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Energy Minimizing States in Adhesion Problems for Elastic Rods

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Abstract

In this paper we analyze and compare two different models for adhesion phenomena, recently proposed by the authors. In the first approach [9] a feasible expression of the adhesion energy is suggested by the existence problem of partially detached equilibrium states. In the second model [10] the macroscopic energy is obtained by performing a multiscale analysis and it is deduced via a macroscopic Γ-limit of the energy at the scale of the microstructure. Interestingly, we obtain that the first model can be deduced by the second one as the limit case when the parameter measuring the relative stiffness of the adhesive layer and the beam diverges.

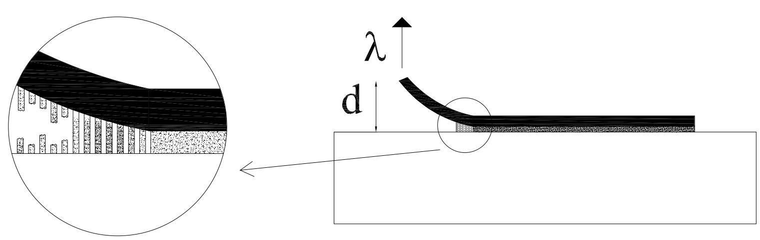
Mathematics Subject Classification: 49J45, 49S05, 74K10

Keywords: Adhesion, Calculus of Variations, Gamma Convergence

1 Introduction

The growing interest in models for adhesion and decohesion phenomena (see [8] and references therein) is due to their important applications in traditional problems of mechanics, such as peeling, cavitation and decohesion behaviors in rubberlike and composite materials, pull out, gecko's fingers, denaturation of DNA, etc., and in modern technological applications such as MEMS and nanotechnologies (see [5],[7],[12], [13]). From the theoretical point of view (see [3], [4], [9],[14]) the difficulties result from the history dependent behavior and the complex energetic exchanges from the multiple involved physical scales. In particular, one of the main parameter regulating the macroscopic behavior is represented by the relative strength of the glue or detaching layer and the connected bodies. The variation of this parameter may result in a dramatic change of the macroscopic response with a transition from ductile to fragile behavior and from diffuse to localized decohesion fronts.

In this paper we compare two different approaches that we recently proposed to study the debonding problem of an elastic beam glued to a rigid substrate and subjected to a force at one of its end-points. In the first model [9] we consider a Bernoulli-Navier linearly elastic beam with a (Griffith type) decohesion energy depending on the measure of the detached zone. In the second model we consider the behavior of a discrete lattice of massless points connected by shear springs and by breakable links to a rigid layer. Through a Γ-convergence procedure (see [1]) we then deduce the macroscopic limit system.



In this work we show that the behavior of the continuous model endowed with the Griffith like energy can be deduced by the Γ-limit system of the discrete model, when the stiffness of the glue layer grows as compared with respect to the stiffness of the beam.

This study describes a possible systematic approach to deduce, starting from the properties at the micro scale, different macroscopic decohesion models, taking care of the crucial role played by the relative stiffness of the adhesion and adhesive layers. In particular we show how through the Γ-limit of our discrete model we may obtain both Griffith type ([6]) and cohesive type energies. This result can also deliver an important tool in analyzing the regularity problems associated to these kinds of energies (see [2] and [11]).

In Section 2 we synthetically expose the results obtained in [9] for a flexural adhering beam. In Section 3 these results are extended to the case of a shear beam. In Section 4 we summarize the results obtained in [10] for a discrete lattice of shear springs and its Γ-limit. Finally in Section 5 we show how the first model can be deduced from the second one as a limit case when the relative stiffness parameter diverges.

