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Norm Bounds for a Transformed Price Vector in Sraffian Systems

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Abstract

This paper gives lower and upper bounds, which are expressed in terms of the 'maximum column sum matrix norm', for the largest and the smallest element of a transformed price vector in Sraffian systems. It is found that the bounds depend on the socio-technical conditions of production, *i.e.*, the vertically integrated coefficients and the ratio of the uniform rate of profit to the maximum rate of profit. On the empirical side, the paper provides an illustration of the bounds using data from the Symmetric Input-Output Tables of the Greek economy.

Mathematics Subject Classification: 91B24, 91B38, 91B66, 91B74

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1 Introduction

It is well known that long-period relative prices in Sraffian systems can change in a complicated way as income distribution changes.[1] This paper does not directly explore the relationship between prices and changes in income distribution, but takes another route: it gives lower and upper bounds for the largest and the smallest element of a price vector, which is (i) normalized with Sraffa's [26, chs 4-5] 'Standard commodity'; and (ii) associated with a transformed system or, more precisely, a

[1] See [26, chs 3 and 6], [24], [20, pp. 82-84] and [13, pp. 99-100]. It should be noted, however, that C. Bidard, H.G. Ehbar, U. Krause and I. Steedman have detected some 'monotonicity laws' for the relative prices (see [5] and the references provided there). Finally, for *empirical* studies of the variation of relative prices, see [18, ch. 7], [21], [6], [25], [32] and [16], *inter alia*.

vertically integrated system in which the technical coefficients matrix is a *stochastic* matrix. The bounds are expressed in terms of the 'maximum column sum matrix norm', and depend on 'commodity i values' (see [9, pp. 18-21]) and the 'relative rate of profit', *i.e.*, the ratio of the uniform rate of profit to the maximum rate of profit.

The remainder of the paper is structured as follows. Section 2 deals with the usual circulating capital model ([26, Part I]) and gives norm bounds for the vector of the transformed prices of production. It is shown that the bounds depend on the vector of vertically integrated labour coefficients (or 'labour values') and the relative rate of profit. Section 3 provides an empirical illustration of the bounds using data from the 19×19 Symmetric Input-Output Tables (SIOT) of the Greek economy for the period 1988-1997.[2] Section 4 extends the theoretical analysis by allowing for (i) alternative value bases; or (ii) joint products and differentiated profit rates (which are of great empirical importance).[3] Section 5 concludes the paper.

2 Norm Bounds

Consider a closed, linear system, involving only single products, basic commodities (in the sense of Sraffa [26, §6]) and circulating capital. Furthermore, assume that (i) the input-output coefficients are fixed; (ii) the system is viable, *i.e.*, the Perron-Frobenius (P-F hereafter) eigenvalue of the irreducible $n\times n$ matrix of input-output coefficients, $\mathbf{A}$, is less than 1;[4] (iii) the rate of profit, r, is uniform; (iv) labour is not an input to the household sector and may be treated as homogeneous because relative wage rates are invariant ([26, §10]); and (v) wages are paid at the beginning of the common production period.[5]

On the basis of these assumptions we can write

$$\mathbf{p}^{\mathrm{T}} = (1+r)(\mathbf{p}^{\mathrm{T}}\mathbf{A} + w\mathbf{l}^{\mathrm{T}}) \tag{1}$$

where $\mathbf{p}$ denotes a vector of prices of production, w the money wage rate and $\mathbf{l}$ ($>\mathbf{0}$) the vector of direct labour coefficients. Relation (1) after rearrangement gives:

$$\mathbf{p}^{\mathrm{T}} = r\mathbf{p}^{\mathrm{T}}\mathbf{H} + (1+r)w\mathbf{v}^{\mathrm{T}} \tag{2}$$

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[2] The SIOT of the Greek economy are provided by the National Statistical Service of Greece (for details, see [32, Appendix]).

[3] Regarding the empirical importance of joint production, see [29] and [8]. It goes without saying that the so-called 'Supply and Use Tables (SUT)' could be considered as the empirical *counterpart* of joint-product systems (see, *e.g.*, [4, pp. 434-436]).

[4] Matrices (and vectors) are denoted by boldface letters. The transpose of an $n\times 1$ vector $\mathbf{x}\equiv[x_i]$ is denoted by $\mathbf{x}^{\mathrm{T}}$. $\lambda_{\mathbf{A}}$ denotes the P-F eigenvalue of a semi-positive matrix $\mathbf{A}\equiv[a_{ij}]$, $(\mathbf{x}_{\mathbf{A}},\mathbf{y}_{\mathbf{A}}^{\mathrm{T}})$ the corresponding eigenvectors, and $\hat{\mathbf{y}}_{\mathbf{A}}$ the diagonal matrix formed from the elements of $\mathbf{y}_{\mathbf{A}}$. Finally, $\mathbf{e}$ denotes the summation vector, *i.e.*, $\mathbf{e}\equiv[1,1,...,1]^{\mathrm{T}}$.

[5] We hypothesize that wages are paid *ex ante* in order to follow most of the *empirical* studies on this topic (for the general case, see [28, pp. 103-105]). However, it would make no relevant difference to what follows to suppose *ex post* payment of wages.

where $\mathbf{H} \equiv \mathbf{A}[\mathbf{I}-\mathbf{A}]^{-1}$ $(>\mathbf{0})$ denotes the 'vertically integrated technical coefficients matrix' ([19]), $\mathbf{I}$ the identity matrix, and $\mathbf{v}^{\mathrm{T}} \equiv \mathbf{l}^{\mathrm{T}}[\mathbf{I}-\mathbf{A}]^{-1}$ $(>\mathbf{0}^{\mathrm{T}})$ the vector of vertically integrated labour coefficients or 'labour values'.

If Sraffa's Standard commodity is chosen as the standard of value or *numéraire*, *i.e.*, $\mathbf{p}^{\mathrm{T}}[\mathbf{I}-\mathbf{A}]\mathbf{x}_{\mathbf{A}}=1$, with $\mathbf{l}^{\mathrm{T}}\mathbf{x}_{\mathbf{A}}=1$, then (1) implies that

$$(1+r)w = 1-(r/R)$$

or

$$(1+r)w = (1-\rho) \tag{3}$$

where $R \equiv (1/\lambda_{\mathbf{A}})-1$ $(=1/\lambda_{\mathbf{H}})$ represents the maximum rate of profit and $\rho \equiv r/R$, $0 \le \rho \le 1$, the 'relative rate of profit', which is no greater than the share of profits in the Standard system.[6] Substituting (3) in (2) yields

$$\mathbf{p}^{\mathrm{T}} = \rho\mathbf{p}^{\mathrm{T}}\mathbf{J} + (1-\rho)\mathbf{v}^{\mathrm{T}} \tag{4}$$

or, if $\rho < 1$,

$$\mathbf{p}^{\mathrm{T}} = (1-\rho)\mathbf{v}^{\mathrm{T}}[\mathbf{I}-\rho\mathbf{J}]^{-1} = (1-\rho)\mathbf{v}^{\mathrm{T}}\sum_{k=0}^{\infty}\rho^{k}\mathbf{J}^{k} \tag{5}$$

where $\mathbf{J} \equiv R\mathbf{H}$ and $\lambda_{\mathbf{J}} = 1$.[7]

Given that

$$\mathbf{e}^{\mathrm{T}}[\hat{\mathbf{y}}_{\mathbf{A}}\mathbf{J}\hat{\mathbf{y}}_{\mathbf{A}}^{-1}] = \mathbf{y}_{\mathbf{A}}^{\mathrm{T}}\mathbf{J}\hat{\mathbf{y}}_{\mathbf{A}}^{-1} = \mathbf{y}_{\mathbf{A}}^{\mathrm{T}}\hat{\mathbf{y}}_{\mathbf{A}}^{-1} = \mathbf{e}^{\mathrm{T}}$$

it follows that $\mathbf{J}$ is similar to the *column stochastic* matrix $\mathbf{K} \equiv [k_{ij}] \equiv \hat{\mathbf{y}}_{\mathbf{A}}\mathbf{J}\hat{\mathbf{y}}_{\mathbf{A}}^{-1}$ $(>\mathbf{0})$, the elements of which are independent of the choice of physical measurement units (and the normalization of $\mathbf{y}_{\mathbf{A}}$). Substituting $\mathbf{J} = \hat{\mathbf{y}}_{\mathbf{A}}^{-1}\mathbf{K}\hat{\mathbf{y}}_{\mathbf{A}}$, with $\mathbf{y}_{\mathbf{A}}^{\mathrm{T}}[\mathbf{I}-\mathbf{A}]\mathbf{x}_{\mathbf{A}}=1$, in (4) yields

$$\boldsymbol{\pi}^{\mathrm{T}} = \rho\boldsymbol{\pi}^{\mathrm{T}}\mathbf{K} + (1-\rho)\boldsymbol{\omega}^{\mathrm{T}} \tag{6}$$

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[6] Relation (3) can be rewritten as:

$$r = R(1-w)(1+wR)^{-1}$$

or

$$\rho = (1-w)(1+wR)^{-1} \le 1-w$$

where $1-w$, $(1+wR)^{-1}$ equal the share of profits and the ratio of the means of production to the total capital, respectively, in the Standard system. If wages are paid *ex post*, then $\rho = 1-w$. See also [26, §§29-32]; [14, pp. 136-138].

