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A Sensitivity Analysis of Linear and Nonlinear Elastic and Viscoelastic Panel Responses

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Abstract

A unified set of governing linear and nonlinear relations are presented, which contain aerodynamic, structural, material and control inclusive contributions. Qualitative analyses are undertaken to determine the influences of the individual terms in the governing dynamic relations of lifting surface panels exposed to inertia, aerodynamic loads and noise. Panel buckling and flutter are considered as well as piezoelectric and aero-servo controls. Nonlinear results are evaluated and compared to linear solutions. Unfortunately the phase relations between the various components of the plate governing relations are sufficiently complicated to prevent analytical sensitivity comparisons. A limited set of numerical simulations are offered instead. Results are applicable to full scale flight vehicles as well as to UAVs and MAVs.

Mathematics Subject Classifications: 45G10, 73B05, 73F055, 73F15, 73F99
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1 Introduction

Due the high structural flexibility of modern flight vehicles, lifting surfaces and fuselages as well as their skin panels deflect during flight. These bending and torsional deflections may lead to instabilities such as buckling, flutter, torsional divergence, etc. The current study focuses on panels (plates) and their responses to aerodynamic noise, although any of the other loading functions such as lift or vibratory mechanical forces would serve equally well. The primary purpose of this

Table 1: List of Symbols

SYMBOL	NAME
$\alpha(x_2)$	flight angle of attack
α_r	rigid body angle of attack
α_0	zero lift angle
α_{ST}	stall angle
$\mathcal{A}_W$	effective angle of attack for deformed wing
C_{ijkl}	elastic or viscoelastic compliance
D_m, D, D_{ij}	plate bending stiffness (rigidity)
E_{ijkl}	elastic or viscoelastic modulus
$f_\alpha(x_2) = \alpha_r(x_2) - \alpha_0(x_2)$	built in wing function of rigid angles of attack
g, g_{ijkl}	structural damping coefficient
h	plate thickness
h_b	piezo-device bond thickness
h_P	piezo-device thickness
m_p	panel mass per unit area
M_{ij}	bending and shear moments
N_{ij}	in plane forces
$\Delta L_i(t)$	s_i direction deflected plate surface length change
$\nu(x,t)$	Poisson's ratio
$\Delta p(\widetilde{x}, t, w, \dot{w}, \ddot{w})$	pressure fluctuation (aerodynamic noise)
$s_i(\widetilde{x}, t),\ i = 1, 2$	tangent coordinate along deflected plate surface
$\theta(x_2, t)$	wing angle of twist
$u_i(x, t)$	displacement components
U_∞	undisturbed velocity
$\mathbf{V}(\widetilde{x}, t)$	piezoelectric voltage
$w(\widetilde{x}, t)$	panel bending deflection
$W(x_2, t)$	wing bending deflection
$x = \{x_i\},\ i = 1, 2, 3$	3-D Cartesian coordinates
$\widetilde{x} = \{\widetilde{x}_i\},\ i = 1, 2$	2-D Cartesian coordinates
^{E}	superscript denotes elastic quantities
$^{\mathbf{EX}}$	superscript denotes externally applied force
$^{\mathbf{P}}$	superscript denotes piezoelectric force
$^{\mathbf{T}}$	superscript denotes thermal force or moment

investigation is to catalogue and quantify in terms of order of magnitude analyses the nature of loading sources and panel responses to determine their linear or nonlinear contributions.

Materials that are considered are elastic metal plates and viscoelastic composite and elevated temperature metal panels. As in all self-excited aeroelastic and aero-viscoelastic problems, the governing drivers are angles of attack and phases relationships between forces and deflections or stresses and strains. A modification of elastic moduli by the inclusion of a 90^0 out phase structural damping is used to represent dry Coulomb friction, such as that found in structural joints.

For a number of years, viscoelastic tapes [2 – 5] have been used to help damp aerodynamic noise, aeroelastic phenomena and vibrations. With the serious and pervasive advent of composites as primary flight structural materials [6 – 8], significant weight savings can be achieved by using their inherent viscoelastic material properties to control aerodynamic noise, deformations and component failures [9 – 11]. Flight structures and their components can be further controlled by light weight piezoelectric devices [9 – 13], magneto-restrictive [14] and smart materials [15] and aero-servo-vicoelasticity [16]. However, experimental evidence in [17 – 21] strongly indicates that piezoelectric devices respond in a viscoelastic fashion by exhibiting creep and relaxation properties and. therefore, the control EMFs will be time dependent.

It should also be understood that the present analysis does not include a formal treatment of large deformations in terms of generalized tensors as covered in [22 – 28]. However, nonlinear phenomena contributed by either or both material behavior and deflections are considered.

2 Analysis

2.1 Flat Plates with Small Deformations

Consider a flat plate (panel) with Cartesian coordinates $x = \{x_1, x_2, x_3\}$, where x_3 is the thickness direction, with one of the following anisotropic viscoelastic constitutive relations

$$\text{elastic} \quad \Longrightarrow \quad \sigma_{ij}^{E}(x,t) \;=\; E_{ijkl}^{E}\,\epsilon_{kl}^{E}(x,t) \;-\; \delta_{ij}\,E_{ij}^{ET}\,\mathcal{A}T(x,t) \tag{1}$$

$$\text{viscoelastic} \quad \Longrightarrow \quad \sigma_{ij}(x,t) \;=$$
$$\int\limits_{-\infty}^{t} \left[E_{ijkl}(x,t,t')\,\epsilon_{kl}(x,t') \;-\; \delta_{ij}\,E_{ij}^{T}(x,t,t')\,\mathcal{A}T(x,t')\right] dt' \tag{2}$$

with the linearity or nonlinearity defined in terms of the moduli E^E or E as shown in Appendix A.

