On Stratification and Domination in Prisms

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Abstract

A graph $G$ is 2-stratified if its vertex set partitioned into two color classes. We color the vertices in one color class red and the other class blue. Let $F$ be a 2-stratified graph with one fixed blue vertex $v$ specified. We say that $F$ is rooted at $v$. The $F$-domination number of a graph $G$ is the minimum number of red vertices of $G$ in a red-blue coloring of the vertices of $G$ such that every blue vertex $v$ of $G$ belongs to a copy of $F$ rooted at $v$. In this paper we investigate the $F$-domination number of prisms when $F$ is 2-stratified 6-cycle rooted at a blue vertex. And we get a new generalization result of stratified domination number for prisms.

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1 Introduction

The study combining stratification and domination in graphs was started by Chartrand et al. [3]. A graph $G = (V,E)$ together with a fixed partition of its vertex set $V$ into nonempty subsets is called a stratified graph. If the partition is $V = \{V_1, V_2\}$ then $G$ is a 2-stratified graph and the sets $V_1$ and $V_2$ are called the strata or sometimes the color classes of $G$. We ordinarily color the vertices of $V_1$ red and the vertices of $V_2$ blue. In [13], Rashidi studied a number of problems involving stratified graphs; while distance in stratified graphs was investigated in [4,5,6]. The concept of domination in graphs, with its many variations, has been well studied in graph theory. The literature on this subject has been surveyed and detailed in the two books by Haynes, Hedetniemi, and Slater [17,18]. In [3] a new mathematical framework for studying domination is presented. Let $F$ be a 2-stratified graph with one fixed $F$ blue vertex $v$
specified. We say that \( F \) is rooted at the blue vertex \( v \). An \( F \)-\textit{colouring} of a graph \( G \) is defined in [3] to be a red-blue coloring of the vertices of \( G \) such that every blue vertex \( v \) of \( G \) belongs to a copy of \( F \) (not necessarily induced in \( G \)) rooted at \( v \). The \( F \)-\textit{domination number} \( \gamma_F(G) \) of \( G \) is the minimum number of red vertices of \( G \) in an \( F \)-\textit{colouring} of \( G \). In [3], an \( F \)-\textit{colouring} of \( G \) that colors \( \gamma_F(G) \) vertices red is called a \( F \)-\textit{colouring} of \( G \). The set of red vertices in a \( F \)-\textit{colouring} is called a \( F \)-set. If \( G \) has order \( n \) and \( G \) has no copy of \( F \), then certainly \( \gamma_F(G) = n \). Let \( F \) be a \( K_2 \) rooted at a blue vertex \( v \) that is adjacent to a red vertex. An \( F \)-\textit{colouring} of \( G \) is then a red-blue coloring of the vertices of \( G \) with the property that every blue vertex is adjacent to a red vertex. Thus the red vertices of \( G \) correspond to a dominating set of \( G \). Hence, \( \gamma(G) \leq \gamma_F(G) \). On the other hand, given a \( \gamma(G) \)-set of \( G \) we color the vertices in this set red and all remaining vertices blue. This red-blue coloring of the vertices of \( G \) has the property that every blue vertex is adjacent to a red vertex and is therefore an \( F \)-\textit{colouring} of \( G \) (where \( F \) is a 2-stratified \( K_2 \)). Thus we have the following observation in [3].

\textbf{Observation 1 ([3])}. If \( F \) is a 2-stratified \( K_2 \) rooted at a blue vertex that is adjacent to a red vertex, then \( \gamma_F(G) = \gamma(G) \).

Observation 1 shows that domination can be interpreted as restricted 2-stratifications or 2-colorings, with the red vertices forming the dominating set. This framework encapsulates many types of domination related parameters, including the domination, total domination, restrained, total restrained, and \( k \)-domination numbers. The framework places the domination number in a new perspective and suggests many other parameters of a graph which are related in some way to the domination number. A detailed discussion of other mathematical frameworks for the domination number of a graph can be found in Chapter 11 in [17]. See [3-16] for related studies about stratification and domination. For notation and graph theory terminology we follow in general [2]. In this paper we investigate the \( F \)-domination number of a prism when \( F \) is a 2-stratified cycle \( C_6 \).

