) | \( 8.03435 \times 10^{-12} \) | \( 1.46994 \times 10^{-13} \) | \( 2.66454 \times 10^{-15} \) |
| 0.7 | \( 7.84702 \times 10^{-11} \) | \( 1.43729 \times 10^{-12} \) | \( 2.63123 \times 10^{-14} \) | \( 5.55112 \times 10^{-16} \) |
| 0.9 | \( 2.79035 \times 10^{-10} \) | \( 5.11080 \times 10^{-12} \) | \( 9.37028 \times 10^{-14} \) | \( 1.77636 \times 10^{-15} \) |
| 1.0 | \( 6.32910 \times 10^{-10} \) | \( 1.15921 \times 10^{-11} \) | \( 2.12275 \times 10^{-13} \) | \( 3.99680 \times 10^{-15} \) |

In figure 3 and 4, we have compared the solution obtained using HPM with four terms and the exact solution.
Figure 3: The absolute error $E_4$ for various $t \in (0, 1)$ at $x = 12$ and $x = 18$.

Figure 4: The absolute error $E_4$ for various $x \in (10, 20)$ at $t = 0.1$ and $t = 0.9$. 
4 Conclusion

In this study, we have successfully developed HPM for solving nonlinear parabolic equations with nonlocal boundary conditions. It is clearly seen that HPM is a very powerful and efficient technique for finding solutions for wide classes of nonlinear partial differential equations in the form of analytical expressions and presents a rapid convergence for the solutions. One of the importance advantage of the HPM is that it solves the nonlinear equations without any need for discretization, perturbation, transformation or linearization. The results of the numerical examples is presented and only with four terms are required to obtain accurate solution. The method was tested on two examples of the nonlinear parabolic equation with nonlocal boundary conditions and it was demonstrated that the HPM is highly accurate and rapidly convergent.

References


Received: May, 2010