Let the variable Ξ be introduced such that

$$\Xi \;\equiv\; x_1, x_2, t, t', T(x_1, x_2, t') \;\equiv\; x_1, x_2, t, t' \;\equiv\; \widetilde{x}, t, t' \tag{3}$$

then the viscoelastic plate (panel) governing relations are [29] with units of [force / length^2]$_U$ = [F / L^2]$_U$

$$\mathcal{L}_P\Big\{ w(x_1, x_2, t) \Big\} \;=\; \mathcal{L}_P\Big\{ w(\widetilde{x}, t) \Big\} \;=\; \underbrace{\int\limits_{-\infty}^{t} \frac{\partial^2}{\partial x_1^2} \left(D_1(\widetilde{x}, t - t') \, \frac{\partial^2 w(\widetilde{x}, t')}{\partial x_1^2} \right) \, dt'}_{\text{viscoelastic bending resistance } (\mathrm{T}_{1\mathrm{A}})}$$

$$+ \underbrace{\int\limits_{-\infty}^{t} \left[2 \, \frac{\partial^2}{\partial x_1 \partial x_2} \left(D_2(\widetilde{x}, t - t') \, \frac{\partial^2 w(\widetilde{x}, t')}{\partial x_1 \partial x_2} \right) + \frac{\partial^2}{\partial x_2^2} \left(D_3(\widetilde{x}, t - t') \, \frac{\partial^2 w(\widetilde{x}, t')}{\partial x_2^2} \right) \right] \, dt'}_{\text{viscoelastic bending resistance } (\mathrm{T}_{1\mathrm{B}} \text{ and } \mathrm{T}_{1\mathrm{C}})}$$

$$- \underbrace{\int\limits_{-\infty}^{t} \int\limits_{0}^{a} D_4(\overline{x}_1, x_2, t - t') \left(\frac{\partial w(\overline{x}_1, x_2, t')}{\partial \overline{x}_1} \right)^2 d\overline{x}_1 \, dt' \, \frac{\partial^2 w}{\partial x_1^2}}_{\text{in plane force due to length change in } \mathrm{x}_1 \text{ direction } (\mathrm{T}_2)}$$

$$- \underbrace{\mathcal{N}_{11}^{\mathbf{EX}}(x_2, t) \, \frac{\partial^2 w}{\partial x_1^2}}_{\text{external force } (\mathrm{T}_3)} \;+\; \underbrace{\rho_p \, h \, \frac{\partial^2 w(\widetilde{x}, t)}{\partial t^2}}_{\text{inertia effects } (\mathrm{T}_4)}$$

$$+ \underbrace{\int\limits_{-\infty}^{t} \int\limits_{0}^{b} D_5(x_1, \overline{x}_2, t - t') \left(\frac{\partial w(x_1, \overline{x}_2, t')}{\partial \overline{x}_2} \right)^2 d\overline{x}_2 \, dt' \, \frac{\partial^2 w}{\partial x_2^2}}_{\text{in plane force due to length change in } \mathrm{x}_2 \text{ direction } (\mathrm{T}_5)} \;+\; \underbrace{\mathcal{N}_{22}^{\mathbf{EX}}(x_1, t) \, \frac{\partial^2 w}{\partial x_2^2}}_{\text{external force } (\mathrm{T}_6)}$$

$$- \underbrace{\int\limits_{-\infty}^{t} \int\limits_{0}^{a} D_6(\overline{x}_1, x_2, t - t') \, \frac{\partial w(\overline{x}_1, x_2, t')}{\partial \overline{x}_1} \frac{\partial w(\overline{x}_1, x_2, t')}{\partial x_2} \, d\overline{x}_1 \, dt' \, \frac{\partial^2 w}{\partial x_1 \partial x_2}}_{\text{in plane force due to angle change between } \mathrm{x}_1 \;\&\; \mathrm{x}_2 \text{ directions } (\mathrm{T}_7)}$$

$$- \underbrace{\mathcal{N}_{12}^{\mathbf{EX}}(t) \, \frac{\partial^2 w}{\partial x_1 \partial x_2}}_{\text{external force } (\mathrm{T}_8)} \;-\; \underbrace{\mathcal{N}_{21}^{\mathbf{EX}}(t) \frac{\partial^2 w}{\partial x_1 \partial x_2}}_{\text{external force } (\mathrm{T}_{10})}$$

$$-\underbrace{\int_{-\infty}^{t}\int_{0}^{a} D_7(x_1,\overline{x}_2,t-t')\;\frac{\partial w(x_1,\overline{x}_2,t')}{\partial x_1}\frac{\partial w(x_1,\overline{x}_2,t')}{\partial \overline{x}_2}\;d\overline{x}_2\,dt'\frac{\partial^2 w}{\partial x_1\partial x_2}}_{\text{in plane force due to angle change between } \mathrm{x}_1 \text{ \& } \mathrm{x}_2 \text{ directions } (\mathrm{T}_9)}$$

$$+\underbrace{a_0\,q\,\sin\left\{\frac{\pi}{2\,\alpha_{ST}}\left[\underbrace{\overbrace{\mathcal{A}_W(\alpha,\theta,W)}^{\substack{\text{wing contribution}\\ \text{Eq. (5)}}}+\overbrace{\arctan\left(\frac{1}{U_\infty}\frac{\partial w(x,t)}{\partial t}+\frac{\partial w(x,t)}{\partial x_1}\right)}^{\text{panel contribution}}}_{\text{combined angle due to attack of deformed wing and panel}}\right]\right\}}_{\mathrm{L}(\alpha,\theta,\mathrm{W},\mathrm{w})\,=\,\text{lift forces } (\mathrm{T}_{11}) \text{ (closed loop)}}$$

$$+\underbrace{\mathcal{N}^{\mathbf{C}}_{11}(\widetilde{x},t)\frac{\partial^2 w}{\partial x_1^2}}_{\substack{\mathrm{x}_1 \text{ piezo, MR and}\\ \text{SM force } (\mathrm{T}_{12})}}+\underbrace{\mathcal{N}^{\mathbf{C}}_{22}(\widetilde{x},t)\frac{\partial^2 w}{\partial x_2^2}}_{\substack{\mathrm{x}_2 \text{ piezo, MR and}\\ \text{SM force } (\mathrm{T}_{13})}}+\underbrace{\mathcal{N}^{\mathbf{T}}_{11}(\widetilde{x},t)\frac{\partial^2 w}{\partial x_1^2}}_{\mathrm{x}_1 \text{ thermal force } (\mathrm{T}_{14})}+\underbrace{\mathcal{N}^{\mathbf{T}}_{22}(\widetilde{x},t)\frac{\partial^2 w}{\partial x_1\partial x_2}}_{\mathrm{x}_2 \text{ thermal force } (\mathrm{T}_{15})}$$