[7] System (5) has been investigated intensively by Bienenfeld [6] and Steedman [30]. Steedman [30, p. 316] notes: 'If, by chance, $\mathbf{H}$ and hence $\mathbf{J}$ should happen to be normal matrices (whether diagonal or otherwise), it would then follow at once that $(\mathbf{v}^{\mathrm{T}}\mathbf{J}^{k}[\mathbf{J}^{k}]^{\mathrm{T}}\mathbf{v}) \le (\mathbf{v}^{\mathrm{T}}\mathbf{v})$. Hence *no* vector $\mathbf{v}^{\mathrm{T}}\mathbf{J}^{k}$, for $k=1,2,...$, would be of greater Euclidean length than $\mathbf{v}^{\mathrm{T}}$ and the decrease in $[(1-\rho)\rho^{k}]$ with k would certainly not be counteracted by any tendency for the vectors $\mathbf{v}^{\mathrm{T}}\mathbf{J}^{k}$ to increase in magnitude with k. More generally, we can of course say that there will be no such tendency provided that every matrix $\mathbf{J}^{k}$ has a norm of unity or less – but this is merely true by definition and we shall not attempt here to characterize the conditions on $\mathbf{H}$ under which this requirement will be met.'. For relevant empirical studies, associated with input-output data from the economies of China, Greece and Japan, see [16], [32] and [31], respectively.

or, if $\rho < 1$,

$$\boldsymbol{\pi}^{\mathrm{T}} = \boldsymbol{\omega}^{\mathrm{T}}[\mathbf{L}(\rho)]^{-1} \tag{7}$$

where $\boldsymbol{\pi}^{\mathrm{T}} \equiv \mathbf{p}^{\mathrm{T}}\hat{\mathbf{y}}_{\mathbf{A}}^{-1}$, $\boldsymbol{\omega}^{\mathrm{T}} \equiv \mathbf{v}^{\mathrm{T}}\hat{\mathbf{y}}_{\mathbf{A}}^{-1}$, $\mathbf{L}(\rho) \equiv (1-\rho)^{-1}[\mathbf{I}-\rho\mathbf{K}]$, and $[\mathbf{L}(\rho)]^{-1}$ is a column stochastic matrix, since $[\mathbf{L}(\rho)]^{-1} \geq \mathbf{0}$ and

$$\mathbf{e}^{\mathrm{T}}[\mathbf{L}(\rho)]^{-1} = (1-\rho)(1-\rho)^{-1}\mathbf{e}^{\mathrm{T}} = \mathbf{e}^{\mathrm{T}}$$

From relations (6)-(7), and the normalization conditions, we derive the following:

(i). $\boldsymbol{\pi} = \boldsymbol{\omega}$ at $\rho = 0$, and $\boldsymbol{\pi} = \mathbf{e}$ at $\rho = 1$. In the trivial case in which $\boldsymbol{\omega} = \mathbf{e}$, then $\boldsymbol{\pi} = \mathbf{e}$ (this corresponds to the 'equal capital-intensity' case). Furthermore, since $(\boldsymbol{\pi}^{\mathrm{T}} - \boldsymbol{\omega}^{\mathrm{T}})(\hat{\mathbf{y}}_{\mathbf{A}}\mathbf{x}_{\mathbf{A}}) = 0$ for each ρ, it follows that $(\mathbf{e}^{\mathrm{T}} - \boldsymbol{\omega}^{\mathrm{T}})(\hat{\mathbf{y}}_{\mathbf{A}}\mathbf{x}_{\mathbf{A}}) = 0$, which in its turn implies

$$\min\{\omega_j\} \leq 1 \leq \max\{\omega_j\} \tag{8}$$

(if $\boldsymbol{\omega} \neq \mathbf{e}$, then both inequalities in (8) are strict).

(ii). Relation (7) implies that π_j, $j = 1, 2, ..., n$, is a convex combination of the elements of $\boldsymbol{\omega}$. Thus, we may write

$$\min\{\omega_j\} \leq \pi_j \leq \max\{\omega_j\}$$

or

$$\left\|\boldsymbol{\pi}^{\mathrm{T}}\right\| = \max\{\pi_j\} \leq \left\|\boldsymbol{\omega}^{\mathrm{T}}\right\| = \max\{\omega_j\} \tag{9}$$

and

$$\min\{\omega_j\} = 1/\left\|\hat{\boldsymbol{\omega}}^{-1}\right\| \leq \min\{\pi_j\} = 1/\left\|\hat{\boldsymbol{\pi}}^{-1}\right\| \tag{10}$$

where $\|\bullet\|$ denotes the 'maximum column sum matrix norm'.

(iii). Relation (6) can be restated as

$$(1-\rho)\boldsymbol{\omega}^{\mathrm{T}} = \boldsymbol{\pi}^{\mathrm{T}}[\mathbf{I}-\rho\mathbf{K}] \tag{11}$$

Taking norms of (11), and using the Hölder's inequality, we obtain

$$(1-\rho)\left\|\boldsymbol{\omega}^{\mathrm{T}}\right\| \leq \left\|\boldsymbol{\pi}^{\mathrm{T}}\right\|(\max\{1-\rho k_{jj} + \sum_{\substack{i=1 \\ i \neq j}}^{n} \rho k_{ij}\}),\ j = 1, 2, ..., n \tag{12}$$

or, given that $\sum_{\substack{i=1 \\ i \neq j}}^{n} k_{ij} = 1 - k_{jj}$,

$$(1-\rho)\left\|\boldsymbol{\omega}^{\mathrm{T}}\right\| \leq \left\|\boldsymbol{\pi}^{\mathrm{T}}\right\|(\max\{1+\rho(1-2k_{jj})\})$$

or

$$(1-\rho)\left\|\boldsymbol{\omega}^{\mathrm{T}}\right\| \leq \left\|\boldsymbol{\pi}^{\mathrm{T}}\right\|[1+\rho(1-2m)]$$

or

$$f(\rho) \leq \left\|\boldsymbol{\pi}^{\mathrm{T}}\right\| / \left\|\boldsymbol{\omega}^{\mathrm{T}}\right\| \tag{13}$$

where $m \equiv \min\{k_{jj}\}$, $0 < m < 1$, and $f(\rho) \equiv (1-\rho)/[1+\rho(1-2m)]$, $0 < f(\rho) \le 1$, a strictly decreasing function of ρ, which is strictly convex to the origin for $m < 0.5$ and tends to 1 as m tends to 1.[8]

(iv). Post-multiplying (6) by $\hat{\boldsymbol{\omega}}^{-1}\hat{\boldsymbol{\pi}}^{-1}$ $(= \hat{\boldsymbol{\pi}}^{-1}\hat{\boldsymbol{\omega}}^{-1})$ gives

$$\mathbf{e}^{\mathrm{T}}\hat{\boldsymbol{\omega}}^{-1} = \rho\boldsymbol{\pi}^{\mathrm{T}}\mathbf{K}\hat{\boldsymbol{\omega}}^{-1}\hat{\boldsymbol{\pi}}^{-1} + (1-\rho)\mathbf{e}^{\mathrm{T}}\hat{\boldsymbol{\pi}}^{-1}$$

Taking norms, and recalling $\|\mathbf{K}\| = 1$, we obtain

$$\|\hat{\boldsymbol{\omega}}^{-1}\| \le \rho\|\boldsymbol{\pi}^{\mathrm{T}}\|\|\hat{\boldsymbol{\omega}}^{-1}\|\|\hat{\boldsymbol{\pi}}^{-1}\| + (1-\rho)\|\hat{\boldsymbol{\pi}}^{-1}\|$$

or, dividing both sides by $\|\hat{\boldsymbol{\pi}}^{-1}\|$ and recalling (9),

$$\|\hat{\boldsymbol{\omega}}^{-1}\| / \|\hat{\boldsymbol{\pi}}^{-1}\| \le g(\rho) \le h(\rho) \tag{14}$$

where

$$g(\rho) \equiv \rho(\|\boldsymbol{\pi}^{\mathrm{T}}\|\|\hat{\boldsymbol{\omega}}^{-1}\| - 1) + 1 \ (\ge 1) \tag{14a}$$

$$g(1) = \|\boldsymbol{\pi}^{\mathrm{T}}(1)\|\|\hat{\boldsymbol{\omega}}^{-1}\| = \|\hat{\boldsymbol{\omega}}^{-1}\| / \|\hat{\boldsymbol{\pi}}^{-1}(1)\| = \|\hat{\boldsymbol{\omega}}^{-1}\| \tag{14b}$$

(since $\boldsymbol{\pi} = \mathbf{e}$ at $\rho = 1$), and

$$1 \le h(\rho) \equiv \rho(\|\boldsymbol{\omega}^{\mathrm{T}}\|\|\hat{\boldsymbol{\omega}}^{-1}\| - 1) + 1 \le \|\boldsymbol{\omega}^{\mathrm{T}}\|\|\hat{\boldsymbol{\omega}}^{-1}\| \tag{14c}$$

Combining (9) and (13) gives

$$f(\rho) \le \|\boldsymbol{\pi}^{\mathrm{T}}\| / \|\boldsymbol{\omega}^{\mathrm{T}}\| \le 1 \tag{15}$$

whilst combining (10) and (14) gives

$$1 \le \|\hat{\boldsymbol{\omega}}^{-1}\| / \|\hat{\boldsymbol{\pi}}^{-1}\| \le h(\rho) \tag{16}$$

We therefore conclude that the upper (lower) bound for $\|\boldsymbol{\pi}^{\mathrm{T}}\| / \|\boldsymbol{\omega}^{\mathrm{T}}\|$ (for $\|\hat{\boldsymbol{\omega}}^{-1}\| / \|\hat{\boldsymbol{\pi}}^{-1}\|$) equals 1 and the lower (upper) bound decreases (increases) with increasing ρ.[9]

3 Empirical Illustration

The application of the previous analysis to the 19×19 SIOT of the Greek economy (1988-1997) gives the results summarized in Tables 1 and 2.[10]

Table 1 (reproduced from [32, p. 428]) shows the maximum, 'actual', and relative rates of profit (R, r^{a}, and $\rho^{\mathrm{a}} \equiv r^{\mathrm{a}} / R$, respectively), which are estimated from (1), with $w = 0$, and the eigenequation

$$[\mathbf{p}^{\mathrm{a}}]^{\mathrm{T}} = (1 + r^{\mathrm{a}})[\mathbf{p}^{\mathrm{a}}]^{\mathrm{T}}\mathbf{C} \tag{17}$$

where $\mathbf{p}^{\mathrm{a}}$ denotes the vector of the actual prices of production, $\mathbf{C} \equiv \mathbf{A} + \mathbf{b}^{\mathrm{a}}\mathbf{l}^{\mathrm{T}}$ the matrix of the 'augmented' input-output coefficients, *i.e.*, each coefficient represents

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[8] It may be noted that the 'condition number' of $\mathbf{L}(\rho)$, defined as $\|\mathbf{L}(\rho)\|\|[\mathbf{L}(\rho)]^{-1}\|$ equals $1/f(\rho)$.

