An Economic Reliability Test Plan for
Marshall-Olkin Extended Exponential Distribution

G. Srinivasa Rao

Department of Basic Sciences
Hamelmalo Agricultural College, Keren, Eritrea

M. E. Ghitany

Department of Statistics and Operations Research
Faculty of Science, Kuwait University, Kuwait
meghitany@yahoo.com

R. R. L. Kantam

Department of Statistics
Acharya Nagarjuna University, Guntur-522510, India

Abstract

The Marshall-Olkin extended exponential distribution is considered as a probability model for the lifetime of the product. Sampling plans in which items that are put to test, to collect the life of the items in order to decide upon accepting or rejecting a submitted lot, are called reliability test plans. A test plan to determine the termination time of the experiment for a given sample size, producer’s risk and termination number is constructed. The preferability of the present test plan over similar plans exists in the literature is established with respect to time of the experiment. Results are illustrated by an example.

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1 Introduction

The variable sampling plans are developed by proposing a decision rule to accept or reject a submitted lot of products on the basis of inspected measurable quality characteristic for a sample products taken from the lot. As required by the principles of statistical inference, it is necessary to specify the probability distribution of variable characteristic. In the absence of such specification, it is taken as the well known normal distribution. However, if normal distribution is not a good fit to the data under consideration, the decision process constructed on this basis would be misleading. At the same time appeal to central limit theorem as a justification to normality assumption is not always valid as the sample size in quality control data is not large enough to adopt normality. In this backdrop, Sobel and Tischendrof (1959) developed reliability test plans for exponential distribution. Goode and Kao (1961) constructed sampling plans for Weibull distribution. Gupta and Groll (1961) constructed sampling plans for Gamma distribution. Sampling plans similar to those of Gupta and Groll (1961) are developed by Kantam and Rosaiah (1998) for half-logistic distribution and Kantam et al. (2001) for Log-logistic distribution, Rosaiah and Kantam (2005) for the inverse Raleigh distribution and Rosaiah et al. (2006) for exponentiated log-logistic distribution. Sampling plans in a new approach for log-logistic distribution are suggested by Kantam et al. (2006). Our interest in this paper is working of a variable sampling plan parallel to the construction of a theoretical parametric test of hypothesis. Of these, the present paper deals with the construction of sampling plan with a new approach and its comparison with similar existing plans are given in Section 2. The operating characteristic is presented in Section 3. The results are illustrated by an example towards the end of Section 3.

In scaled densities, a null hypothesis about scale parameter such as "the scale parameter is greater than or equal to a specified value" is equivalent to saying that the 'average life of a product governed by the given scaled density exceeds a specified average life'. Acceptance of this hypothesis by a test procedure means that the sample life times used for testing indicate that the lot from which the sample is drawn is a good lot. Similarly rejection of the hypothesis implies that the lot is a bad lot. In this paper, we discussed the parallel between the testing of hypothesis in scaled densities and sampling plans.
2 The Sampling Plan

We assume that the lifetime of a product follows Marshall-Olkin extended exponential distribution. The cumulative distribution function and probability density function of the Marshall-Olkin extended exponential distribution are given by [Marshall and Olkin (1997)],

\[ G(t; \alpha, \sigma) = \frac{1 - e^{-t/\sigma}}{1 - \bar{\alpha} e^{-t/\sigma}}, \quad t > 0, \quad \alpha, \sigma > 0, \quad \bar{\alpha} = 1 - \alpha, \quad (2.1) \]

and

\[ g(t; \alpha, \sigma) = \frac{\alpha e^{-t/\sigma}}{(1 - \bar{\alpha} e^{-t/\sigma})^2}, \quad t > 0, \quad \alpha, \sigma > 0, \quad \bar{\alpha} = 1 - \alpha, \quad (2.2) \]

where \( \sigma \) is the scale parameter and \( \alpha \) is the shape parameter.

Marshall-Olkin extended exponential distribution can be considered as a model for lifetimes, if the lifetimes show a large variability and is shown to be an decreasing or increasing failure rate model according as \( 0 < \alpha \leq 1 \) or \( \alpha \geq 1 \).

Consider a null hypothesis "\( H_0 : \sigma > \sigma_0 \)". If Marshall-Olkin extended exponential distribution is assumed as the model of a variable representing lifetimes of some items that have life and eventual failure, the above hypothesis is regarding the average life of those items in the population. If the \( H_0 \) is accepted on the basis of some sample lifetimes collected through a life testing experiment from out of a submitted lot of such items using any admissible statistical test procedure, we may conclude that the submitted lot has a better average life than what is specified accordingly the lot that can be termed as a good lot and can be accepted. Srinivasa Rao et al. (2009) constructed the minimum sample size required to make a decision about the lot given the waiting time in terms of \( \sigma_0 \) (i.e., \( t/\sigma_0 \)) and acceptance number \( c \), some risk probability, say \( \alpha^* \). With a specified \( \sigma_0 \) of \( \sigma \), the probability of detecting \( c \) or less failures (probability of accepting the lot) in a sample of size \( n \) is given by

\[ \sum_{i=0}^{c} \binom{n}{i} p^i (1-p)^{n-i}, \quad (2.3) \]

where \( p = G(t; \alpha, \sigma_0) \).