$$+\underbrace{2\,\mathcal{N}^{\mathbf{T}}_{12}(\widetilde{x},t)\frac{\partial^2 w}{\partial x_1\partial x_2}}_{\substack{\text{thermal shear}\\ \text{force } (\mathrm{T}_{16})}}+\underbrace{\frac{\partial M^{\mathbf{T}}_{11}(\widetilde{x},t)}{\partial x_2^2}}_{\substack{\mathrm{x}_3 \text{ load due}\\ \text{to } \mathrm{M}^{\mathbf{T}}_{11}(\mathrm{T}_{17})}}+\underbrace{\frac{\partial M^{\mathbf{T}}_{22}(\widetilde{x},t)}{\partial x_1^2}}_{\substack{\mathrm{x}_3 \text{ load due}\\ \text{to } \mathrm{M}^{\mathbf{T}}_{22}(\mathrm{T}_{18})}}+\underbrace{2\,\frac{\partial M^{\mathbf{T}}_{12}(\widetilde{x},t)}{\partial x_1\partial x_2}}_{\substack{\mathrm{x}_3 \text{ load due}\\ \text{to } \mathrm{M}^{\mathbf{T}}_{12}(\mathrm{T}_{19})}}$$

$$+\underbrace{\Delta p\left(\widetilde{x},t,\overbrace{\alpha,\theta,W,\frac{\partial W}{\partial t},\frac{\partial^2 W}{\partial t^2}}^{\text{wing contributions}}\overbrace{w,\frac{\partial w}{\partial \widetilde{x}},\frac{\partial w}{\partial t},\frac{\partial^2 w}{\partial t^2}}^{\text{panel contributions}}\right)}_{\text{aerodynamic noise pressure } (\mathrm{T}_{20}) \text{ (closed loop)}}+\underbrace{F_V(\widetilde{x},t)}_{\substack{\text{external } \mathrm{x}_3-\text{direction}\\ \text{vibratory force } (\mathrm{T}_{21})\\ \text{(open loop)}}} \tag{4}$$

$$+\overbrace{\underbrace{F_{ASV}\left(\widetilde{x},t,w(\widetilde{x},t),\frac{\partial w(\widetilde{x},t)}{\partial t},\frac{\partial^2 w(\widetilde{x},t)}{\partial t^2},\frac{\partial^3 w(\widetilde{x},t)}{\partial t^3},\int_0^t w(\widetilde{x},t')\,dt'\right)}_{\text{external closed loop servo control force } (\mathrm{T}_{22})}}^{\text{proportional } (\mathrm{T}_{22\mathrm{P}}),\text{ differential } (\mathrm{T}_{22\mathrm{D}}) \text{ and/or integral } (\mathrm{T}_{22\mathrm{I}}) \text{ servo controller}}$$

$$+\underbrace{F_C\left(\overbrace{\boldsymbol{V}(x,t,w)}^{\substack{\text{piezo}-\\ \text{electric}\\ \text{voltage } (\mathrm{T}_{25\mathrm{PZ}})}},\overbrace{\boldsymbol{I}(x,t,w)}^{\substack{\text{MR}\\ \text{current}\\ (\mathrm{T}_{25\mathrm{MR}})}},\overbrace{\boldsymbol{\sigma}^{\mathrm{SM}}(x,t,w)}^{\substack{\text{smart}\\ \text{materials}\\ (\mathrm{T}_{25\mathrm{SM}})}}\right)}_{\text{external open or closed loop control force } (\mathrm{T}_{23})}=0$$

where the two distinct deflections are for the plate $w = w(\widetilde{x}, t) = w(x_1, x_2, t)$ and for the wing $W = W(x_2, t)$, unless otherwise indicated. Terms T_{12} and T_{13} refer to piezoelectric, magneto-restrictive [14] and smart material [15] induced control forces. Additional control forces stemming servo-aero-viscoelastic actions [16] are included in T_{22}. In the absence of chordwise bending the panel effective angle of attack due to wing contributions is

$$\mathcal{A}_W(\alpha, \theta, W) \;=\; \underbrace{\overbrace{\alpha_r(x_2) - \alpha_0(x_2)}^{\substack{f_\alpha(x_2)=\text{built in}\\ \text{function}}} \,+\, \overbrace{\alpha(x_2)}^{\substack{\text{angle of}\\ \text{attack}}}}_{\text{rigid wing contribution}}$$

$$+\; \underbrace{\overbrace{\theta(x_2, t)}^{\substack{\text{wing angle}\\ \text{of twist}}} \,+\, \overbrace{\arctan\left(\frac{1}{U_\infty}\frac{\partial W(x_2, t)}{\partial t}\right)}^{\text{wing bending velocity}}}_{\text{wing aero}-\text{viscoelastic contributions}} \tag{5}$$

For elastic plates based on Eqs. (1) similar relations can be derived without the time integrals and with

$$D_m^E(\widetilde{x}) \;\equiv\; D_m(\widetilde{x}, t, t') \tag{6}$$

If the elastic or viscoelastic material is isotropic then in terms T_1, the bending rigidities become

$$D_1 \;=\; D_2 \;=\; D_3 \;\equiv D \tag{7}$$

and the T_1 terms reduce to

$$\text{Term } T_1 \text{ viscoelastic} \;\Longrightarrow$$

$$\int_{-\infty}^{t} \vec{\boldsymbol{\nabla}} D(\widetilde{x}, t - t') \,\boldsymbol{\cdot}\, \vec{\boldsymbol{\nabla}} w(\widetilde{x}, t')\, dt' \quad \text{or} \quad \int_{-\infty}^{t} \vec{\boldsymbol{\nabla}} D(\widetilde{x}, t, t') \,\boldsymbol{\cdot}\, \vec{\boldsymbol{\nabla}} w(\widetilde{x}, t')\, dt' \tag{8}$$

$$\text{Term } T_1 \text{ elastic} \;\Longrightarrow\; \vec{\boldsymbol{\nabla}} D^E(\widetilde{x}) \,\boldsymbol{\cdot}\, \vec{\boldsymbol{\nabla}} w^E(\widetilde{x}, t) \tag{9}$$

with $\vec{\boldsymbol{\nabla}}$ defined in terms of unit vectors by

$$\vec{\boldsymbol{\nabla}} \;=\; \vec{\boldsymbol{i}}\,\frac{\partial^2}{\partial x_1^2} + \sqrt{2}\,\vec{\boldsymbol{j}}\,\frac{\partial^2}{\partial x_1\,\partial x_2} + \vec{\boldsymbol{k}}\,\frac{\partial^2}{\partial x_2^2} \tag{10}$$