[9] The monotonicity of $g(\rho)$ is *a priori* unknown.

[10] The analytical results of this paper are available on request from the author.

the sum of the respective material and wage good input per unit of output, and $\mathbf{b}^{\mathrm{a}}$ the vector of the actual real wage rate, which is estimated on the basis of the available input-output data.

Table 1. Maximum, actual and relative rates of profit; Greek economy, 1988-1997

	1988	1989	1990	1991	1992	1993	1994	1995	1996	1997
R	0.817	0.851	0.874	0.917	1.076	1.026	1.006	0.903	0.977	0.882
r^{a}	0.211	0.220	0.218	0.243	0.275	0.236	0.254	0.230	0.247	0.238
ρ^{a}	0.258	0.259	0.249	0.265	0.255	0.230	0.252	0.254	0.252	0.270

Table 2 shows (i) the values of m and the sectors in which they occur (the sector numbers are indicated by $[\bullet]$);[11] (ii) the values of $\|\boldsymbol{\omega}^{\mathrm{T}}\|\|\hat{\boldsymbol{\omega}}^{-1}\|$ and the sectors in which $\|\boldsymbol{\omega}^{\mathrm{T}}\|$ and $1/\|\hat{\boldsymbol{\omega}}^{-1}\|$ occur (the sector numbers are indicated by $[\bullet,\bullet]$, where the first (second) number refers to $\|\boldsymbol{\omega}^{\mathrm{T}}\|$ (to $1/\|\hat{\boldsymbol{\omega}}^{-1}\|$); (iii) the values of $\|\boldsymbol{\pi}^{\mathrm{T}}\|/\|\boldsymbol{\omega}^{\mathrm{T}}\|$ and $\|\hat{\boldsymbol{\omega}}^{-1}\|/\|\hat{\boldsymbol{\pi}}^{-1}\|$, at $\rho=\rho^{\mathrm{a}}$ and $\rho=0.9$ (*i.e.*, a high, somewhat unrealistic value), and the sectors in which $\|\boldsymbol{\pi}^{\mathrm{T}}\|$ and $1/\|\hat{\boldsymbol{\pi}}^{-1}\|$ occur, respectively (the sector numbers are indicated by $[\bullet]$); and (iv) the relevant values of $f(\rho)$, $g(\rho)$ and $h(\rho)$ (see relations (13) and (14)).

[11] Sectoral Classification: **[1]** Agriculture, hunting and related service activities, products of forestry: logging related service; **[2]** Fish and other fishing products; **[3]** Mining of coal and lignite; extraction of peat, extraction of crude oil and natural gas, mining of nuclear materials; **[4]** Mining of metal ores, other mining and quarrying products; **[5]** Manufacture of food products and beverages, tobacco products; **[6]** Manufacture of textiles, manufacture of clothes process and dyeing of fur, manufacture of tanning and dressing of leather; **[7]** Wood and wood products; **[8]** Pulp, paper and paper products publishing printing and reproduction of recorded media; **[9]** Manufacture of coke: refined petroleum products and nuclear fuel; **[10]** Manufacture of chemicals and chemical products, manufacture of rubber and plastic products; **[11]** Manufacture of other non-metallic mineral products; **[12]** Basic metals and fabricated metal products; **[13]** Fabricated metal products except machinery and equipment; **[14]** Machinery and equipment, office machinery and computers, electrical machinery and apparatus, radio, television and communication equipment and apparatus, medical precision and optical instruments, watches and clocks, motor vehicles trailers and semi-trailers; **[15]** Electricity, gas, steam and hot water, collection purification and distribution of water; **[16]** Construction work; **[17]** Whole sale and retail sale of motor vehicles, whole sale and retail sale except vehicles and retail trade; **[18]** Hotel and restaurant services; and **[19]** Transports, water transport services, air transport services, post and telecommunications.

Table 2. Norm bounds for the vector of prices of production; Greek economy, 1988-1997

		1988	1989	1990	1991	1992	1993	1994	1995	1996	1997
	$m \times 10^3$	0.534 **[18]**	0.590 **[18]**	0.585 **[18]**	0.593 **[18]**	0.680 **[18]**	0.755 **[18]**	0.773 **[18]**	0.788 **[18]**	0.813 **[18]**	0.678 **[18]**
	$\lVert \boldsymbol{\omega}^{\mathrm{T}} \rVert \lVert \hat{\boldsymbol{\omega}}^{-1} \rVert$	7.342 **[1,10]**	7.591 **[19,12]**	7.612 **[4,12]**	6.814 **[4,12]**	5.069 **[19,10]**	5.895 **[2,12]**	6.562 **[19,12]**	8.049 **[2,12]**	6.833 **[2,12]**	6.728 **[2,12]**
$\rho = \rho^{\mathrm{a}}$	$f(\rho)$	0.590	0.589	0.601	0.581	0.594	0.626	0.598	0.595	0.598	0.575
	$\lVert \boldsymbol{\pi}^{\mathrm{T}} \rVert / \lVert \boldsymbol{\omega}^{\mathrm{T}} \rVert$	0.840 **[1]**	0.832 **[19]**	0.824 **[4]**	0.820 **[4]**	0.840 **[19]**	0.869 **[2]**	0.834 **[19]**	0.830 **[2]**	0.838 **[2]**	0.824 **[2]**
	$\lVert \hat{\boldsymbol{\omega}}^{-1} \rVert / \lVert \hat{\boldsymbol{\pi}}^{-1} \rVert$	1.128 **[10]**	1.178 **[12]**	1.157 **[12]**	1.175 **[12]**	1.101 **[10]**	1.134 **[12]**	1.126 **[12]**	1.152 **[12]**	1.143 **[12]**	1.155 **[12]**
	$g(\rho)$	2.333	2.377	2.313	2.216	1.833	1.949	2.127	2.442	2.191	2.227
	$h(\rho)$	2.636	2.707	2.646	2.541	2.038	2.126	2.402	2.790	2.470	2.547
$\rho = 0.9$	$f(\rho)$	0.053	0.053	0.053	0.053	0.053	0.053	0.053	0.053	0.053	0.053
	$\lVert \boldsymbol{\pi}^{\mathrm{T}} \rVert / \lVert \boldsymbol{\omega}^{\mathrm{T}} \rVert$	0.336 **[1]**	0.360 **[19]**	0.337 **[4]**	0.366 **[1]**	0.395 **[19]**	0.404 **[2]**	0.355 **[2]**	0.323 **[2]**	0.350 **[2]**	0.354 **[2]**
	$\lVert \hat{\boldsymbol{\omega}}^{-1} \rVert / \lVert \hat{\boldsymbol{\pi}}^{-1} \rVert$	1.628 **[10]**	1.880 **[12]**	1.786 **[12]**	1.799 **[12]**	1.503 **[10]**	1.707 **[12]**	1.608 **[12]**	1.707 **[12]**	1.654 **[12]**	1.669 **[12]**
	$g(\rho)$	2.324	2.560	2.408	2.346	1.902	2.244	2.194	2.441	2.255	2.247
	$h(\rho)$	6.708	6.932	6.951	6.233	4.662	5.405	6.006	7.344	6.250	6.155

On the basis of these estimates we may remark the following:

(i). m always occurs in the same sector, *i.e.*, sector 18, and it is much less than 0.5, which implies that $f(\rho)$ is always strictly convex to the origin.[12]

(ii). $g(\rho)$ may be a non-monotonic function (consider the years 1988 and 1995).

(iii). Not quite unexpected, the relative errors in $f(\rho^{\mathrm{a}})$ and $h(\rho^{\mathrm{a}})$ (as bounds for $\lVert \boldsymbol{\pi}^{\mathrm{T}} \rVert / \lVert \boldsymbol{\omega}^{\mathrm{T}} \rVert$ and $\lVert \hat{\boldsymbol{\omega}}^{-1} \rVert / \lVert \hat{\boldsymbol{\pi}}^{-1} \rVert$, respectively), are less than the relative errors in $f(0.9)$ and $h(0.9)$, respectively. The relative error in $g(\rho^{\mathrm{a}})$ (as a bound for $\lVert \hat{\boldsymbol{\omega}}^{-1} \rVert / \lVert \hat{\boldsymbol{\pi}}^{-1} \rVert$) is greater than the relative error in $g(0.9)$.

(iv). Within each year, $1/\lVert \hat{\boldsymbol{\omega}}^{-1} \rVert$ and $1/\lVert \hat{\boldsymbol{\pi}}^{-1} \rVert$ occur in the same sector. However, this does not hold true for $\lVert \boldsymbol{\omega}^{\mathrm{T}} \rVert$ and $\lVert \boldsymbol{\pi}^{\mathrm{T}} \rVert$. More precisely, they occur in different sectors in the years 1991 and 1994, and at $\rho = 0.9$.

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[12] It may be noted that also $\max\{k_{jj}\}$ always occurs in the same sector, *i.e.*, sector 10, and it is in the range of 0.619 (year 1989)-0.740 (year 1988).

For reasons of clarity of presentation and economy of space, the following set of figures is associated with the year 1991: Figure 1 displays π_j as functions of ρ. Numerical calculations show that (i) prices change more often than not in a strictly monotonic way; (ii) π_1 and π_4, *i.e.*, the largest elements of $\{\pi_j\}$, are strictly decreasing and equal to each other at $\rho \cong 0.846$, whilst π_{12}, *i.e.*, $1/\left\|\hat{\boldsymbol{\pi}}^{-1}\right\|$, is strictly increasing; and (iii) the case of a maximum point is observed in sectors 5, 9 and 18, where $\omega_j > 1$, whilst the case of a minimum point is observed in sector 14, where $\omega_j < 1$ (see the dashed lines in Figure 1). Figure 2 displays $f(\rho)$, π_1 / ω_4 and π_4 / ω_4 as functions of ρ. Finally, Figure 3 displays (i) π_{12} / ω_{12} as function of ρ; (ii) the functions $g(\rho)$, associated with π_1 / ω_{12} and π_4 / ω_{12}; and (iii) $h(\rho)$.