\section*{2 Stratification and domination in prisms}

A prism is the cartesian product \( G = C_n \times K_2, n \geq 3 \), of a cycle \( C_n \) and a \( K_2 \). Throughout this paper, our prism \( G \) consist of two \( n \)-cycles \( v_1, v_2, \ldots, v_n, v_1 \) and \( u_1, u_2, \ldots, u_n, u_1 \) with \( u_i v_i \) an edge for all \( i = 1, 2, \ldots, n \). By a claw mean the graph \( K_{1,3} \). “The claw domination” for prisms has been studied in [1]. There are eight possible choices for a 2-stratified claw rooted at a blue vertex \( v \). These graphs are shown in
On stratification and domination in prisms

Theorem 2. ([11]). For \( n \geq 3 \), let \( G \) be the prism \( C_n \times K_2 \). Then
(a) \( \gamma_{X_1}(G) = 2\left\lceil \frac{n}{4} \right\rceil \).
(b) \( \gamma_{X_2}(G) = 2\left\lfloor \frac{n}{3} \right\rfloor \).
(c) \( \gamma_{X_3}(G) = n \).
(d) \( \gamma_{X_4}(G) = 2\left\lfloor \frac{n}{5} \right\rfloor \) if \( n \equiv 0, 3, 4 \pmod{5} \) or \( n \equiv 2, 6 \pmod{10} \),
\( \gamma_{X_4}(G) = 2\left\lfloor \frac{n}{5} \right\rfloor - 1 \) for \( n \equiv 1, 7 \pmod{10} \).
(e) \( \gamma_{X_5} = 2\left\lfloor \frac{n}{2} \right\rfloor \).
\( \gamma_{X_5} = 2 \) if \( n = 3 \) or \( n \equiv 2, 6 \pmod{10} \),
\( \gamma_{X_5} = 2\left\lfloor \frac{n}{4} \right\rfloor + i \) if \( n \geq 4 \) and \( n \equiv i \pmod{4} \).
(g) \( \gamma_{X_6}(G) = 2\left\lceil \frac{n}{2} \right\rceil \).

Let \( X \) be a 2-stratified \( C_4 \) rooted at a blue vertex \( v \). The five possible choices for the graph \( X \) are shown in Fig.2. (The red vertices in Fig.2 are darkened.)

The relationship between the \( X \)-domination number of a prism and domination...
type parameters is also determined in [9].

![Figure 3: The nine 2-stratified $C_5$](image)

Let $S$ be a 2-stratified $C_5$ rooted at a blue vertex $v$. The nine possible choices for the graph $S$ are shown in Fig. 3. The $S$-domination number was studied in [15]. Also in [15] a theorem which generalize to find 2-stratified $C_{2k+1}$-domination number ($k \geq 2$) for a prism of $C_n \times K_2$ ($n \geq 3$), was introduced.

**Theorem 4.**([15]). For $n \geq 3$, let $G$ be a prism $C_n \times K_2$ ($n \neq 2k-1, n \neq 2k+1$) and $S$ be one of the 2-stratified $C_{2k+1}$ rooted at a blue vertex $v$. Then $\gamma_S(G) = 2n$.

### 3 Stratified $C_6$-Domination in Prisms

In this section we determine the Z-domination number of a prism when $Z$ is a 2-stratified cycle $C_6$. There are eighteen 2-stratified $C_6$ which are shown Fig.4. Throughout in this section we let $G = C_n \times K_2$ ($n \geq 3$).
Proposition 5. For \( i \in \{1, 2, 3, 6, 7, 8, 10, 11, 12, 13, 17, 18\} \), \( \gamma_{Z_i}(G) = 2n \).

Proof. There is no \( Z_i \)-coloring of \( G \) in which any vertex can be colored blue. Therefore \( \gamma_{Z_i}(G) = 2n \).

Theorem 6.(a) \( \gamma_{Z_1}(G) = \frac{2n}{3} + 4, \ n \equiv 0 \pmod{3} \).

(b) \( \gamma_{Z_1}(G) = \left\lceil \frac{2n}{3} \right\rceil + 1, \ n \equiv 1 \pmod{6} \).

(c) \( \gamma_{Z_1}(G) = 2 \left\lceil \frac{n}{3} \right\rceil + 2, \ n \equiv 2 \pmod{3} \).

(d) \( \gamma_{Z_1}(G) = 2 \left\lceil \frac{n}{3} \right\rceil, \ n \equiv 4 \pmod{6} \).