It is to be noted that in corresponding elastic isotropic cases, the bending rigidity D^E is defined as

$$D^E(\widetilde{x}) \;=\; \frac{E^E(\widetilde{x})\; h^3}{12\left[1 - \left(\nu^E(\widetilde{x})\right)^2\right]} \tag{11}$$

Table 2: Governing Relations and BC Contributions

TERM	FUNCTION	CONTRIBUTION
T_1	internal bending resistance	linear for small deflections & linear materials
$T_2,\ T_5$ $T_7,\ T_9$	in plane traction due to length change in plane shear due change in angle	nonlinear, closed loop
T_3	external force in x_1 direction	linear, open loop
$T_4,\ T_{32}$	panel inertia force	linear
T_6	external force in x_2 direction	linear, open loop
$T_8,\ T_{10}$	external shear force on x_1 and x_2 edge	linear, open loop
T_{11}	wing lift force	linear or nonlinear, c-l
$T_{12},\ T_{13}$	x_1, x_2 direction piezo, MR & SM force	linear or nonlinear, o/c-l
$T_{14} - T_{19}$	thermal forces and bending moments	linear or nonlinear, o-l
T_{20}	aerodynamic noise pressure	linear or nonlinear, c-l
T_{21}	external x_3-direction vibratory load	linear, open loop
T_{22}	servo-aero-viscoelastic forces [16]	linear or nonlinear, c-l
T_{23}	piezo, MR & SM force	linear or nonlinear, o/c-l
T_{24}	s_i length change of deflected surface	nonlinear, closed loop
T_{25}	large deformations	nonlinear
T_{26}, T_{27}	boundary support constitutive relations	linear or nonlinear
T_{30}	stress function	linear
T_{31}	large deformation stress-strain relations	nonlinear, closed loop
T_{33}	large deformation in plane forces	nonlinear, closed loop

whereas for equivalent isotropic viscoelastic materials the use of Poisson's ratio is inappropriate and counter-indicated [33 - 36], since there is no viscoelastic counterpart for the elastic expression (11), i.e.

$$\overline{\overline{D}}(\widetilde{x},\omega) \neq \frac{\overline{\overline{E}}(\widetilde{x},\omega)\, h^3}{12\left[1-\overline{\overline{\nu^2}}(\widetilde{x},\omega)\right]} \tag{12}$$

except under severely restricted conditions, such as incompressible viscoelastic materials with $\nu(x,t) = 0.5$. In all other more general cases, viscoelastic Poisson ratios are time, loading history and stress dependent. For viscoelastic materials, therefore, only relaxation moduli or creep compliances or relaxation/creep functions can be used to define bending rigidities, i. e.

$$D_{ijkl}(\widetilde{x},t) = \frac{E_{ijkl}(\widetilde{x},t)\, h^3}{12} \tag{13}$$

The linearity or nonlinearity of terms in the governing relation of Eq. (4) is independently influenced by geometry (size of deflections), material property characterizations. boundary conditions and failure conditions (Eqs. (1), (2), and (47) to (49)). The contributions of each linear or nonlinear term of Eqs. (4) are displayed in Table 2. It should be noted that if a single term of the governing relations exhibits nonlinear properties, then integral transforms, such as Laplace and Fourier, are inapplicable and solutions to the governing DEs or integral-differential relations must be obtained by other means.

Details regarding various moduli as well as structural damping effects are discussed in Appendix A. For nonhomogeneous materials Eqs. (13) must be returned to their fundamental roots and become

$$D_{ijkl}(\widetilde{x},t) = D_{ijkl}(x_1,x_2,t) = \int_{-h/2}^{h/2} E_{ijkl}(x_1,x_2,x_3,t)\, x_3^2\, dx_3 \tag{14}$$

rather than the simple moment and product of inertia expressions associated with homogeneous elastic and/or viscoelastic material plates.

The change in length ΔL_i of the deflected surface perimeter in the the s_i direction from one boundary to its opposite is given by

$$\Delta L_i(\widetilde{x},t) = \underbrace{\int_{-\infty}^{t} \left[\int_{0}^{\Lambda_{\underline{i}}(s,t')} C_{\underline{i}}(s,t,t')\, \sigma_{\underline{ii}}(s,t')\, ds_{\underline{i}} \right] dt'}_{\text{length change of deflected plate surface } (\mathrm{T}_{24})} \qquad i \neq j = 1,2 \tag{15}$$

with C_i the compliance of the plate in the x_i direction and where

$$\Lambda_i(\widetilde{x},t) \;=\; 1 \;\pm\; \Delta L_i(\widetilde{x},t) \qquad i \neq j = 1,2 \tag{16}$$

and

$$ds_i \;=\; \sqrt{dx_i^2 + dw^2} \qquad i = 1,2 \tag{17}$$

The coordinates $s = \{s_1(x_1,x_2,t), s_2(x_1,x_2,t)\}$ are tangent to the deflected surface and are given by

$$\frac{\partial x_{\underline{i}}(s,t)}{\partial s_{\underline{i}}} \;=\; \cos\left[\arcsin \frac{\partial w(s_{\underline{i}}, s_j, t)}{\partial s_{\underline{i}}}\right] \qquad i \neq j = 1,2 \tag{18}$$

or

$$x_i(s,t) \;=\; \int\limits_0^{s_{\underline{i}}} \cos\left[\arcsin \frac{\partial w\left(s'_{\underline{i}}, s_j, t\right)}{\partial s'_{\underline{i}}}\right] ds'_i \quad i \neq j = 1,2 \quad 0 \leq s_i \leq \Lambda_i(\widetilde{x},t) \tag{19}$$

It should be noted that with appropriate boundary conditions one can maintain zero length changes ΔL_{x_i} in either or both x_i directions, while the ΔL_i in the s_i directions will generally have nonzero values due to surface curvatures. Only for small deflections and linear analysis can the approximate relations $\Delta L_i \approx \Delta L_{x_i} \approx 0$ be justified, simultaneously implying that $s_i \approx x_i$.

Eqs. (15) for the ΔL_i is further complicated and nonlinear by having the unknown variable in the upper limit of the r.h.s. integral.

2.2 Boundary Conditions

Any diverse BCs can be prescribed on any of the plate boundaries or portions thereof. A number of these BCs for elastic plates can be found in [29] . The surrounding structure may provide flexible supports in the form of elastic and/or viscoelastic translational and rotational restraints. Additionally, certain deflections may be prescibed. In these cases and for some extra rigid BCs, the problem becomes statically indeterminate with the boundary displacements becoming part of the solution as well as prescribing boundary conditions *per se*.