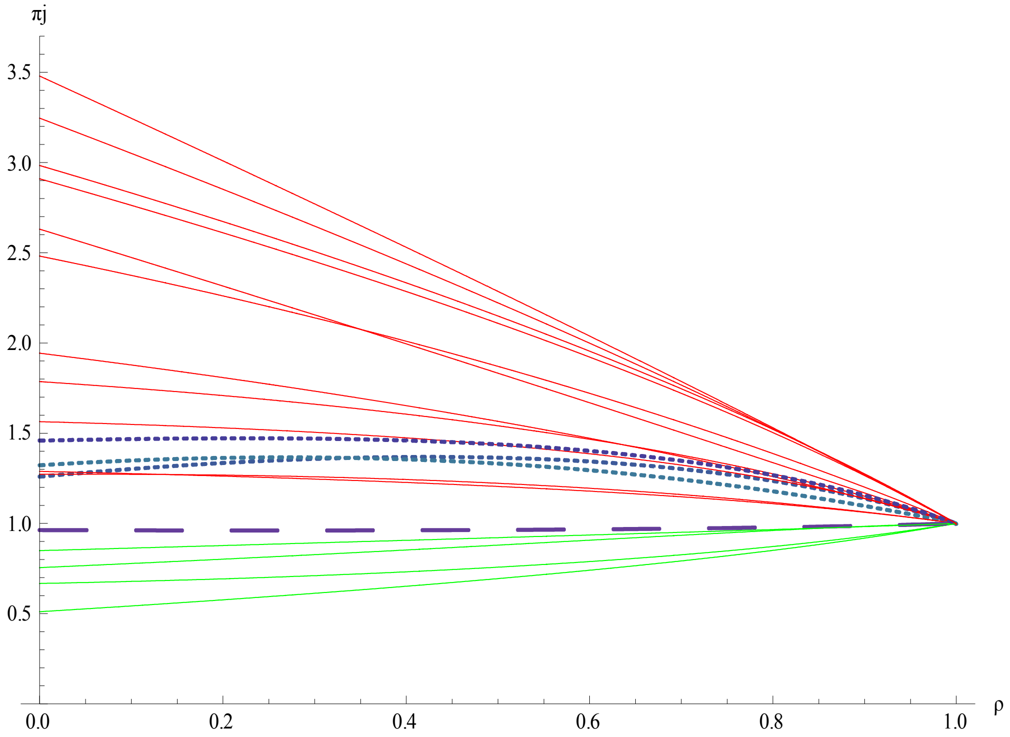

Figure 1. Transformed prices of production as functions of the relative rate of profit; Greek economy, 1991

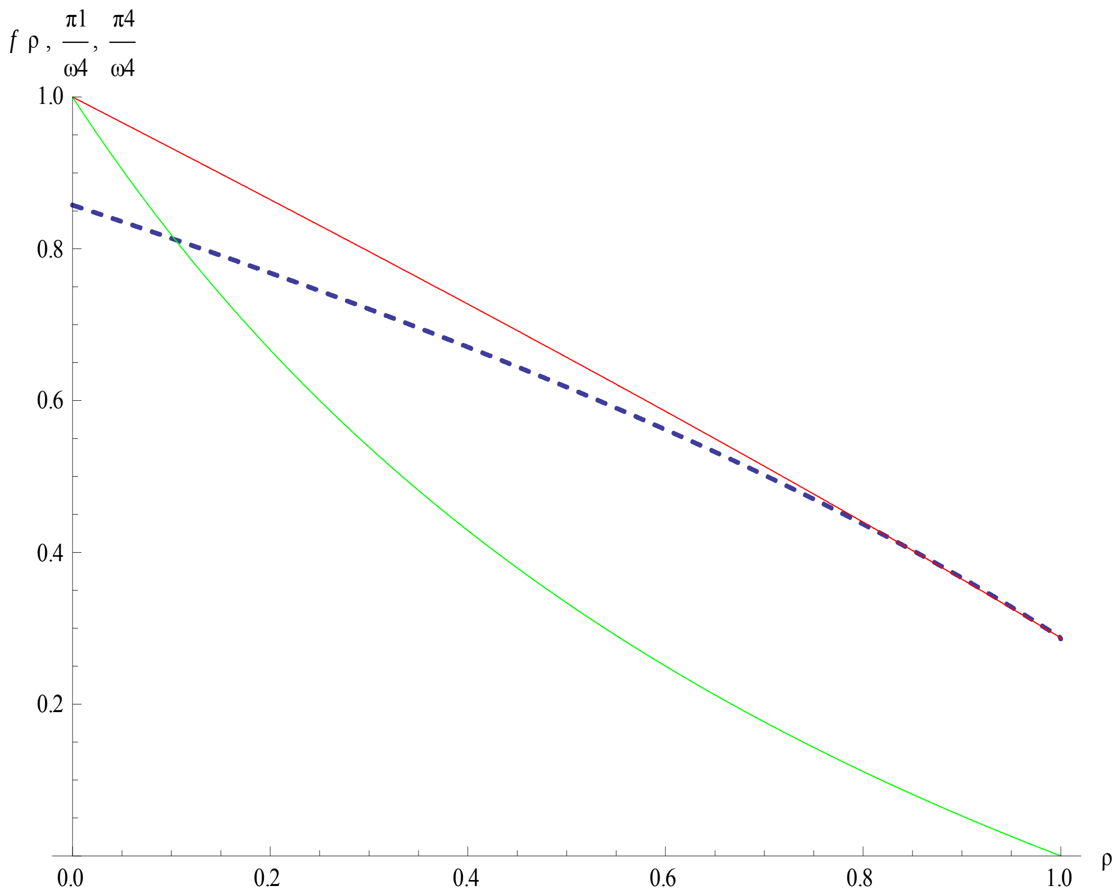

Figure 2. $\|\boldsymbol{\pi}^{T}\| / \|\boldsymbol{\omega}^{T}\|$ and their lower bound; Greek economy, 1991

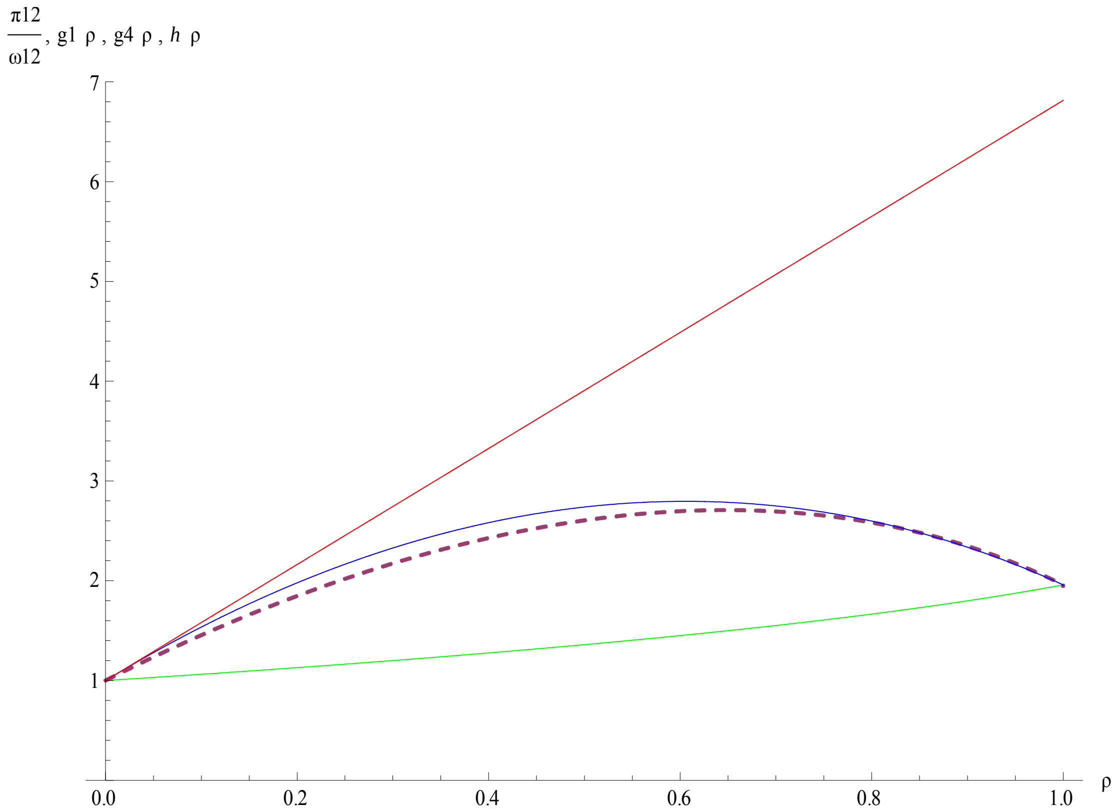

Figure 3. $\|\hat{\boldsymbol{\omega}}^{-1}\|/\|\hat{\boldsymbol{\pi}}^{-1}\|$ and its upper bounds; Greek economy, 1991

We may conclude this section by noting that, to our knowledge, there is *no* relevant empirical study where the actual value of ρ is less than 0.17 and greater than 0.40.[13] Thus, it is reasonable to expect that, in the 'real' world, the vectors of the actual prices of production tend to obey the following inequalities (see (15) and (16)):

$$F(m) \leq \|\boldsymbol{\pi}^{\mathrm{T}}\|/\|\boldsymbol{\omega}^{\mathrm{T}}\| \leq 1$$

and

$$1 \leq \|\hat{\boldsymbol{\omega}}^{-1}\|/\|\hat{\boldsymbol{\pi}}^{-1}\| \leq H(\boldsymbol{\omega})$$

where

$$[3/(7-4m)] \leq F(m) \leq [83/(117-34m)]$$

and

[13] See, *e.g.*, the empirical studies mentioned in footnote 1. Taking into account that ρ is no greater than the share of profits in the Standard system (see footnote 6), this seems to be in accordance with many well-known estimations of the share of profits in actual economies. For example, a recent study by Ellis and Smith [7] find that the share of profits in a sample of 20 advanced countries (for the period 1960-2005) only in a few years and a few countries has exceeded the 40%, by some percentage points, and, typically, fluctuates around an average of 30% (see also [10]).

$$0.17(\|\boldsymbol{\omega}^{\mathrm{T}}\|\|\hat{\boldsymbol{\omega}}^{-1}\|)+0.83 \leq H(\boldsymbol{\omega}) \leq 0.4(\|\boldsymbol{\omega}^{\mathrm{T}}\|\|\hat{\boldsymbol{\omega}}^{-1}\|)+0.6$$

4 Extensions

In this section we shall (i) reformulate the norm bounds in terms of alternative value bases; and (ii) consider the cases of joint production and differentiated profit rates.