2 Peeling of a linear elastic bending beam

Let $u : [0, L] \to \mathbb{R}$ be the vertical displacement of a linear elastic beam endowed, in the Bernoulli-Navier hypothesis, with the elastic energy

$$W_e(u) = \frac{1}{2}\int_0^L k_{\mathrm{B}|u''|^2 dx.} \tag{2.1}$$

Let

$$\Sigma_u = \{x \in [0, L] \, |u(x) > 0\}.$$

We define the *adhesion potential*

$$W_a(u) = \vartheta(L^{-1}|\Sigma_u|), \tag{2.2}$$

where $|\Sigma_u|$ is the Lebesgue measure of Σ_u and $\vartheta : \mathbb{R}_+ \to \mathbb{R}$ is continuous and strictly increasing, with $\vartheta(0) = 0$. We assume the beam is subjected to a given load f applied at the right end $x = L$. Then, the load potential is given by

$$W_l(u) = -fu(L). \tag{2.3}$$

Therefore, the total potential energy is

$$\mathcal{F}(u) = W_e(u) + W_a(u) + W_l(u)$$

and so we are led to study the following variational problem:

$$\mathcal{F}(u) \to \min$$

$$u \in \mathcal{A} = \{u \in H^2(0, L) \, |u(0) = 0, \;\; u(x) \geq 0\}.$$

2.1 Existence and structure of the minimizers

Theorem 2.1. *$\mathcal{F}$ admits a minimum in $\mathcal{A}$. Moreover, let $u \in \operatorname{argmin} \mathcal{F}$, then there exists $\xi \in [0, L]$ such that $\Sigma_u = [L-\xi, L]$. In particular, we have $u'(L-\xi) = 0$ and $u(x) = 0$ in $[0, L-\xi]$.*

Theorem 2.2. *Let $u \in \operatorname{argmin} \mathcal{F}$ then*

$$u(x) = \frac{f}{k_B}\left(x - (L-\bar{\xi})\right)^2 \left(-\frac{x}{6} + \frac{L}{2} - \frac{L-\bar{\xi}}{3}\right) \mathbf{1}_{[L-\bar{\xi},L]}(x) \tag{2.4}$$

where $\bar{\xi}$ is a global minimizer of

$$F(\xi) = \vartheta(\xi) - \frac{f^2 \xi^3}{6k_B} \tag{2.5}$$

in $[0, L]$.

By introducing the normalized variable $\zeta = \left(\frac{\xi}{L}\right)$, (2.5) becomes

$$G(\zeta) = \vartheta(\zeta) - \frac{f^2 L^3}{6k_B}\zeta^3$$

and so the detachment region is determined by solving the problem

$$\min_{\zeta \in [0,1]} G(\zeta).$$

2.2 Growth conditions and stability analysis

Different growth assumptions on ϑ give rise to different constitutive properties of the peeling model. The stability of the equilibrium configurations is characterized by the growth properties of ϑ.

Proposition 2.3. *Let $\alpha \in]0, 3]$, we assume that for every $\zeta \in [0, 1]$*

$$\vartheta(\zeta) = c|\zeta|^\alpha,$$

for some constant $c > 0$. We set $f_{cr} = \sqrt{\frac{6ck_B}{L^3}}$. Then, for $0 \le f < f_{cr}$, $G(\zeta)$ achieves the minimum at $\zeta = 0$ and for $f > f_{cr}$ $G(\zeta)$ achieves the minimum at $\zeta = 1$.

Proposition 2.4. *Let $\alpha > 3$, we assume that for every $\zeta \in [0, 1]$*

$$\vartheta(\zeta) = c|\zeta|^\alpha,$$

for some constant $c > 0$. We set $f_{cr}^\alpha = \sqrt{\frac{2\alpha c k_B}{L^3}}$. Then, for every $0 < f < f_{cr}^\alpha$ there exists a unique $\bar{\zeta} \in]0, 1[$ such that $G(\zeta)$ achieves the minimum at $\bar{\zeta}$. For every $f \ge f_{cr}^\alpha$, $G(\zeta)$ achieves the minimum at $\zeta = 1$.

By Proposition 2.3 we can deduce that for $1 < \alpha < 3$, if ϑ is such that $c_1\zeta^\alpha < \vartheta(\zeta) < c_2\zeta^\alpha$, for some $0 < c_1 < c_2$, then for $0 \leq f < \sqrt{\frac{6c_1k_B}{L^3}}$ the minimum of G is attained at $\zeta = 0$.

According with previous results we may have two different possible regimes of adhesion behavior.

- *Slow Growth* $\alpha \in]0, 3]$

  In this case, there is no stable partially detached equilibrium solution and the system exhibits a *catastrophe-like* behavior, i.e. it undergoes a bifurcation between the totally attached to the totally debonded configurations.