Linear or nonlinear elastic and viscoelastic moduli for boundary supports may have distinct properties from those of the plate and their actions may be augmented by additional piezoelectric devices providing control tractions and shears along any portion of the boundaries. Structural damping in the form of Coulomb friction [56] – [58] may also be present in the plate joints depending on connecting methods. (See Appendix A.)

For instance, if inplane elastic or viscoelastic boundary restraints are present at $x = 0$, then the

$$\text{elastic BCs} \quad \Longrightarrow \quad \epsilon^E_{11}(\Xi_0) \; = \; \frac{\partial u^E_1(\Xi_0)}{\partial x_1} \; + \; \underbrace{\frac{1}{2}\left[\frac{\partial u^E_i(\Xi_0)}{\partial x_1}\;\frac{\partial u^E_i(\Xi_0)}{\partial x_1}\right]}_{\text{large deformations (T}_{25}\text{)}} \; =$$

$$\underbrace{C^E_{11kl}[0, x_2, 0, I(\Xi_0)] \;\; \sigma^E_{kl}(\Xi_0)}_{\text{elastic constitutive relations (T}_{26}\text{)}} \tag{20}$$

$$\text{viscoelastic BCs} \quad \Longrightarrow \quad \epsilon_{11}(\Xi_0) \; = \; \frac{\partial u_1(\Xi_0}{\partial x_1} \; + \; \underbrace{\frac{1}{2}\left[\frac{\partial u_i(\Xi_0)}{\partial x_1}\;\frac{\partial u_i(\Xi_0)}{\partial x_1}\right]}_{\text{large deformations (T}_{25}\text{)}} \; =$$

$$\underbrace{\int\limits_{-\infty}^{t} C_{11kl}\Big[\Xi_0, t', I(\Xi'_0), \dot{I}(\Xi'_0)\Big] \;\; \sigma_{kl}(\Xi'_0)\; dt'}_{\text{viscoelastic constitutive relations (T}_{27}\text{)}} \tag{21}$$

with $\Xi_0 = (0, x_2, 0, t)$ and $\Xi'_0 = (0, x_2, 0, t')$ and where $I = \{I_1, I_2, I_3\}$ are the principal invariants of the strain tensor ϵ_{ij}, defined by

$$I_1 \; = \; \frac{1}{3}\epsilon_{ii} \qquad I_2 \; = \; \frac{1}{3}\epsilon_{ij}\;\epsilon_{ij} \qquad I_3 \; = \; \frac{1}{3}\epsilon_{ij}\;\epsilon_{ik}\;\epsilon_{jk} \tag{22}$$

2.3 Thermal Expansions and Stress Contributions

Temperature distributions $T(x,t)$ relative to a reference rest temperature T_0 produce thermal expansions which may lead thermal stresses and/or thermal bending moments. Materials exhibit thermal expansion coefficients $\mathcal{A}$, which lead thermal strains

$$\text{anisotropic} \quad \Longrightarrow \quad \epsilon^{\mathbf{T}}_{ij}(x,t) \; = \; \mathcal{A}_{ij}\, T(x,t) \tag{23}$$

$$\text{isotropic} \quad \Longrightarrow \quad \epsilon^{\mathbf{T}}_{\underline{ii}}(x,t) \; = \; \mathcal{A}_{\underline{ii}}\, T(x,t) \; = \; \mathcal{A}T(x,t) \tag{24}$$

If the thermal coefficients are independent of the temperature then the thermal strains are directly proportional to the temperature and linear w.r.t. deformations.

The resulting thermal stresses involve moduli and are given by

$$\text{elastic} \quad \Longrightarrow \quad \sigma^{ET}_{ij}(x,t) \; = \; E^{ET}_{\underline{ij}}(x)\, \mathcal{A}_{\underline{ij}} T(x,t) \tag{25}$$

$$\text{viscoelastic} \quad \Longrightarrow \quad \sigma^T_{ij}(x,t) \;=\; \int\limits_{-\infty}^{t} E^T_{\underline{ij}}(x,t,t')\,\mathcal{A}_{\underline{ij}} T(x,t')\,dt' \tag{26}$$

which in turn produce thermal tractions

$$\mathcal{N}^T_{ij}(x_1,x_2,t) \;=\; \int\limits_{-h^*/2}^{h^*/2} \sigma^T_{ij}(x,t)\,dx_3 \qquad i,j<3 \tag{27}$$

and bending moments given by

$$M^T_{\underline{ii}}(x_1,x_2,t) \;=\; \int\limits_{-h^*/2}^{h^*/2} \sigma^T_{\underline{jj}}(x,t)\,x_3\,dx_3 \qquad i\neq j<3 \tag{28}$$

and

$$M^T_{12}(x_1,x_2,t) \;=\; \int\limits_{-h^*/2}^{h^*/2} \sigma^T_{12}(x,t)\,x_3\,dx_3 \tag{29}$$

with $h^* = h + h_b + h_P/2$ and where $h_b \ll h_P \ll h$. For a general fundamental coverage of elastic thermal stresses and their effects see [37].

2.4 Stability Issues

Elastic and viscoelastic wings are subject to torsional divergence and to bending/torsion flutter leading to catastrophic instabilities. Plates or panels exhibit flutter and additionally may buckle under in or out of plane loads. In elastic lifting structures these aeroelastic instabilities occur at characteristic velocities, while buckling is associated with a critical load.

In viscoelastic structures these four phenomena lead to distinct survival times due to increased deformations (creep) and decreased moduli (stiffness) in time, both decreasing the length of structural life times.

As noted in Table 2, panel flutter is a nonlinear phenomenon and can only occur in the presence of one or more terms T_5, T_7 or T_9. (For a detailed formulation and discussion of panel flutter see [38 – 40].) On the other hand, wing flutter and torsional divergence and panel buckling can manifest themselves in either linear or nonlinear systems. See Table 3 for the definitions of the various stability criteria.