Proof. (a) For \( n \equiv 0 \pmod{3} \), \( G \) has \( n/3 \) disjoint 6-cycles and therefore has at least \( 2n/3 \) red vertices. Thus \( \gamma_{Z_1}(G) \geq 2n/3 \). Further suppose that exactly \( 2n/3 \) vertices are colored red. Then every 6-cycle in \( G \) contains exactly two red vertices. In particular \( v_i \) and \( u_1 \) are the only red vertices in the 6-cycle \( u_1 v_1 u_2 v_2 u_3 u_4 \). Since \( u_3 \) and \( v_3 \) are rooted in a copy of \( Z_4 \), the vertices \( u_4 \) and \( v_4 \) are colored red, and so \( u_4 \) and \( v_4 \) are the only red vertices in the 6-cycle \( u_4 v_4 u_5 v_5 u_6 u_7 \). Since \( u_6 \) and \( v_6 \) are rooted in a copy of \( Z_4 \), the vertices \( u_7 \) and \( v_7 \) are colored red, and so \( u_7 \) and \( v_7 \) are the only red vertices in the 6-cycle \( u_7 v_7 u_8 v_8 u_9 u_{10} \). Proceeding in this manner, \( u_{n-2} \) and \( v_{n-2} \) are the only
red vertices in the 6-cycle \( u_{n-2}v_nu_{n-1}v_{n-1}u_nu_{n-1}u_{n-2} \). But then \( u_n \) and \( v_n \) are not rooted at a copy of \( Z_4 \) and \( u_n \) and \( v_n \) have red colors. Therefore \( u_{n-1} \) and \( v_{n-1} \) have red colors too. \( u_{n-1} \) and \( v_{n-1} \) are not rooted at a copy of \( Z_4 \). This is a contradiction. Hence if \( n \equiv 0 \pmod{3} \), then at least \( 2n/3 + 4 \) vertices are colored red.

For \( n \equiv 0 \pmod{3} \), let \( S = \bigcup_{i=0}^{n/3-1} \{ v_{3i+1}, u_{3i+1} \} \) and \( D = S \cup \{ v_{n-1}, v_n, u_{n-1}, u_n \} \). In all cases, coloring the vertices in \( D \) red and coloring all remaining vertices blue, produces a \( Z_4 \)-coloring of \( G \) and so \( \gamma_{Z_4}(G) \leq 2n/3 + 4 \).

(b) Consider any given \( Z_4 \)-coloring of \( G \). If every vertex of \( G \) is colored red, then the required lower bound follows. \( G - \{ u_n, v_n \} \) can be partitioned into \( (n-1)/3 \) disjoint 6-cycles, each of which contains at least two red vertices, and so our given \( Z_4 \)-coloring of \( G \) colors at least \( 2 + 2(n-1)/3 = \lceil 2n/3 \rceil + 1 \) vertices red. Thus \( \gamma_{Z_4}(G) \geq \lceil 2n/3 \rceil + 1 \).

We show next that \( \gamma_{Z_4}(G) \leq \lceil 2n/3 \rceil + 1 \). Let \( S = \bigcup_{i=0}^{(n-1)/3} \{ v_{3i+1}, u_{3i+1} \} \). Coloring the vertices of \( S \) red and coloring all remaining vertices of \( G \) blue produces an \( Z_4 \)-coloring of \( G \). Thus \( \gamma_{Z_4}(G) \leq |S| = \lceil 2n/3 \rceil + 1 \).

(c) \( G - \{ v_{n-1}, v_n, u_{n-1}, u_n \} \) has \( (n-2)/3 \) disjoint 6-cycles, each of which contains at least two red vertices, and so our given \( Z_4 \)-coloring of \( G \) colors at least \( 4 + 2(n-2)/3 = 2 \lceil n/3 \rceil + 2 \) vertices red. Thus \( \gamma_{Z_4}(G) \geq 2 \lceil n/3 \rceil + 2 \).

We show next that \( \gamma_{Z_4}(G) \leq 2 \lceil n/3 \rceil + 2 \). Let \( S = \bigcup_{i=0}^{(n-2)/3} \{ v_{3i+1}, u_{3i+1} \} \) and \( D = \{ u_n, v_n \} \). Coloring the vertices of \( S \cup D \) red and coloring all remaining vertices of \( G \) blue produces an \( Z_4 \)-coloring of \( G \).

Therefore \( \gamma_{Z_4}(G) \leq |S \cup D| = 2 \lceil n/3 \rceil + 2 \).