- *Fast Growth* $\alpha > 3$

  In this case the free boundary ξ is a continuous and increasing function of f and the debonding front quasi-statically evolves as the force is increased.

3 Peeling of a linear elastic shear beam

In this section we extend previous results to the case when a shear beam is considered. The same arguments used in [9] lead to prove the corresponding results stated in this section. Under this hypothesis the elastic energy is given by

$$W_e(u) = \frac{1}{2}\int_0^L k_{\mathrm{S}|u'|^2 dx}. \tag{3.1}$$

As previous Theorem 2.2 we have the following result

Theorem 3.1. *Let* $u \in \operatorname{argmin} \mathcal{F}$ *then*

$$u(x) = \frac{f}{k_S}\left(x - (L - \bar{\xi})\right) \mathbf{1}_{[L-\bar{\xi},L]}(x) \tag{3.2}$$

where $\bar{\xi}$ *is a global minimizer of*

$$F(\xi) = \vartheta\,(\xi) - \frac{f^2\xi}{2k_S} \tag{3.3}$$

in $[0, L]$.

By introducing again the normalized variable $\zeta = \frac{\xi}{L}$, (3.3) becomes

$$G(\zeta) = \vartheta(\zeta) - \frac{f^2 L}{2k_S}\zeta \tag{3.4}$$

and so the detachment region is determined by solving the problem

$$\min_{\zeta\in[0,1]} G(\zeta).$$

By following the same steps followed in [9] we obtain the following results.

Proposition 3.2. *Let $\alpha \in]0,1]$, we assume that for every $\zeta \in [0,1]$*

$$\vartheta(\zeta) = c|\zeta|^\alpha,$$

for some constant $c > 0$. We set $f_{cr} = \sqrt{\frac{2ck_S}{L}}$. Then, for $0 \le f < f_{cr}$, $G(\zeta)$ achieves the minimum at $\zeta = 0$ and for $f > f_{cr}$ $G(\zeta)$ achieves the minimum at $\zeta = 1$.

Proposition 3.3. *Let $\alpha > 1$, we assume that for every $\zeta \in [0,1]$*

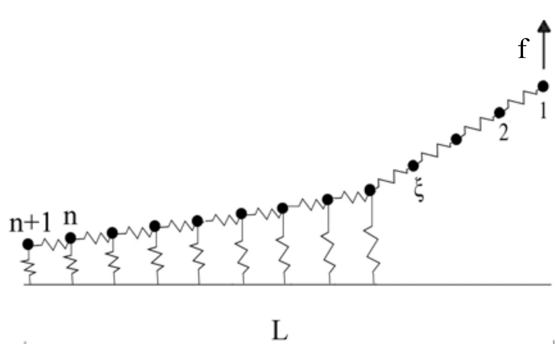
$$\vartheta(\zeta) = c|\zeta|^\alpha,$$

for some constant $c > 0$. We set $f^\alpha_{cr} = \sqrt{\frac{2\alpha ck_S}{L}}$. Then, for every $0 < f < f^\alpha_{cr}$ there exists a unique $\bar\zeta \in]0,1[$ such that $G(\zeta)$ achieves the minimum at $\bar\zeta$. For every $f \ge f^\alpha_{cr}$, $G(\zeta)$ achieves the minimum at $\zeta = 1$.

We obtain, thus, the same conclusion regarding the interplay between growth assumptions and stability as in the previous section, after substituting $\alpha = 3$ with $\alpha = 1$.

4 Multiscale elastic model

To the aim of studying how *microscopic* and *macroscopic* responses affect the adhesion phenomena, here we introduce a simple discrete model constituted by a chain of $n + 1$ massless points at a distance $l = L/n$.