2.5 Flat Plates with Large Deformations

Von Kármán [41] and Donnell [42] derived nonlinear governing relations for plates with linear elastic materials with large deformations in terms of a stress function $\phi(\widetilde{x},t)$ and the deformation $w(\widetilde{x},t)$. The function ϕ is defined in the usual manner as

$$\sigma_{11} \;=\; \frac{\partial^2\phi}{\partial x_2^2} \qquad \sigma_{22} \;=\; \frac{\partial^2\phi}{\partial x_1^2} \qquad \sigma_{12} \;=\; \frac{\partial^2\phi}{\partial x_1\,\partial x_2} \tag{30}$$

The constitutive relations for a linear isothermal viscoelastic material then become with units of $[\mathrm{F}\,/\,\mathrm{L}^4]_U$

$$\mathcal{L}_{LD1}\Big\{w(x_1,x_2,t)\Big\} \;=\; \mathcal{L}_{LD1}\Big\{w(\widetilde{x},t)\Big\} \;=$$

$$\underbrace{\frac{\partial^4\phi(\widetilde{x},t)}{\partial x_1^4} \;+\; 2\,\frac{\partial^4\phi(\widetilde{x},t)}{\partial x_1^2\partial x_2^2} \;+\; \frac{\partial^4\phi(\widetilde{x},t)}{\partial x_2^4}}_{\text{stress functions } (\mathrm{T}_{30\mathrm{A}},\ \mathrm{T}_{30\mathrm{B}}\ \&\ \mathrm{T}_{30\mathrm{C}})=\nabla^4\phi}$$

$$\underbrace{-\int\limits_{-\infty}^{t} E(\widetilde{x},t-t')\left[\left(\frac{\partial^2 w(\widetilde{x},t')}{\partial x_1\,\partial x_2}\right)^2 \;-\; \frac{\partial^2 w(\widetilde{x},t')}{\partial x_1^2}\,\frac{\partial^2 w(\widetilde{x},t')}{\partial x_2^2}\right] dt'}_{\text{in plane forces } (\mathrm{T}_{31\mathrm{A}}\ \&\ \mathrm{T}_{31\mathrm{B}})} \;=\; 0 \tag{31}$$

The transverse equilibrium with units of $[\mathrm{F}\,/\,\mathrm{L}^2]_U$ and without rotary inertia, control and thermal terms are

$$\mathcal{L}_{LD2}\Big\{w(x_1,x_2,t)\Big\} \;=\; \mathcal{L}_{LD2}\Big\{w(\widetilde{x},t)\Big\} \;=$$

$$+\;\underbrace{\rho_p\,h\,\frac{\partial^2 w(\widetilde{x},t)}{\partial t^2}}_{\text{inertia effects } (\mathrm{T}_4)} \;-\; \underbrace{\rho_p\,I\,\overbrace{\left\{\frac{\partial^2}{\partial x_1^2} \;+\; 2\,\frac{\partial^2}{\partial x_1\partial x_2} \;+\; \frac{\partial^2}{\partial x_2^2}\right\}}^{=\,\nabla^2}\left\{\frac{\partial^2 w(\widetilde{x},t)}{\partial t^2}\right\}}_{\text{rotary inertia } (\mathrm{T}_{32})}$$

$$+\;\underbrace{\int\limits_{-\infty}^{t}\frac{\partial^2}{\partial x_1^2}\left(D_1(\widetilde{x},t-t')\,\frac{\partial^2 w(\widetilde{x},t')}{\partial x_1^2}\right) dt'}_{\text{viscoelastic bending resistance } (\mathrm{T}_{1\mathrm{A}})}$$

$$+\;\underbrace{\int\limits_{-\infty}^{t}\left[2\,\frac{\partial^2}{\partial x_1\partial x_2}\left(D_2(\widetilde{x},t-t')\,\frac{\partial^2 w(\widetilde{x},t')}{\partial x_1\partial x_2}\right) + \frac{\partial^2}{\partial x_2^2}\left(D_3(\widetilde{x},t-t')\,\frac{\partial^2 w(\widetilde{x},t')}{\partial x_2^2}\right)\right] dt'}_{\text{viscoelastic bending resistance } (\mathrm{T}_{1\mathrm{B}} \text{ and } \mathrm{T}_{1\mathrm{C}})}$$

$$- \underbrace{h \left[\frac{\partial^2 \phi(\widetilde{x},t)}{\partial x_2^2} \frac{\partial^2 w(\widetilde{x},t)}{\partial x_1^2} + \frac{\partial^2 \phi(\widetilde{x},t)}{\partial x_1^2} \frac{\partial^2 w(\widetilde{x},t)}{\partial x_2^2} - 2 \frac{\partial^2 \phi(\widetilde{x},t)}{\partial x_1 \partial x_2} \frac{\partial^2 w(\widetilde{x},t)}{\partial x_1 \partial x_2} \right]}_{\text{in plane forces } (\mathrm{T_{33A}},\ \mathrm{T_{33B}}\ \&\ \mathrm{T_{33C}})}$$

$$+ \underbrace{L(x,t,\alpha,\theta,W,w)}_{\text{lift forces } (\mathrm{T_{11}})} + \underbrace{\Delta p(x,t,\alpha,\theta,W,w)}_{\text{aerodynamic noise } (\mathrm{T_{20}})} = 0 \tag{32}$$

For detailed definitions of terms T_{11} and T_{20} see Eqs. (4) and (5).

Unsteady aerodynamics theory produces expressions which may be linear (or nonlinear) in the wing $W(x_2,t)$ and plate $w(\widetilde{x},t)$, but without exception are nonlinear in the response frequencies and velocities.

2.6 Structural Control

Structural control can be provided in many forms, such as material damping [43], [44], piezoelectric [9 – 13], structural damping in joints [56 – 58], magneto/electro/restrictive materials [14], smart materials [15], servo-aero-viscoelasticity [16] and designer materials [61]. Such controls can be accomplished through energy extraction and/or by supplying additional forces. An examination of the governing relations (4) reveals that the action of these control forces to be essentially similar. However, an investigation of the operational efficiencies of the various control devices is beyond the intended scope of this paper.

2.7 Deterministic and Probabilistic Failure Conditions

Elastic or plastic failure criteria are conventionally expressed in terms of uniaxial ultimate stresses and then combined in a suitable failure rule, such as von Misses, octahedral shear stress, Shanley-Ryder, etc. Viscoelastic failure criteria, such as ultimate stresses, degrade in time independently of relaxation moduli and failures may occur before or after any creep buckling instabilities manifest themselves (Fig. 2). These are material failures which are independent of creep buckling and define the life time of the structure designated as t_{LF}. Consequently, t_{cr} may be greater than, smaller than or equal to t_{LF}.