(d) \( G - \{ u_1, v_1 \} \) can be partitioned into \( (n-1)/3 \) disjoint 6-cycles, each of which contains at least two red vertices and so our given \( Z_4 \)-coloring of \( G \) colors at least; \( 2 + 2(n-1)/3 = 2 \lceil n/3 \rceil \) vertices red. Thus \( \gamma_{Z_4} \geq 2 \lceil n/3 \rceil \). We show next that \( \gamma_{Z_4} \leq 2 \lceil n/3 \rceil \). Let \( S = \bigcup_{i=0}^{(n-1)/3} \{ v_{3i+1}, u_{3i+1} \} \). Coloring the vertices of \( S \) red and coloring all remaining vertices blue produces an \( Z_4 \)-coloring of \( G \). Therefore; \( \gamma_{Z_4}(G) \leq |S| = 2 \lceil n/3 \rceil \).
On stratification and domination in prisms

Theorem 7: (a) $\gamma_{Z_s}(G) = 2n/3 + 4, \ n \equiv 0(\text{mod} \ 3)$.
(b) $\gamma_{Z_s}(G) = 2\left\lfloor n/3 \right\rfloor + 2, \ n \equiv 1(\text{mod} \ 6)$.
(c) $\gamma_{Z_s}(G) = 2\left\lfloor n/3 \right\rfloor + 2, \ n \equiv 2(\text{mod} \ 3)$.
(d) $\gamma_{Z_s}(G) = 2\left\lfloor n/3 \right\rfloor + 1, \ n \equiv 4(\text{mod} \ 6)$.

Proof. We prove only (b). Let $G \setminus \{u_n,v_n\}$ have $(n-1)/3$ disjoint 6-cycles, each of which contains at least two red vertices, and so our given $Z_s$-coloring of $G$ colors at least $2 + 2(n-1)/3 = 2\left\lfloor n/3 \right\rfloor + 2$ vertices red. Therefore, $\gamma_{Z_s}(G) \geq 2\left\lfloor n/3 \right\rfloor + 2$.

Let $S = \bigcup_{i=0}^{(n-1)/3} \{u_{3i+1},v_{3i+1}\}$. Coloring the vertices of $S$ red and coloring all remaining vertices of $G$ blue produces an $Z_s$-coloring of $G$. Thus $\gamma_s(G) \leq 2\left\lfloor n/3 \right\rfloor + 2$.

Because of the proofs of Theorems 8, 9, 10, 11 are similar we omit the proof of next four theorems.

Theorem 8. (a) $\gamma_{Z_s}(G) = 2n/3, \ n \equiv 0(\text{mod} \ 3)$.
(b) $\gamma_{Z_s}(G) = 2\left\lfloor n/3 \right\rfloor + 1, \ n \equiv 1(\text{mod} \ 3)$.
(c) $\gamma_{Z_s}(G) = 2\left\lfloor n/3 \right\rfloor + 4, \ n \equiv 2(\text{mod} \ 6)$.
(d) $\gamma_{Z_s}(G) = 2\left\lfloor n/3 \right\rfloor + 2, \ n \equiv 5(\text{mod} \ 6)$

Theorem 9. $\gamma_{Z_{2s}}(G) = n$.

Theorem 10. (a) $\gamma_{Z_{3s}}(G) = n+1$, if $n$ is odd.
(b) $\gamma_{Z_{3s}}(G) = n+2$, if $n$ is even.

Theorem 11. (a) $\gamma_{Z_{6s}}(G) = n$ if $n \neq 6$ and $n$ is even.
(b) $\gamma_{Z_{6s}}(G) = n+1$ if $n$ is odd.

We conclude this section a theorem which generalize that ‘Which 2-stratified prisms domination number equals $n$?’.

Theorem 12. For $n \geq 4$ and $n$ is even, let $G = C_n \times K_2$ be a prism and $Z$ be one of the 2-stratified $C_{2k}$ in which the vertices of $A = \{u_1,u_3,\ldots,u_{n-1}\}$ and $B = \{v_2,v_4,\ldots,v_n\}$ are colored red, $\gamma_s(G) = n$.

Proof. $G$ can be partitioned into $n/k$ disjoint $2k$ cycles each of which contains at least $k$ red vertices and so given $Z$-coloring of $G$ colors at least
$k \times n/k = n$ vertices red. Thus, $\gamma_z(G) \geq n$. Let $D = \bigcup_{i=1}^{n/2} \{u_{2k-1}, v_{2k}\}$. Coloring the vertices of $D$ red and coloring all remaining vertices of $G$ blue produces an $Z$-coloring of $G$. Thus, $\gamma_z(G) \leq n$.

References


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