Let u_i be the displacement orthogonal to the chain of the i-th point. Each point interacts with the rigid substrate through a breakable elastic spring, with u_r the breaking displacement, endowed with the following potential energy:

$$\varphi(w_i) = \begin{cases} \frac{1}{2}Ew_i^2 & \text{if } w_i < 1 \\ \\ \frac{1}{2}E & \text{if } w_i \geq 1, \end{cases}$$

where we used the normalized variables $w_i = \frac{u_i}{u_r}$. We assume the points are connected by n linear shear spring whose potential energy is

$$\phi(\delta_i) = \frac{1}{2}k_S\,\delta_i^2,$$

where

$$\delta_i = \frac{u_{i+1} - u_i}{l}.$$

Then the total averaged potential energy of the system takes the form

$$\Phi_n = \frac{1}{n}\left[\sum_{i=1}^{n+1}\varphi(w_i) + \sum_{i=1}^{n}\phi(\delta_i)\right]. \tag{4.1}$$

In this perspective the detachment region is given by the discrete set constituted by the debonded springs identified through the internal variables

$$\chi^f(i) = \begin{cases} 0 & \text{if } w_i < 1 \\ \\ 1 & \text{if } w_i \geq 1. \end{cases}$$

After setting

$$\mathbf{D} = \text{diag}(\chi^f(i), \ldots, \chi^f(n+1)) \text{ and } \quad \nu^2 = \frac{u_r^2\,k_S}{L^2\,E},$$

where ν represents our fundamental parameter, measuring the relative stiffness of the adhesion and shear chain, we can write the total energy (4.1) in the following compact form

$$\Phi = \frac{E}{2n}(\mathbf{B}_e\mathbf{w}\cdot\mathbf{w} + \xi), \tag{4.2}$$

where

$$\mathbf{B}_e = \mathbf{I} - \mathbf{D} + n^2\nu^2\mathbf{A},$$

and $\mathbf{A}$ is the $(n+1)\times(n+1)$ tri-diagonal matrix whose entries are given by $A_{11} = A_{nn} = 1$, $A_{ii} = 2$ for $i = 2,\dots,n$, $A_{(i+1)i} = A_{i(i+1)} = -1$ for $i = 1,\dots,n$. Thus we can study the following discrete variational problem

$$\Phi(\mathbf{w},\mathbf{D}) \to \min$$

$$\mathbf{w} \in \mathcal{A}_n = \{\mathbf{u} \in \mathbb{R}^{n+1}, \mathbf{u}\cdot\mathbf{i}_1 = d > 0\}.$$

The Euler-Lagrange equations give the equilibrium condition

$$E\mathbf{B}_e\mathbf{w} - fn\nu\mu\mathbf{i}_1 = \mathbf{0}, \tag{4.3}$$

where f is the external force acting on the first mass to impose the assigned displacement (Lagrange multiplier) and $\mu^2 = E/k_S$.

Without entering in all the details of the structure of the metastable states and of the possible strategies of transition among the multiple local minimizers that are discussed in [10], together with an explicit thermodynamic discussion evidencing the insurgence of adhesion hysteresis, here we pass immediately to the analysis of the *macroscopic limit* deduced via Γ-convergence.

For every $n \in \mathbb{N}$ let

$$\mathcal{A}_n(0,L) =: \left\{ \sum_{i=1}^{n} a_i^n \mathbf{1}_{[i-1,i)\frac{L}{n}} : a_i^n \in \mathbb{R},\ a_1^n = \frac{d}{u_r} \right\}.$$

We define then the energy functional

$$J_n(w) = \begin{cases} \frac{1}{nE}\left(\sum_{i=1}^{n} \phi\left(\frac{w(i\frac{L}{n}) - w((i-1)\frac{L}{n})}{L/n} u_r \right) + \sum_{i=1}^{n+1} \varphi\left(w\left(i\frac{L}{n}\right)\right)\right) & \text{if } w \in \mathcal{A}_n(0,L) \\ +\infty \quad \text{otherwise in} \quad L^2(0,L). \end{cases}$$

Moreover, we define $\mathcal{A}_n^*$ as the subset of the functions $u \in H^1(0,L)$ such that there exists $w \in \mathcal{A}_n$ such that

$$u' = \sum_{i=1}^{n} u_r \frac{w\left(i\frac{L}{n}\right) - w\left((i+1)\frac{L}{n}\right)}{L/n} \mathbf{1}_{[i,i+1)\frac{L}{n}}, \quad u(0) = u_r\, d. \tag{4.4}$$

Thus, for every $w \in \mathcal{A}_n$ we have

$$J_n(w) = \frac{1}{EL}\int_0^L \phi(u_r w')\,dx + \frac{1}{E}\sum_{i=1}^{n+1}\frac{1}{n}\varphi\left(w\left(i\frac{L}{n}\right)\right).$$

