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Differential Transform Solution of Some Linear Wave Equations with Mixed Nonlinear Boundary Conditions and its Blow up

Ümmügülsüm CANSU

Department of Mathematics, Rize University
53050, Campus, Rize, Turkey
ummugulsumcansu@yahoo.com

Ozan ÖZKAN

Department of Mathematics, Selcuk University
42031 Campus, Konya, Turkey
oozkan@selcuk.edu.tr

Abstract

Using Differential Transform to solve blow up solutions of some linear wave equation with mixed non-linear boundary conditions is proposed in this study.Non-linear boundary conditions cause the finite time blow up in solutions even if the inital data is smooth and well-defined for all times.

In this article, Differential Transform (DT) method solutions of some linear wave equation with mixed nonlinear boundary conditions , its blow up and blow up of the energy equations are investigated.

Mathematics Subject Classifications: 35C10, 35L05, 74S30

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1 Introduction

The finite time blow up solutions of linear or non-linear partial differential equations have been the subject of analytical and numerical studies [2,3,4,8,9,11].

The behaviour of the blow up solution are investigated comprehensively by lots of researchers [7,10,11].In some sufficient conditions blow up occurs in a finite time is obtained from these investigations. Thanks to them, Budd[4] achieved to describe simple cellular aggregation and self-focusing in lasers and described combustion models, have solutions which form strong singularities

or blow up in a finite time. A point x_0 is called blow up point if there exists sequence (x_n, t_n) such that

$$x_n \to x_0, \qquad t_n \to T, \qquad u(x_n, t_n) \to \infty, \qquad as\ n \to \infty \tag{1}$$

In [6], the wave equation with non-linear damping and source terms, on a bounded domain Ω of R^n :

$$u_{tt} - \Delta u + au_t \left|u_t\right|^{n-1} = bu \left|u\right|^{p-1}, \qquad x \in \Omega, \qquad t > 0 \tag{2}$$

with the conditions

$$\begin{aligned} u(x,t) &= 0, \quad x \in \partial\Omega, \quad t > 0 \\ u(x,t) &= u_0(x), \quad u_t(x,0) = u_1(x), \qquad x \in \Omega \end{aligned} \tag{3}$$

where $a, b > 0$, $p, m > 1$ has been studied. The interaction between the damping and source terms [8,9] was examined and a global existence for $p > m$ and a blow up result for $p < m$ were established. Moreover, it was formally shown that the source term causes a finite time blow-up for $a = 0$, and the nonlinear damping term develops global existence for small initial data for $b = 0$.

Budd et al.[4], dealed with the semilinear parabolic equation for combustion theory

$$u_t = u_{xx} + f(u) \qquad for \quad (x,t) \in (-L, L) \times R_+ \tag{4}$$

with the Dirichlet boundary and initial equations

$$\begin{aligned} u(-L,t) &= u(L,t) = 0, \quad t > 0, \\ u(x,0) &= u_0(x) \geq 0, \quad in\ (-L, L). \end{aligned} \tag{5}$$

The function $f(u) > 0$ for $u > 0$ is superlinear term for $u \gg 1$ and satisfies the well-known necessary blow-up condition

$$\int_1^\infty \frac{du}{f(u)} < \infty. \tag{6}$$

The two important non-linear functions

$$f(u) = e^u, \qquad \text{and} \qquad f(u) = u^n, \qquad n > 1 \tag{7}$$

were studied in [4].The blow up behaviour of two reaction-diffussion problems and quasilinear degenerate diffusion was investigated comprehensively.

Bandle[3], provided a useful survey on blow-up for diffussion equations. Thanks to that survey, well-known algorithms and results were compiled and provided to pay attention to some open problems. Moreover, the main goal of the survey was to highlight important results and tools in the analysis of the blow-up. Recently, Messaoudi[8,9] studied the following linear partial differential equation

$$u_{tt} = u_{xx}, \qquad x \in I, \qquad t > 0 \tag{8}$$

with mixed nonlinear boundary conditions

$$\begin{aligned} u_x(0,t) &= |u(0,t)|^\alpha \; u(0,t), \qquad u_x(1,t) = |u(1,t)|^\alpha u(1,t), \;\; t > 0 \quad (9)\\ u(x,0) &= f(x), \qquad u_t(x,0) = g(x) \qquad x \in I \quad for \;\; \alpha > 0. \end{aligned}$$

A local existence theorem was established by Messaoudi [8,9] and for suitable chosen initial conditions, it was shown that the solution blows-up in a finite time.

In [8], if $f(x) \in H^2(I)$ and $g(x) \in H^1(I)$ be given satisfying

$$E_0 = \int_0^1 (g^2(x) + (f'(x))^2)dx - \frac{2}{\alpha+2}(|f(1)|^{\alpha+2} - |f(0)|^{\alpha+2}) < 0 \tag{10}$$

then any solution of Eq.(2) blows-up in a finite time. For more details about the proof of the local existence theorem and blow-up solutions see [8,9].At last, Wazwaz solved Messaoudi's blow-up problem with Adomian Decomposition method [11].