For \( \sigma > \sigma_0 \), the above probability of acceptance should increase. Therefore, if \( \alpha^* \) is a prefixed risk probability this means

\[ \sum_{i=0}^{c} \binom{n}{i} p^i (1-p)^{n-i} \geq 1 - \alpha^*. \quad (2.4) \]
For a given $\sigma_0$ and hence of $t/\sigma_0$, this is a single inequality in two unknowns $n$ and $c$ assuming that the parameter $\alpha$ is known. Because $c$ is always less than $n$, inequality (2.4) can be solved for $n$ with successive values of $c$ from zero onwards. The earliest value of $n$ that satisfies the inequality (2.4) are given for $\alpha = 2$, $1 - \alpha^* = 0.75, 0.90, 0.95, 0.99$ and $t/\sigma_0 = 0.241, 0.361, 0.482, 0.602, 0.903, 1.204, 0.505, 1.806$ by Srinivasa Rao et al. (2009) along with the associated performance characteristics like operating characteristics, producer’s risk, scope for variability of $\sigma$ etc. A typical portion of tables of Srinivasa Rao et al. (2009) for Marshall-Olkin extended exponential distribution are reproduced in the Table 1 for $\alpha = 2$.

In the present investigation, inequality (2.4) can be considered in a different way. Let us fix $n$ and let $r$ be a natural number less than $n$, so that as soon as the $r^{th}$ ($r = c + 1$) failure is observed, the process is stopped and the lot is rejected. Given $\sigma = \sigma_0$, the probability of such a rejection should be as small as possible. That is

$$\sum_{i=r}^{n} \binom{n}{i} p^i (1-p)^{n-i} < \alpha^*. \quad (2.5)$$

Specifying $n$ as a multiple of $r$, say $kr$ ($k = 1, 2, \ldots$), inequality (2.5) can be regarded as an inequality in a single unknown in terms of $t/\sigma$ with known $\alpha$. With the choice of $r, k, \alpha^*$, inequality (2.5) can be solved for the earliest $p$, say $p_0$, from which the value of $t/\sigma_0$ can be obtained by inverting the $G(t; \alpha, \sigma)$ given by (2.1). The specified population average in terms of $\sigma_0$ can be used here to get the value of $t$ called the termination time. These are presented in Table 2 for various values of $n, r = 1(1)10, \alpha = 2$ at $\alpha^* = 0.05, 0.01$.

Table 1. Minimum sample size necessary to assert that the average life $\sigma_0$, with probability $1 - \alpha^*$ and the corresponding acceptance number $c$, using binomial probabilities for $\alpha = 2$. 

<table>
<thead>
<tr>
<th>$n$</th>
<th>$c$</th>
<th>$t/\sigma_0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>1</td>
<td>0.241</td>
</tr>
<tr>
<td>5</td>
<td>2</td>
<td>0.361</td>
</tr>
<tr>
<td>5</td>
<td>3</td>
<td>0.482</td>
</tr>
<tr>
<td>5</td>
<td>4</td>
<td>0.602</td>
</tr>
<tr>
<td>5</td>
<td>5</td>
<td>0.903</td>
</tr>
<tr>
<td>5</td>
<td>6</td>
<td>1.204</td>
</tr>
<tr>
<td>5</td>
<td>7</td>
<td>0.505</td>
</tr>
<tr>
<td>5</td>
<td>8</td>
<td>1.806</td>
</tr>
</tbody>
</table>
Table 2. Life test termination in units of scale parameter \((t/\sigma_0)\) for Marshall-Olkin extended exponential distribution with \(\alpha = 2\).
Comparative Study

In order to compare the present sampling plan with that of Srinivasa Rao et al. (2009), the entries common for both the approaches are presented for \( \alpha = 2; \alpha^* = 0.05, 0.01 \) in Table 3. The entries given in the first row are corresponding to present test plan and those given in the second row are obtained by Srinivasa Rao et al. (2009). All the entries in Table 3 show that for a given \( n, r(r = c + 1) \), the values of \( t/\sigma_0 \)-the scaled termination time is uniformly smaller for the present reliability test plans than those of Srinivasa Rao et al. (2009), resulting in savings in experimental time.