If we define

$$J(w) = \begin{cases} \dfrac{1}{EL}\displaystyle\int_0^L (\phi(u_r w') + \varphi(w))\,dx \text{ if } w \in \mathcal{A}(0,L) \\ +\infty \text{ otherwise in } L^2(0,L) \end{cases}$$

in [10] we proved the following result

Theorem 4.1. *Let $\bar{w}_n \in \mathcal{A}_n^*$ such that*

$$J_n(\bar{w}_n) - \inf_{\mathcal{A}_n^*} J_n \to 0 \tag{4.5}$$

then up to subsequences $\bar{w}_n \rightharpoonup \bar{w}$ in $H^1(0,L)$ and

$$J_n(\bar{w}_n) \to J(\bar{w}) = \inf_{\mathcal{A}} J.$$

After introducing the following notation:

$$x := i/n, \quad \text{coordinate of the } i\text{-th spring},$$
$$\zeta = \xi/n, \quad \text{fraction of debonded springs},$$

the solutions of the Euler Lagrange equations give the following relations

$$d(\zeta) = u_r + \frac{L\,f}{k_S}\zeta, \tag{4.6}$$

$$u_\zeta(x) = \begin{cases} u_r + \dfrac{Lf}{k_S}(\zeta - x) & \text{if } x \in (0,\zeta), \\ u_r\dfrac{\cosh(\frac{1}{\nu}(1-x))}{\cosh(\frac{1}{\nu}(1-\zeta))} & \text{if } x \in (\zeta,1). \end{cases} \tag{4.7}$$

$$J(\zeta) = \frac{1}{2}(\zeta\,(1 + \frac{f^2}{Ek_S}) + \frac{\nu f \mu}{E}). \tag{4.8}$$

5 The limit for small relative stiffness

In this section we compare the results of the two models under the hypothesis that the stiffness of the beam is negligible with respect to the stiffness of the adhesive.

In the following figure we represent the different behavior of the system in dependence of the relative stiffness ν.

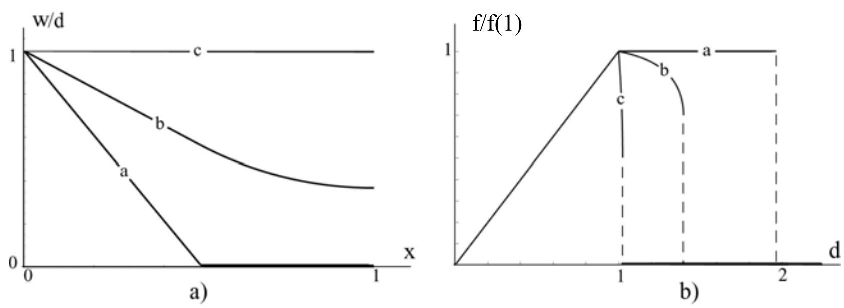


Figure 5.1: a) displacement fields at $\zeta = 0.5$ and b) force-displacement diagrams. Here $\mu = 1$ and for the curves labeled with a) $\nu = 0.01$, with b) $\nu = 0.5$, with c) $\nu = 10$.

Observe that as the parameter ν increases, the system shows a localization of the decohesion region (see [10] for details). In particular we obtain that as the strength of the adhesive layer grows (i.e. $u_r \to 0$ and thus $\nu \to 0$), previous relations give

$$d(\zeta) = \frac{L\,f}{k_S}\zeta, \tag{5.1}$$

$$u_\zeta(x) = \begin{cases} \dfrac{Lf}{k_S}(\zeta - x) & \text{if } x \in (0, \zeta), \\ 0. & \end{cases} \tag{5.2}$$

$$J(\zeta) = \frac{1}{2}\zeta\,(1 + \frac{f^2}{k_S}). \tag{5.3}$$

The continuous limit obtained in this section thus behaves as the system proposed in Section 2 after setting in (2.2) and (3.4) $\vartheta(\zeta) = \frac{1}{2}EL\zeta$ and $G(\zeta) = EJ(\zeta) - fd$, when we consider the limit $\nu \to 0$.

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