4.1 Alternative value bases

System (17) may be rewritten as

$$\mathbf{p}^{\mathrm{T}} = (1+r)(\mathbf{p}^{\mathrm{T}}\mathbf{C}_i + p_i\mathbf{c}_i^{\mathrm{T}}) \tag{18}$$

or

$$\mathbf{p}^{\mathrm{T}} = r\mathbf{p}^{\mathrm{T}}\mathbf{H}_i + (1+r)p_i\mathbf{v}_i^{\mathrm{T}} \tag{19}$$

where $\mathbf{C}_i$ denotes the matrix derived from $\mathbf{C}$ by replacing all the elements on its ith row by zeroes, $\mathbf{c}_i^{\mathrm{T}}$ the ith row of $\mathbf{C}$, $\mathbf{H}_i \equiv \mathbf{C}_i[\mathbf{I}-\mathbf{C}_i]^{-1}$, and $\mathbf{v}_i^{\mathrm{T}} \equiv \mathbf{c}_i^{\mathrm{T}}[\mathbf{I}-\mathbf{C}_i]^{-1}$ the vector of 'commodity i values' (see also, *e.g.*, [22, pp. 24-26]).[14] If prices are normalized by setting $\mathbf{p}^{\mathrm{T}}[\mathbf{I}-\mathbf{C}_i]\mathbf{x}_{\mathbf{C}_i} = 1$, with $\mathbf{c}_i^{\mathrm{T}}\mathbf{x}_{\mathbf{C}_i} = 1$, then (18) implies that

$$(1+r)p_i = (1-\rho_i) \tag{20}$$

where $\rho_i \equiv r/R_i$, $0 \leq \rho_i \leq 1$, and $R_i \equiv (1/\lambda_{\mathbf{C}_i})-1$ $(=1/\lambda_{\mathbf{H}_i})$. Substituting (20) in (19) yields

$$\mathbf{p}^{\mathrm{T}} = \rho_i\mathbf{p}^{\mathrm{T}}\mathbf{J}_i + (1-\rho_i)\mathbf{v}_i^{\mathrm{T}} \tag{21}$$

where $\mathbf{J}_i \equiv R_i\mathbf{H}_i$ is semi-positive and $\lambda_{\mathbf{J}_i} = 1$. By letting $\mathbf{K}_i \equiv \hat{\mathbf{y}}_{\mathbf{C}_i}\mathbf{J}_i\hat{\mathbf{y}}_{\mathbf{C}_i}^{-1}$, with $\mathbf{y}_{\mathbf{C}_i}^{\mathrm{T}}[\mathbf{I}-\mathbf{C}_i]\mathbf{x}_{\mathbf{C}_i} = 1$, (21) can be transformed to

$$\boldsymbol{\pi}^{\mathrm{T}} = \rho_i\boldsymbol{\pi}^{\mathrm{T}}\mathbf{K}_i + (1-\rho_i)\boldsymbol{\omega}_i^{\mathrm{T}} \tag{22}$$

or, if $\rho_i < 1$,

$$\boldsymbol{\pi}^{\mathrm{T}} = \boldsymbol{\omega}_i^{\mathrm{T}}[\mathbf{L}_i(\rho_i)]^{-1} \tag{23}$$

where $\boldsymbol{\pi}^{\mathrm{T}} \equiv \mathbf{p}^{\mathrm{T}}\hat{\mathbf{y}}_{\mathbf{C}_i}^{-1}$, $\boldsymbol{\omega}_i^{\mathrm{T}} \equiv \mathbf{v}_i^{\mathrm{T}}\hat{\mathbf{y}}_{\mathbf{C}_i}^{-1}$ and $\mathbf{L}_i(\rho_i) \equiv (1-\rho_i)^{-1}[\mathbf{I}-\rho_i\mathbf{K}_i]$. Since $[\mathbf{L}_i(\rho_i)]^{-1}$ $(\geq \mathbf{0})$ is a column stochastic matrix and the smallest element on the main diagonal of $\mathbf{K}_i$ equals zero, by following the same steps as above (see Section 2) we obtain

$$f_i(\rho_i) \leq \|\boldsymbol{\pi}^{\mathrm{T}}\|/\|\boldsymbol{\omega}_i^{\mathrm{T}}\| \leq 1 \tag{24}$$

and

$$1 \leq \|\hat{\boldsymbol{\omega}}_i^{-1}\|/\|\hat{\boldsymbol{\pi}}^{-1}\| \leq g_i(\rho_i) \leq h_i(\rho_i) \tag{25}$$

where

$$f_i(\rho_i) \equiv (1-\rho_i)/(1+\rho_i)$$

[14] It should be noted that $\mathbf{C}_i$ is a 'Sraffa matrix' ([12, pp. 177-178]) and, therefore, $\mathbf{x}_{\mathbf{C}_i}$ is semi-positive, *i.e.*, its ith element equals zero, whilst the corresponding left-hand-side eigenvector is positive.

$$g_i(\rho_i) \equiv \rho_i(\|\boldsymbol{\pi}^{\mathrm{T}}\|\|\hat{\boldsymbol{\omega}}_i^{-1}\| - 1) + 1$$

and

$$h_i(\rho_i) \equiv \rho_i(\|\boldsymbol{\omega}_i^{\mathrm{T}}\|\|\hat{\boldsymbol{\omega}}_i^{-1}\| - 1) + 1$$

4.2 Joint production and differential profit rates

Let $\mathbf{B}$ denote the $n \times n$ matrix of outputs and let $\hat{\mathbf{r}}$ denote the diagonal matrix of sectoral rate of profits, $r_i \geq 0$. Then, with wages paid *ex post*, the price system becomes

$$\mathbf{p}^{\mathrm{T}}\mathbf{B} = \mathbf{p}^{\mathrm{T}}\mathbf{A}[\mathbf{I} + \hat{\mathbf{r}}] + w\mathbf{l}^{\mathrm{T}} \tag{26}$$

where $\mathbf{p}^{\mathrm{T}}$ now denotes a vector of 'disequilibrium prices'.[15] Provided that (i) $[\mathbf{B} - \mathbf{A}]$ is non-singular; and (ii) r_i exhibit a stable structure in relative terms, which implies that $\hat{\mathbf{r}}$ can be written as $r\hat{\bar{\mathbf{r}}}$, where r now represents the 'overall level' ([28, p. 180]) of the rates of profit, (26) may be written as

$$\mathbf{p}^{\mathrm{T}} = r\mathbf{p}^{\mathrm{T}}\mathbf{H} + w\mathbf{v}^{\mathrm{T}} \tag{27}$$

where $\mathbf{H} \equiv \mathbf{A}\hat{\bar{\mathbf{r}}}[\mathbf{B} - \mathbf{A}]^{-1}$ and $\mathbf{v}^{\mathrm{T}} \equiv \mathbf{l}^{\mathrm{T}}[\mathbf{B} - \mathbf{A}]^{-1}$ now denotes the vector of 'additive labour values' ([27]).

It need hardly be said that, setting aside the well-known 'all-productive (all-engaging)' systems, which are characterized by $[\mathbf{B} - \mathbf{A}]^{-1} \geq (>) \mathbf{0}$ (see [23, pp. 34-36]; [2]), the main difference introduced here is that $\mathbf{H}$ and/or $\mathbf{v}$ can contain one or more *negative* elements (for a general treatment of joint production, see [13, chs 7-9]; [3]). Nevertheless, the system retains *certain* essential properties of our basic system (2) if the following conditions are fulfilled:

(C1) $\mathbf{H}$ has a positive eigenvalue, λ_1, which is simple and greater in modulus that any other eigenvalue, and the corresponding eigenvectors, $(\mathbf{x}_1, \mathbf{y}_1^{\mathrm{T}})$, are positive;

(C2) all the off-diagonal elements of $\mathbf{H}$ are semi-positive; and

(C3) $\mathbf{v}^{\mathrm{T}}\mathbf{x}_1 \neq 0$.