Some failure mechanisms observed in composites are substantially different from those observed in metals [45 – 46] as delamination is a phenomenon unique to composites (Fig. 3). From a design analysis point of view, one needs only to consider delamination onset because at that stage a flight structure has for all practical purposes failed, particularly if it is a light weight flight structure. In Ref. [45] an expression has been formulated for the temperature, moisture and time dependency of uniaxial composite failure stresses and in [48] an extensive review of available experimental composite failure data is presented and using this data, deterministic and stochastic delamination failure analyses are formulated. Experimental results indicate that uniaxial deterministic delamination onset stresses in tension and shear obey laws of the type

$$\sigma_{ij}^{\mathbf{F}}(t) \;=\; \begin{cases} \sigma_{ij0}^{\mathbf{F}} & -\infty \leq t \leq t_2^{\mathbf{F}} \\[2ex] \sigma_{ij0}^{\mathbf{F}} \;-\; D_{ij}\;\log\left(t/t_4^{\mathbf{F}}\right) & t_2^{\mathbf{F}} \leq t \leq t_3^{\mathbf{F}} \\[2ex] 0 & t \geq t_3^{\mathbf{F}} \end{cases} \tag{33}$$

where all parameters are material, temperature, moisture and load (tension, shear, etc.) dependent.

In [49] deterministic and stochastic invariant combined load failure criteria are formulated in terms of two relations

$$\frac{1}{3}\sum_{i=1}^{3}\left[\frac{\widetilde{J}_i(x,t)}{\mathcal{J}_i(x,t)}\right]^{c_i} \;=\; \widetilde{V}(x,t) \qquad \text{and} \qquad \frac{1}{3}\sum_{i=1}^{3}\left[\frac{\widetilde{\mathcal{J}}_i(x,t)}{\mathcal{J}_i(x,t)}\right]^{c_i} \;=\; \widetilde{v}(x,t) \tag{34}$$

where J_i, $\mathcal{J}_i$ and c_i are mean values and random variables are indicated with a $\widetilde{\ }$. The upper summation limit q is the number uniaxial loads. Typical deterministic failure stress surfaces may be found in [49] and [50]. The applied and failure stress invariants are defined by

$$\widetilde{J}_1 \;=\; \widetilde{\sigma}_{ii} \qquad \widetilde{J}_2 \;=\; \widetilde{\sigma}_{ij}\,\widetilde{\sigma}_{ij} \qquad \widetilde{J}_3 \;=\; \widetilde{\sigma}_{ij}\,\widetilde{\sigma}_{ik}\,\widetilde{\sigma}_{kj} \tag{35}$$

$$\widetilde{\mathcal{J}}_1 \;=\; \widetilde{F}_{ii} \qquad \widetilde{\mathcal{J}}_2 \;=\; \widetilde{F}_{ij}\,\widetilde{F}_{ij} \qquad \widetilde{\mathcal{J}}_3 \;=\; \widetilde{F}_{ij}\,\widetilde{F}_{ik}\,\widetilde{F}_{kj} \tag{36}$$

where F_{ij} are uniaxial failure stresses.

Failure occurs whenever

$$\widetilde{U}(x,t) \;=\; \widetilde{V}(x,t) \;-\; \widetilde{v}(x,t) \;\leq\; 0 \tag{37}$$

For deterministic applied loads and random failure stresses, or vice versa, one needs only to apply one probability density function (PDF) to either Eqs. (34). In [46] it is shown that experimental delamination data which can be represented by a Weibull type probability density function (PDF) [51 – 52]. Upon integrating this PDF, one obtains the failure probability $\widetilde{P}_F$ as

$$\widetilde{P}_F(x,t) \;=\; 1 \;-\; \exp\left\{-\left[\frac{\widetilde{U}(x,t)}{\kappa}\right]^{\gamma}\right\} \tag{38}$$

where the material property parameters γ and κ and their values were discussed in detail in [46] and [48]. Since for plate problems the stresses σ_{11}, σ_{22} and σ_{12} are functions of x and t, it follows that the failure probabilities $\widetilde{P}_F$ are also dependent on position within the plate and on time.

The time t_{LF} corresponding to the largest value of $\widetilde{P}$ at a point $x_i = c_i$ in a structure is the life time or survival time. It is defined by

$$\widetilde{P}(c, t_{LF}) = \max\left\{\widetilde{P}(x,t)\right\} \leq 1 \tag{39}$$

In stochastic probabilistic structural failure analysis, one seeks similar points or regions where $\widetilde{P}_F(x, t_{LF}) = 1$ or alternately the maximum probability value $\widetilde{P}_F(x, t_{LF}) < 1$ to indicate column survival probabilities under a prescribed load $P(t) < P_E$ and a given initial imperfection $w_o(x)$. A similar but distinct class of problems arises from the imposition of the specification of design survival times t_{LFD} each corresponding to a design failure probability $\widetilde{P}_{FD}(t_{LFD}) \leq 1$, or conversely the prescription of a t_{LFD} with an attendant $\widetilde{P}_{FD}(t_{LFD})$.

It must, of course, be remembered that the five decisive times listed in Table 3 (t_{cr}, t^{ϵ}_{cr}, t^{*}_{cr}, t_{LF}, t_{LFD}) are unrelated to each other and each represents a distinct definition of instability or failure conditions. Consequently, depending on the design and/or analysis *a priori* choices distinct lifetime predictions will result.

2.8 Computational Issues and Protocols

Despite the fact that in last seventy years viscoelasticity has achieved a high degree of maturity, the most vexing issue remains the efficient evaluation of the time integrals stemming from the constitutive relations. Isothermal linear cases are amenable to solutions by Laplace or Fourier transforms and their attendant elastic-viscoelastic correspondence principles [30] , [31] . However, difficulties may be encountered with analytical integral transform inversions and one may have to resort to numerical inversions by fast Fourier transform procedures. Alternately, one can appeal to a number of approximate time evaluations as summarized in [47] .

The spatial variations can be determined through finite element approaches, Galerkin's method or successive approximation techniques [22] . Galerkin's approach has the advantage of reducing partial-integral equations to total differential-integral relations in time. The Poincaré method reduces nonlinear PDEs to series of linear PDEs, whose total number is determined by the user based on the desired rapidity of convergence of the successive approximation terms. These PDEs are only backward coupled in the previously determined series terms, thus requiring only solutions of one series term from one linear PDE at a time.