Table 3. Proportions of life test termination time for sampling plans of Srinivasa Rao et al. (2009) and the present sampling plans with producer’s risk \( \alpha^* = 0.05, 0.01 \).
Table 4. Operating characteristic (O.C.) values of sampling plans \((n, r, t/\sigma_0)\) for \(\alpha = 2\).
3 Operating Characteristic Function

If the true but unknown life of the product deviates from the specified life of the product it should result in a considerable change in the probability of acceptance of the lot based on the sampling plan. Hence the probability of acceptance can be regarded as a function of the deviation of specified average from the true average. This function is called operating characteristic function of the sampling plan. Specifically if \( G(t) \) is the cumulative distribution function of the life time random variable of the product, \( \sigma_0 \) corresponds to specified life, we can write

\[
G(t/\sigma) = G[(t/\sigma_0).(\sigma_0/\sigma)],
\]

where \( \sigma \) corresponds to true but unknown average life. The ratio \( \sigma_0/\sigma \) in the R.H.S. of equation (3.1) can be taken as a measure of changes between true and specified lives.

For instance \((\sigma_0/\sigma) < 1 \) implies that the true mean life is more than the declared life leading to more acceptance probability or less failure risk. Similarly, \((\sigma_0/\sigma) > 1 \) implies less acceptance probability or more failure risk. Hence giving a set of hypothetical values, say \( \sigma_0/\sigma = 0.1(0.1)1.9 \), we can have the corresponding acceptance probability for the given sampling plan. Here we have selected some plans and the operating characteristic (O.C.) values of these plans are given in Table 4. Note that the O.C. values for sampling plans: \((n, r, t/\sigma_0) : (6, 2, 0.1260), (10, 2, 0.0736), \) for \( \alpha = 2 \) and \( 1 - \alpha^* = 0.95 \), are almost identical. Also, for sampling plans: \((n, r, t/\sigma_0) : (6, 2, 0.0536), (10, 2, 0.0312), \) for \( \alpha = 2 \) and \( 1 - \alpha^* = 0.99 \), are almost identical.

Example:
Consider the following ordered failure times of the release of a software given in terms of hours from the starting of the execution of the software denoting the times at which the failure of the software is experienced (Wood, 1996). This data can be regarded as an ordered sample of size \( n = 16 \) with observations:

\[
\{x_i : i = 1, 2, \ldots 16\} = \{519, 968, 1430, 1893, 2490, 3058, 3625, 4422, 5218, 5823, 6539, 7083, 7485, 7846, 8205, 8564\}.
\]

The confidence level of the decision processes assured by the sampling plan only if the lifetimes follows Marshall-Olkin extended exponential distribution. We have verified this for the above sample data by \( Q - Q \) plot with \( \alpha = 2 \).

Case I:
Let the required average lifetime be 1000 hours and the testing time be \( t = 903 \) hours, this leads to ratio of \( t/\sigma_0 = 0.903 \) with a corresponding sample size \( n = 16 \) and an acceptance number \( c = 3 \), which are obtained from Table 1 for \( 1 - \alpha^* = 0.95 \). Therefore, the sampling plan for the above sample data is \( (n = 16, c = 3, t/\sigma_0 = 0.903) \). Based on the observations, we have to decide whether to accept the product or reject it. We accept the product only if the number of failures before 903 hours is less than or equal to 3. In the above sample of 16 only one failure occurred at 519 hours before \( t = 903 \) hours. Therefore we accept the product using the sampling plan constructed by Srinivasa Rao et al. (2009).

**Case II:**

From Table 2, the entry against \( r = 4 \) \( (r = c + 1) \) under the column \( 4r \) is 0.18109. Since the acceptable mean life is given to be 1000 hours for Marshall-Olkin extended exponential distribution. If the termination time is given by \( t_0 \) the table value says that \( t_0 = 0.18109 \) that is \( t_0 = 0.18109 \times 1000 = 181.09 = 181 \) hours (approx.).

Using the present sampling plan, this test plan will be implemented as follows: Select 16 items from the submitted lot and put them to test. If the 3\(^{rd} \) failure is realized before 181\(^{th} \) hour of the test, reject the lot otherwise accept the lot in either case terminating the experiment as soon as the 3\(^{rd} \) failure is reached or 181\(^{th} \) hour of the test time is reached whichever is earlier. In the case of acceptance, the assurance is that the average life of the submitted products is at least 1000 hours.

In this approach, we see that in the sample of 16 failures there is no failure before 181\(^{th} \) hour, therefore we accept the product.

In both of these approaches the sample size, acceptance number (termination number), the risk probability and the final decision about the lot are the same. But the decision on the first approach can be reached at the 519\(^{th} \) hour and that in the second approach reached at the 181\(^{th} \) hour, thus second approach (the present sampling plan) requiring a less waiting time and also minimum experimental cost. Hence, the present sampling plan be preferred.

**References**


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