The method that we used here is the differential transform method (DT) [1,5,7,10]. It gives exact values of the nth derivative of an analytical function at a point in terms of initial or boundary conditions in a fast manner. This method transforms the initial and boundary value problems for integro-differential equations into a algebraic equations which can be solved by simple systematic procedures. Furthermore, the accuracy of approximate solution can be improved by using a varying grid size. The grid-size adaptation of the differential transformation keeps the truncation error within specified bounds, since it is possible to adapt the number and position of the grid points [10].

In this study solutions of the linear partial differential equations with mixed non-linear boundary condition is obtained with a different method. DT which is based on Taylor series expansion is used to get solution of the Eq.(2).Unknown solution function $u(x,t)$ is found with a series solution by DT method. Unlike the other method which requires boundary and initial conditions as the decomposition method, DT method provides analytic solution by only using initial data. The boundary conditions can be used to control accuracy of the obtained solution.

As proven by Budd[4] and Wazwaz[11] to show the peak of the solution and energy functional will be investigated numerically.

2 Two-Dimensional Differential Transform Method

In this section; in order to solve partial IDE by two- dimensional differential transform its basic theory is stated in [1,5,7] as follows:

We define two dimensional differential transform of the (k,h)th derivative of bi variate function $f(x,y)$ in (x_0,y_0) as

$$W(x,y) = \frac{1}{k!h!}[\frac{\partial^{k+h} w(x,y)}{\partial x^k \partial y^h}]_{\substack{x=x_0 \\ y=y_0}} \tag{11}$$

then the differential inverse transform of $W(k,h)$ is defined as follows:

$$w(x,y) = \sum_{k=0}^{\infty}\sum_{h=0}^{\infty} W(k,h)(x-x_0)^k(y-y_0)^h \tag{12}$$

The relations (11) vand (12) imply that

$$w(x,y) = \sum_{k=0}^{\infty}\sum_{h=0}^{\infty} \frac{1}{k!h!}[\frac{\partial^{k+h} w(x,y)}{\partial x^k \partial y^h}]_{\substack{x=x_0 \\ y=y_0}}(x-x_0)^k(y-y_0)^h \tag{13}$$

which is the Taylor series of bivariate function $f(x,t)$.With (11) and (12),the fundamental mathematical operations given by the two-dimensional differential transform can be obtained and listed in Table 1[1,5].

The terms which is required arbitrary can be obtained for unknown solution with DT method. For the numerical solutions, by increasingly the number of terms calculated it can be approached to the solutions with less roarers. To turn given problem to algebraic ones and the algorithm used to solve the algebraic problem is quietly simple and feasible in computer environment are

Table 1: Some operations of the two dimensional differential transformation.

Original function	*Transformed function*
$w(x,y)=u(x,y)\pm v(x,y)$	$W(k,h)=U(k,h)\pm V(k,h)$
$w(x,y)=\alpha u(x,y)$	$W(k,h)=\alpha U(k,h) \qquad (\alpha = cons\tan t)$
$w(x,y)=e^{\lambda(x+y)}$	$W(k,h)=\frac{\lambda^{(k+h)}}{k!h!}$
$w(x,y)=u(x,y)v(x,y)$	$W(k,h)=\sum_{r=0}^{k}\sum_{s=0}^{h}U(r,h-s)V(k-r,s)$
$w(x,y)=\frac{\partial^{r+s}}{\partial x^r \partial y^s}u(x,y)$	$W(k,h)=\frac{(k+r)!(h+s)!}{r!s!}U(k+r,h+s)$
$w(x,y)=x^m y^n$	$W(k,h)=\delta(k-m,h-n)=\begin{cases}1 & k=m \ and \ h=n \\ 0 & otherwise\end{cases}$

the most feasible feature of DT method. Furthermore it demonstrates unknown solution as series solution. Another remarkable treat of DT method is getting series solution only depends on initial data so that, difficulty of using boundary condition disappears.The boundary conditions can be used only to justify the obtained result. This is the most significant feature of DT method.

Two conclusion can be made here. The pointwise blow-up of the solution can be examined closely through the obtained series solution. Second is the following definition introduced from Messaoudi [8,9] about the energy functional.

$$F(t)=\frac{1}{2}\int_0^1 u^2(x,t)dx+\frac{1}{2}\beta(t+t_0)^2, \qquad t>0 \tag{14}$$

or $t_0 > 0$, $\beta > 0$ where both will be chosen so small. Having determined $u(x,t)$, the blow-up in energy solution can be examined by using $F(t)$ (14).To show these two conclusions the example introduced in [8,9] will be investigated.