Condition (C1) implies that (i) there exists a positive integer k_0 such that $\mathbf{H}^k > \mathbf{0}$ for all $k \geq k_0$ (see [17, pp. 134-136]); (ii) $\lim_{k\to\infty}[(1/\lambda_1)\mathbf{H}]^k = [1/(\mathbf{y}_1^{\mathrm{T}}\mathbf{x}_1)]\mathbf{x}_1\mathbf{y}_1^{\mathrm{T}}$ (*ibid.*); and (iii)

---

[15] It goes without saying that matrix $\mathbf{B}$ allows both for pure joint products and utilized fixed capital goods (Sraffa-von Neumann approach). Setting aside the pure joint products and writing $\mathbf{B}$ as $\mathbf{I} + \mathbf{A}^{\mathrm{F}} - \mathbf{D}$, where $\mathbf{A}^{\mathrm{F}}$ denotes the matrix of capital-output coefficients and $\mathbf{D}$ the matrix of depreciation-output coefficients, (26) becomes

$$\mathbf{p}^{\mathrm{T}}[\mathbf{I} + \mathbf{A}^{\mathrm{F}} - \mathbf{D}] = \mathbf{p}^{\mathrm{T}}\mathbf{A}^{\mathrm{F}}[\mathbf{I} + \hat{\mathbf{r}}] + \mathbf{p}^{\mathrm{T}}\mathbf{A}[\mathbf{I} + \hat{\mathbf{r}}] + w\mathbf{l}^{\mathrm{T}}$$

or

$$\mathbf{p}^{\mathrm{T}} = \mathbf{p}^{\mathrm{T}}\mathbf{A}^{\mathrm{F}}\hat{\mathbf{r}} + \mathbf{p}^{\mathrm{T}}\mathbf{A}[\mathbf{I} + \hat{\mathbf{r}}] + \mathbf{p}^{\mathrm{T}}\mathbf{D} + w\mathbf{l}^{\mathrm{T}}$$

which is used, more often than not, for the analysis of price-labour value deviations in actual economies (see also [19, pp. 3-4 and 20-21]).

there exists an interval of r, $r > -1$, in which the system behaves as basic single-product systems (see [23, p. 35]; [2, p. 328]). Since $\mathbf{x}_1 > \mathbf{0}$ and $[\mathbf{I} - r\mathbf{H}]\mathbf{x}_1 = (1 - r\lambda_1)\mathbf{x}_1 > \mathbf{0}$ for $r < 1/\lambda_1$, condition (C2) implies that $[\mathbf{I} - r\mathbf{H}]$, $0 \le r < 1/\lambda_1$, is a non-singular *M*-matrix (see, *e.g.*, [1, pp. 391-394]), *i.e.*, $[\mathbf{I} - r\mathbf{H}]^{-1} \ge \mathbf{0}$. Finally, regarding condition (C3), if $\mathbf{v}^{\mathrm{T}}\mathbf{x}_1 = 0$, then the system $[\mathbf{H}, \mathbf{v}^{\mathrm{T}}]$ is 'irregular' *à la* Schefold [23, pp. 11-23]) or, equivalently, 'uncontrollable' *à la* Kalman [11] (for the relationships between these concepts, see [15]).

If conditions (C1), (C2) and (C3) are satisfied, and prices are normalized by setting $\mathbf{p}^{\mathrm{T}}\mathbf{x}_1 = 1$, with $\mathbf{v}^{\mathrm{T}}\mathbf{x}_1 = 1$, then (27) implies that

$$w = 1 - \rho_1 \tag{28}$$

where $\rho_1 \equiv r / R_1$, $0 \le \rho_1 \le 1$, and $R_1 \equiv 1/\lambda_1$. Substituting (28) in (27), and letting $\mathbf{J} \equiv R_1\mathbf{H}$ and $\mathbf{K} \equiv \hat{\mathbf{y}}_1\mathbf{J}\hat{\mathbf{y}}_1^{-1}$, with $\mathbf{y}_1^{\mathrm{T}}\mathbf{x}_1 = 1$, yields

$$\boldsymbol{\pi}^{\mathrm{T}} = \rho_1\boldsymbol{\pi}^{\mathrm{T}}\mathbf{K} + (1 - \rho_1)\boldsymbol{\omega}^{\mathrm{T}} \tag{29}$$

or, if $\rho_1 < 1$,

$$\boldsymbol{\pi}^{\mathrm{T}} = \boldsymbol{\omega}^{\mathrm{T}}[\mathbf{L}(\rho_1)]^{-1} \tag{30}$$

where $\boldsymbol{\pi}^{\mathrm{T}} \equiv \mathbf{p}^{\mathrm{T}}\hat{\mathbf{y}}_1^{-1}$, $\boldsymbol{\omega}^{\mathrm{T}} \equiv \mathbf{v}^{\mathrm{T}}\hat{\mathbf{y}}_1^{-1}$, $\|\mathbf{K}\| \ge 1$,

$$\|\mathbf{I} - \rho_1\mathbf{K}\| = 1 + \rho_1(1 - 2m) \tag{31}$$

$m \equiv \min\{k_{jj}\}$, $|m| < 1$, $\mathbf{L}(\rho_1) \equiv (1 - \rho_1)^{-1}[\mathbf{I} - \rho_1\mathbf{K}]$ and $[\mathbf{L}(\rho_1)]^{-1}$ ($\ge \mathbf{0}$) is a column stochastic matrix. Thus, taking norms, we finally get

$$f(\rho_1) \le \|\boldsymbol{\pi}^{\mathrm{T}}\| / \|\boldsymbol{\omega}^{\mathrm{T}}\| \le 1 \tag{32}$$

$$\|\hat{\boldsymbol{\omega}}^{-1}\| / \|\hat{\boldsymbol{\pi}}^{-1}\| \le \gamma(\rho_1) \le \eta(\rho_1) \tag{33}$$

and, iff $\boldsymbol{\omega} \ge \mathbf{0}$,

$$1 \le \|\hat{\boldsymbol{\omega}}^{-1}\| / \|\hat{\boldsymbol{\pi}}^{-1}\| \tag{34}$$

where

$$f(\rho_1) \equiv (1 - \rho_1)/[1 + \rho_1(1 - 2m)],$$

$$\gamma(\rho_1) \equiv \rho_1(\|\boldsymbol{\pi}^{\mathrm{T}}\|\|\hat{\boldsymbol{\omega}}^{-1}\|\|\mathbf{K}\| - 1) + 1$$

and[16]

$$\eta(\rho_1) \equiv \rho_1(\|\boldsymbol{\omega}^{\mathrm{T}}\|\|\hat{\boldsymbol{\omega}}^{-1}\|\|\mathbf{K}\| - 1) + 1$$

5 Concluding Remarks

It has been shown that, when a production system can be transformed (via a diagonal similarity matrix formed from the elements of the Perron-Frobenius eigenvector) into a vertically integrated system in which the technical coefficients matrix is a stochastic matrix, the largest and the smallest element of the transformed (and normalized with

[16] Two numerical examples presented in the appendix to this paper illustrate the points made above.

Sraffa's Standard commodity) price vector admit norm bounds that depend on the vertically integrated coefficients and the relative rate of profit. Furthermore, it has been stated that in 'actual' single-product systems, where the value of the relative rate of profit is rather low (as many empirical studies show), these bounds are not so weak.

Future work should (i) investigate the possibility of improving these (and/or related) bounds; (ii) delve further into the case of pure joint production; and (iii) concretize the model by including the presence of fixed capital and the degrees of its utilization, taxes-subsidies, and additional primary factors (such as labour of different kinds and non-competitive imports).

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Appendix: Numerical Examples for the Case of Joint Production

A.1. Example 1

Consider a system where *all* the off-diagonal elements of $\mathbf{H}$ are positive and $\mathbf{v} \text{ y } \mathbf{0}$, *i.e.*,

$$\mathbf{H} = \begin{bmatrix} 20 & 1 & 1.1 \\ 1 & -10 & 1 \\ 1 & 1 & -10 \end{bmatrix}, \ \mathbf{v}^{\mathrm{T}} = [20, 10, -1]$$

It is obtained that $\lambda_1 \cong 20.072$, $(\mathbf{x}_1, \mathbf{y}_1) > \mathbf{0}$, $\mathbf{v}^{\mathrm{T}}\mathbf{x}_1 \neq 0$, $\mathbf{H}^k > \mathbf{0}$ for $k \geq 10$,

$$\mathbf{K} \cong \begin{bmatrix} 0.996 & 1.444 & 1.452 \\ 0.002 & -0.498 & 0.046 \\ 0.002 & 0.054 & -0.498 \end{bmatrix}, \ \|\mathbf{K}\| \cong 1.996, \ m \cong -0.498$$

$\boldsymbol{\omega}^{\mathrm{T}} \cong [0.987, 14.304, -1.308]$, $\|\mathbf{I} - \rho_1\mathbf{K}\| \cong 1 + \rho_1 1.996$ and $[\mathbf{L}(\rho_1)]^{-1} \geq \mathbf{0}$ for $0 \leq \rho_1 < 1$. Moreover, (i) $\|\boldsymbol{\pi}^{\mathrm{T}}\| = \pi_2$ decreases with increasing ρ_1, whilst π_1 and π_3 increase, and $\pi_3 > 0$ for $\rho_1 > \rho_1^* \cong 0.4242$ (see Figure A.1.1); and (ii) $1/\|\hat{\boldsymbol{\pi}}^{-1}\| = \pi_1$ for $0 \leq \rho_1 < \rho_1^{**} \cong 0.0837$ and $1/\|\hat{\boldsymbol{\pi}}^{-1}\| = |\pi_3|$ for $\rho_1^{**} < \rho_1 < 1$. Finally, Figures A.1.2 and A.1.3 represent relations (32) and (33), respectively.