3 Numerical Experiments

Example

In this example we consider the linear wave equation which also solved in the study of Wazwaz[11]:

$$u_{tt}=u_{xx}, \qquad x\in(0,1), \qquad t>0 \tag{15}$$

with the mixed non-linear boundary conditions

$$\begin{aligned} u_x(x,t) &= |u(x,t)|^{1/3}\, u(x,t), \qquad x=0,1, \quad t>0, \\ u(x,0) &= 27(2-x)^{-3}, \qquad u_t(x,0)=81(2-x)^{-4}. \end{aligned} \tag{16}$$

Taking the two dimensional differential transform Eq.(15) by related theorems we get ,

$$(k+1)(k+2)U(k+2,h) = (h+1)(h+2)U(k,h+2) \tag{17}$$

and by applying the DT to initial conditions Eq.(16) is obtained as follows:

$$U(k,0) = 27\frac{(k+2)!2^{-k-3}}{2!k!} \qquad U(k,1) = 81\frac{(k+2)!2^{-k-4}}{3!k!} \tag{18}$$

By using (18) into (17) recurrence equation, the following series solution is obtained up to$n = 3$

$$U(k,h) = \frac{27}{8} + \frac{81}{16}x + \frac{81}{16}x^2 + \frac{81}{16}t + \frac{81}{8}tx + \frac{81}{16}t^2 + \frac{135}{32}x^3 + ... \tag{19}$$

for $n \to \infty$. It can be seen that the expansion of the

$$U(x,t) = 27(2-x-t)^{-3} \tag{20}$$

This result is full agreement with the result obtained in [8,11].It obvious that the given boundary conditions justify the exact solution (20).As mentioned in [4,8,11] to compute the energy functional, Eq.(20) substituted into (14)

$$F(t) = \frac{729}{5}\left(\frac{1}{(1-t)^5} - \frac{1}{(2-t)^5}\right) \tag{21}$$

is obtained. We easily see that the blow-up occurs at $t = 1 < t_m = 2.362$ where t_m defined as upper bound of blow-up time[3].All calculations in this paper is carried out in Maple environment.

4 Blow-Up

Substituting the initial condition in (16) into the hypothesis (10) indicates the condition for a finite time blow- up in the solution (20) holds. When (21) is examined carefully,it is seen that the blow-up of the energy functional occurs in t=1.It is obvious that, $x \to 1$ and $t \to 1$,then (19) exact solution blows-up. This means as

$$x \to 1 \qquad \text{and} \qquad t \to 1, \quad u(1,1) \to \infty \tag{22}$$

Fig.1 blows-up shows the pointwise blow-up in the solution $u(x,t)$.It was demonstrated by Budd et al.[1] and it can be seen that the solution $u(x,t)$ peak is increasingly narrow.However, Fig.2 shows the blow-up in the energy solution (21).

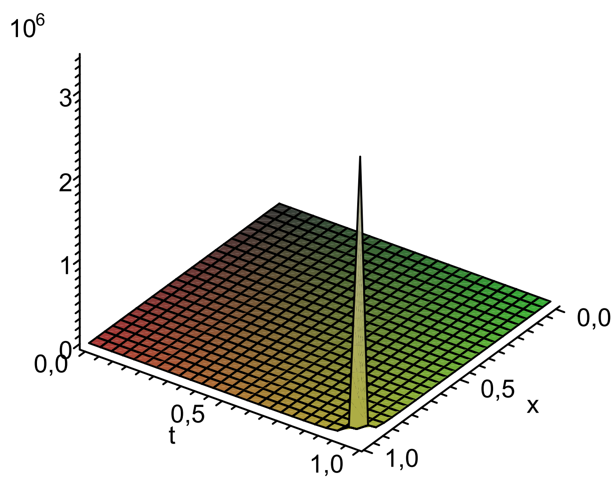

Figure 1: The point-wise blow-up in the exact solution u(x,t)

5 Conclusion

In this work, it was investigated that blow-up in energy solution and DT solution of linear wave equation with mixed nonlinear boundary conditions. DT method provided to get solution without require boundary conditions. The point- wise blow-up and energy function was investigated. Results are full agreement with other methods used in [8,9,11].However, obtained results are also presented as graphics.

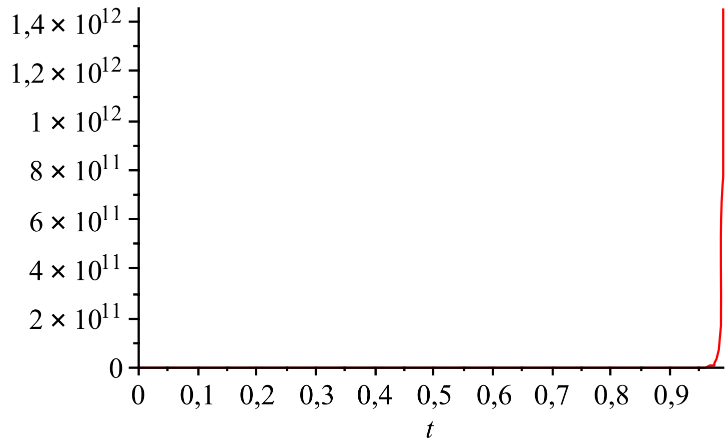

Figure 2: The blow-up in the energy solution F(t)

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