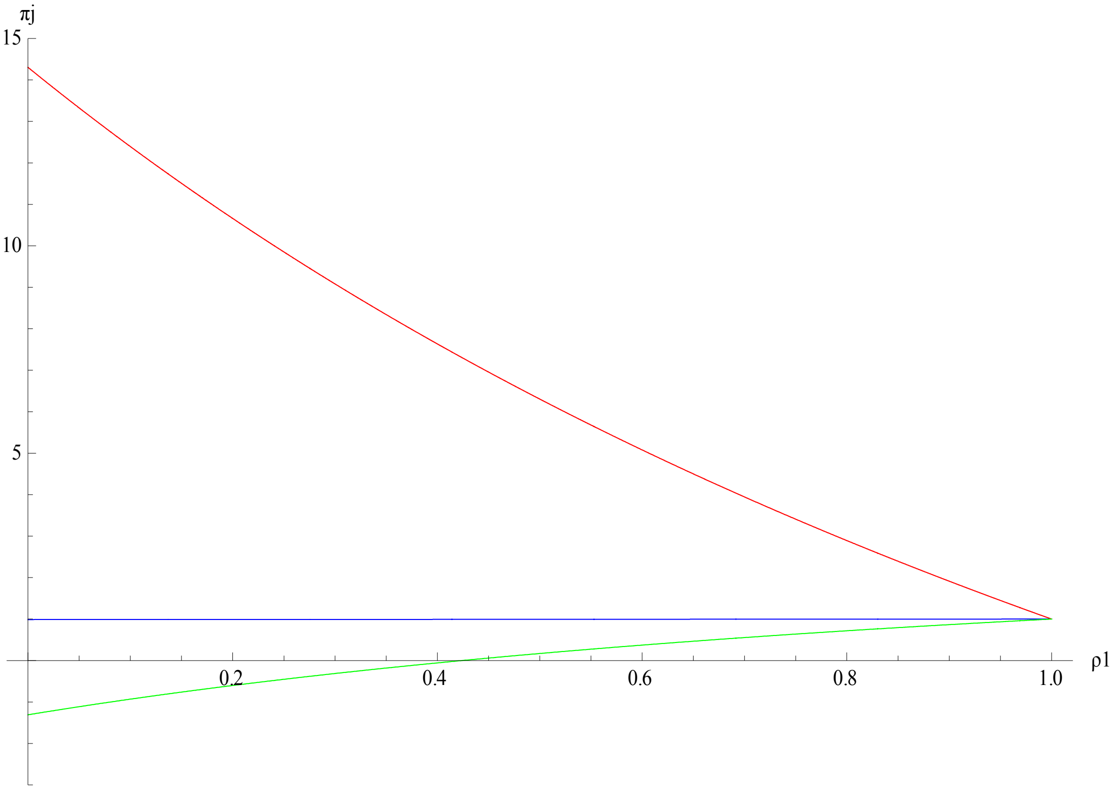

Figure A.1.1. Transformed prices of production as functions of the relative rate of profit

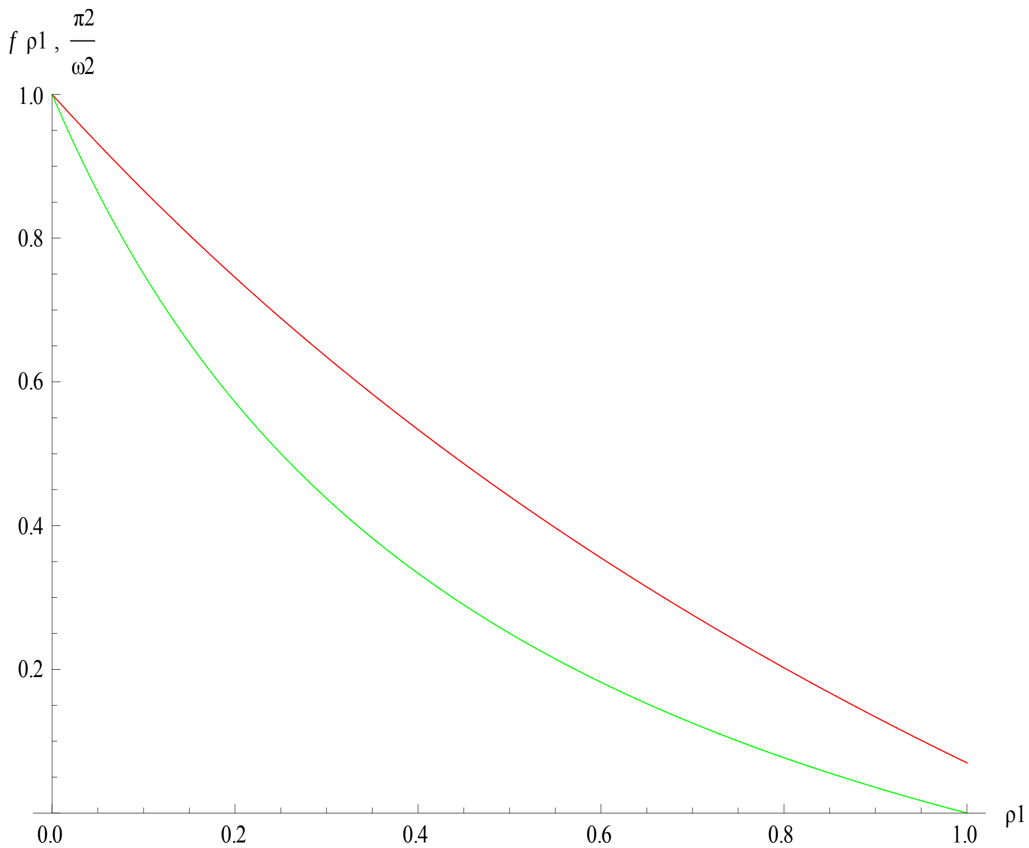

Figure A.1.2. $\left\|\boldsymbol{\pi}^{\mathrm{T}}\right\| / \left\|\boldsymbol{\omega}^{\mathrm{T}}\right\|$ and its lower bound

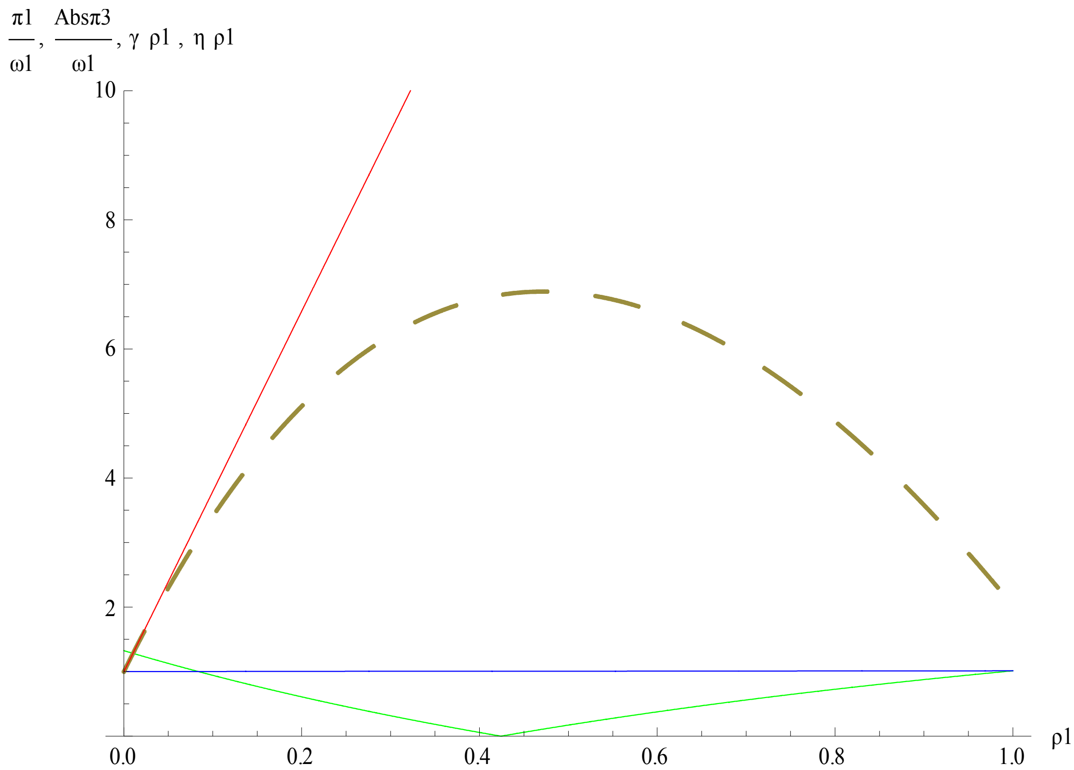

Figure A.1.3. $\|\hat{\boldsymbol{\omega}}^{-1}\| / \|\hat{\boldsymbol{\pi}}^{-1}\|$ and their upper bounds

A.2. Example 2

Now consider a system where $\mathbf{H}$ has a *negative* off-diagonal element and $\mathbf{v} > \mathbf{0}$, *i.e.*,

$$\mathbf{H} = \begin{bmatrix} -1 & 1 & -1 \\ 10 & 8 & 12 \\ 1 & 1 & -1 \end{bmatrix}, \ \mathbf{v}^{\mathrm{T}} = [30, 0.2, 0.3]$$

It is obtained that $\lambda_1 = 10$, $(\mathbf{x}_1, \mathbf{y}_1) > \mathbf{0}$, $\mathbf{H}^k > \mathbf{0}$ for $k \geq 2$, $\mathbf{K} = \mathbf{H}/10$, $\|\mathbf{K}\| = 1.4$,

$$\boldsymbol{\omega}^{\mathrm{T}} \cong [13.171, 0.088, 0.132]$$

$$1 - \rho_1 \leq \|\mathbf{I} - \rho_1 \mathbf{K}\| = 1 + \rho_1 1.4 \leq 1 - \rho_1 + \rho_1 2M \ , \ M \equiv \max\{\sum_{\substack{i=1 \\ i \neq j}}^{n} |k_{ij}|\} = 1.3$$

and $[\mathbf{L}(\rho_1)]^{-1} > \mathbf{0}$ (and, therefore, $\|[\mathbf{L}(\rho_1)]^{-1}\| = 1$) for $0.5 < \rho_1 < 1$. Moreover, (i) $\|\boldsymbol{\pi}^{\mathrm{T}}\| = \pi_1$ decreases with increasing ρ_1, π_2 (> 0) increases, and π_3 (which is positive for $0 \leq \rho_1 \leq \rho_1^* \cong 0.141$ and $\rho_1^{**} \cong 0.359 \leq \rho_1 \leq 1$) increases for $\rho_1 \geq \bar{\rho}_1 \cong 0.247$ (see Figure A.2.1); and (ii) $1/\|\hat{\boldsymbol{\pi}}^{-1}\| = \pi_2$ ($1/\|\hat{\boldsymbol{\pi}}^{-1}\| = |\pi_3|$) for $0 \leq \rho_1 < \rho_1^{***} \cong 0.017$ (for $\rho_1^{***} < \rho_1 < 1$). Finally, taking norms, we deduce that (i) $\phi(\rho_1) \leq \|\boldsymbol{\pi}^{\mathrm{T}}\| / \|\boldsymbol{\omega}^{\mathrm{T}}\|$ (compare with relation (32)), where

$$\phi(\rho_1) \equiv (1-\rho_1)/[1-\rho_1(1-2M)]$$

and relation (33) hold true; and (ii) $\|\boldsymbol{\pi}^{\mathrm{T}}\|/\|\boldsymbol{\omega}^{\mathrm{T}}\|<1$ (see relation (32)) and $1<\|\hat{\boldsymbol{\omega}}^{-1}\|/\|\hat{\boldsymbol{\pi}}^{-1}\|$ (see relation (34)), for $0.5<\rho_1 \le 1$. Numerical calculations show, however, that the former relation holds true for $0<\rho_1 \le 1$ (Figure A.2.2 displays $\|\boldsymbol{\pi}^{\mathrm{T}}\|/\|\boldsymbol{\omega}^{\mathrm{T}}\|$ and $\phi(\rho_1)$), whilst the latter holds true for $0<\rho_1<\tilde{\rho}_1 \cong 0.035$ and $\tilde{\tilde{\rho}}_1 \cong 0.478<\rho_1 \le 1$ (Figure A.2.3 displays $\|\hat{\boldsymbol{\omega}}^{-1}\|/\|\hat{\boldsymbol{\pi}}^{-1}\|$).

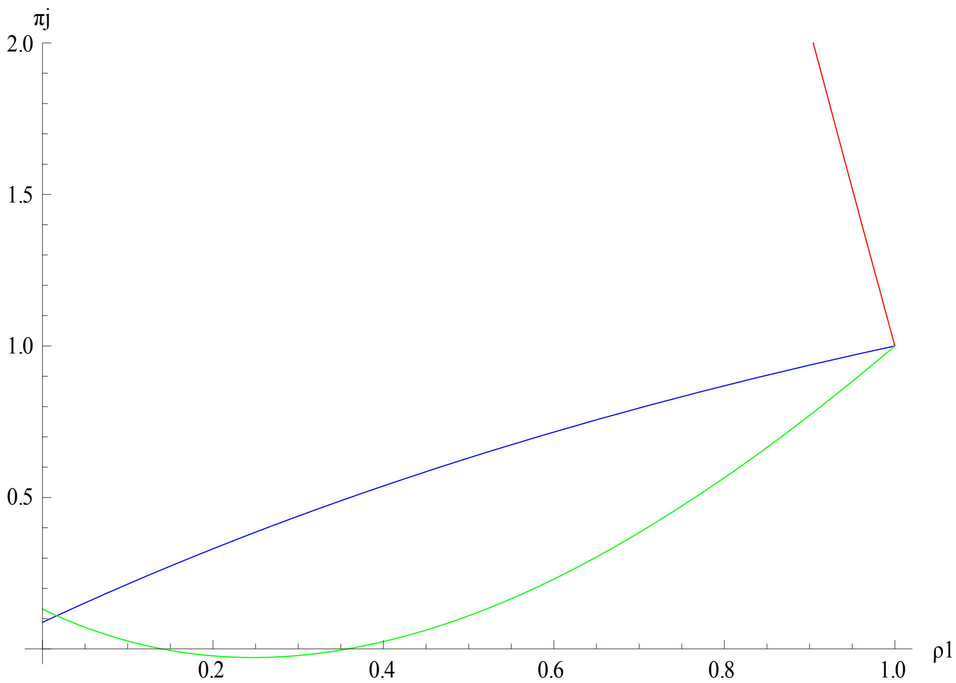

Figure A.2.1. Transformed prices of production as functions of the relative rate of profit

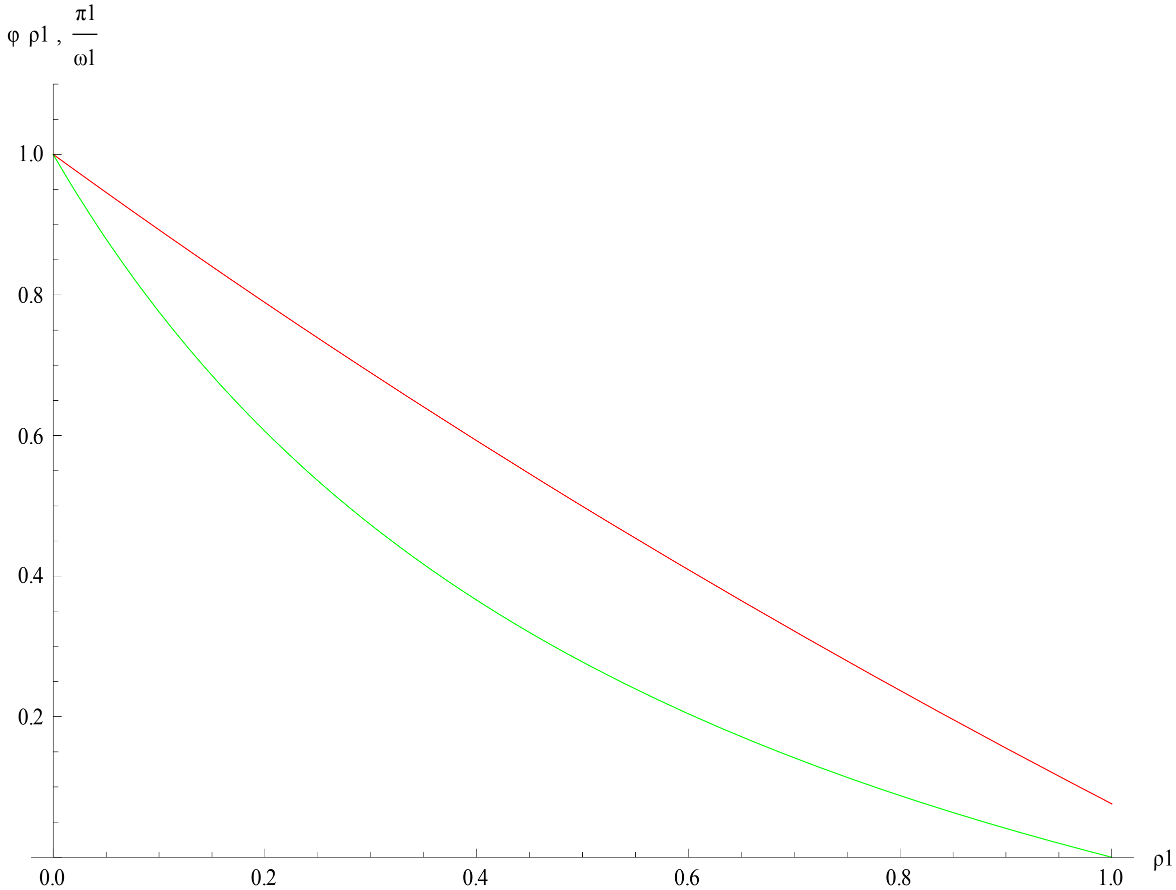

Figure A.2.2. $\|\boldsymbol{\pi}^{T}\| / \|\boldsymbol{\omega}^{T}\|$ and its lower bound

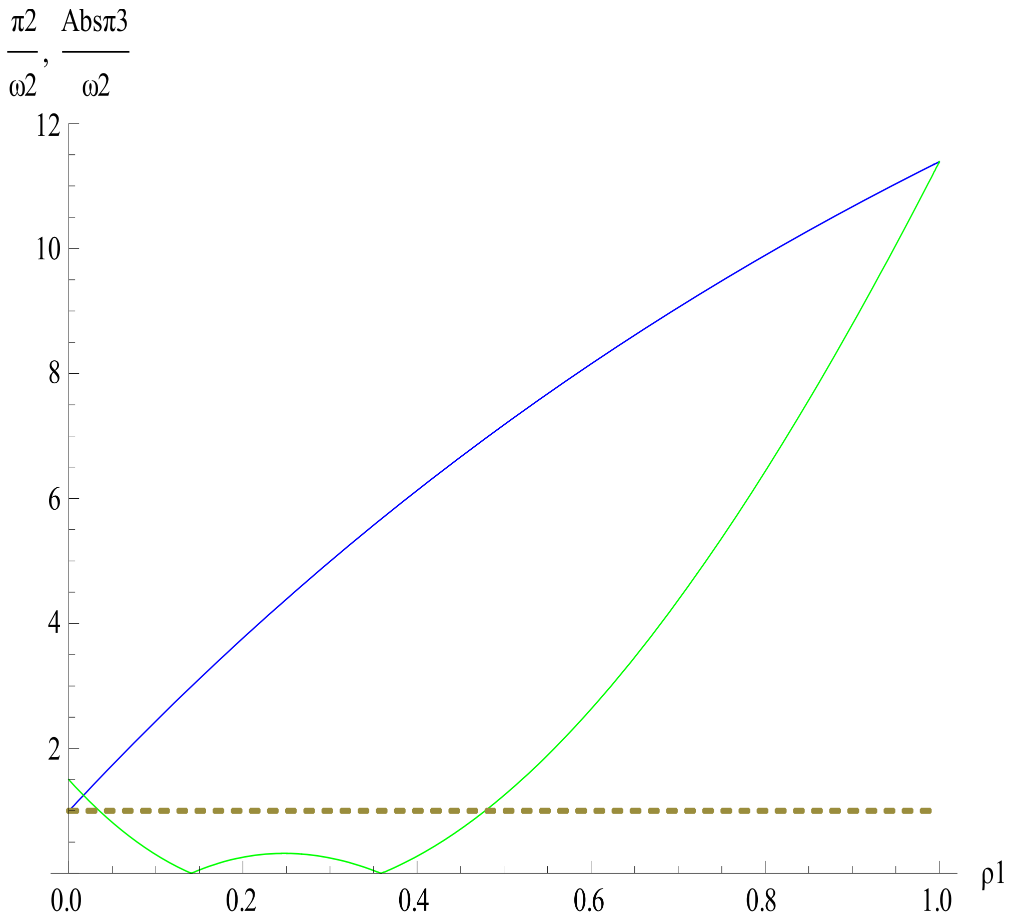

Figure A.2.3. $\|\hat{\boldsymbol{\omega}}^{-1}\| / \|\hat{\boldsymbol{\pi}}^{-1}\|$ as functions of the relative rate of profit

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