However, formal solution and efficient computational protocols for integral-diferential remain a fertile and timely area for much needed research.

2.9 A Qualitative Analysis of Nonlinearities in the Governing Relations and Boundary Conditions

Table 3: List of Critical Plate Parameters

VARIABLE	DEFINITION	CRITERION
$N^{\mathbf{EX}}_{x_1 CR}$	external or boundary restrain	elastic buckling load
$U^{\mathbf{F}}_{\infty}$	$\lim\limits_{U_\infty \to U^{\mathbf{F}}_\infty} w(x,t) \longrightarrow \infty$	panel flutter velocity
$U^{\mathbf{FW}}_{\infty}$	$\lim\limits_{U_\infty \to U^{\mathbf{FW}}_\infty} W(x,t) \longrightarrow \infty$	wing flutter velocity
DB_{max}	prescribed sound pressure	maximum allowable sound intensity
t_{cr}	$\lim\limits_{t \to t_{cr}} \{w(x,t)\} \longrightarrow \infty$	classical creep buckling definition
t^{ϵ}_{cr}	$\lim\limits_{t \to t^{\epsilon}_{cr}} \left\{ \dfrac{\partial \epsilon^T(x,t)}{\partial t} \right\} \longrightarrow 0$	compressive + bending strain reversal
t^{*}_{cr}	$\lim\limits_{t \to t^{*}_{cr}} \left\{ \log \left[\dfrac{\partial w(x,t)}{\partial t} \right] \right\} \longrightarrow$ const.	constant log[deformation] rate
t_{LF}	$F_1(\sigma_{applied}, t_{LF}) \leq F_2(\sigma_{failure}, t_{LF})$ prescribed $\widetilde{P}(t_{LF}) \leq 1$	deterministic or stochastic material failure
t_{LFD}	prescribed design life time t_{LFD}	deterministic or stochastic material failure

Titles of the terms under evaluation here are listed in Eqs. (4), (15), (20) and (21), and in Table 2.

Term T_1

The nonlinear nature of the internal resisting moments is highly dependent on the size of the bending deflections w and/or on material property characteristics. Either nonlinear source can take place in the absence of the other one.

Terms T_2 , T_5, T_7 and T_9

These are the in plane tractions and shears due to boundary constraints. If included in the governing relation, their contributions are always nonlinear.

Terms T_3, T_6, T_8 and T_{10}

These terms represent externally supplied in plane tractions and shears. Their contributions are always linear and independent of deflections unless follower loads are included.

Term T_4

The panel inertia under Newton's law will always be linear. The only exception, which is not applicable here, are cases where the deformation affects the material volume seriously enough to influence changes in the mass m_p to make it a function of the strains and hence of the deflection w. Even for quasi-static conditions of creep torsional divergence, flight control reversal and creep buckling, it has been shown that the dynamic term is most important due to the material creep influence [60]. Its inclusion in the governing relations significantly changes the time stability boundaries.

Term T_{11}

This term represents the lift forces on the panel, which are influenced by the wing bending and torsional deformations $W(x_2, t)$ and $\theta(x_2, t)$, and the built-in rigid zero lift and trim angles of attack $\alpha_r(x_2)$, $\alpha_0(x_2)$ and $\alpha(x_2)$ as well as the panel deflection $w(\widetilde{x}, t)$. The possible nonlinearities are geometric depending on angle sizes which in turn define the four independent approximations

$$\vartheta_{total} \approx \sin(\vartheta_{total}) \quad \text{or} \quad \frac{\partial w}{\partial x_1} \approx \arctan\left(\frac{\partial w}{\partial x_1}\right) \quad \text{or} \quad \frac{\partial w}{\partial t} \approx \arctan\left(\frac{\partial w}{\partial t}\right)$$

$$\text{or} \quad \frac{1}{U_\infty}\frac{\partial W}{\partial t} \approx \arctan\left(\frac{1}{U_\infty}\frac{\partial W}{\partial t}\right) \tag{40}$$

Terms T_{12} and T_{13}

External piezoelectric control forces, if present, are independent of wing and panel deflections and dependent only on the applied EMF. The elastic or viscoelastic piezoelectric constitutive relations for devices in the plane of the plate are

$$\text{elastic} \quad \Longrightarrow \quad \epsilon^{E\mathbf{P}}_{\underline{ii}}(x,t) = E^{E\mathbf{P}}_{\underline{ii}}(x,t)\, \mathbf{V}_{\underline{ii}}(x,t) \qquad i = 1,2 \tag{41}$$

$$\text{viscoelastic} \quad \Longrightarrow \quad \epsilon^{\mathbf{P}}_{\underline{ii}}(x,t) \;=\; \int\limits_{-\infty}^{t} E^{\mathbf{P}}_{\underline{ii}}(x,t,t')\, \mathbf{V}_{\underline{ii}}(x,t')\, dt' \qquad i=1,2 \tag{42}$$

or conversely

$$\text{elastic} \quad \Longrightarrow \quad \mathbf{V}_{\underline{ii}}(x,t) \;=\; C^{E\mathbf{P}}_{\underline{ii}}(x,t)\, \epsilon^{E\mathbf{P}}_{\underline{ii}}(x,t) \qquad i=1,2 \tag{43}$$

$$\text{viscoelastic} \quad \Longrightarrow \quad \mathbf{V}_{\underline{ii}}(x,t) \;=\; \int\limits_{-\infty}^{t} C^{\mathbf{P}}_{\underline{ii}}(x,t,t')\, \epsilon^{\mathbf{P}}_{\underline{ii}}(x,t')\, dt' \qquad i=1,2 \tag{44}$$

where the piezoelectric voltages $\mathbf{V}$ are either externally applied or power is extracted through resistors to effect control. In the latter case the piezoelectric forces depend on the strains through Eqs. (43) and (44). The piezoelectric contributions may be linear or nonlinear depending on the nature of material properties.

Terms T_{14} through T_{19}

These thermal contributions are due to thermal expansions and bending moments produced by thermal stress. They may be linear or nonlinear depending on material properties and/or deflections. (See next section.)

Term T_{20}

Forces due to pressure fluctuations (aerodynamic noise) are subject to the same angular sizing arguments which govern the lift forces T_{11} and are discussed in Eq. (40) although the trigonometric function arguments are different.

In-depth quantitative sensitivity investigations of these nonlinear effects have been conducted and are evaluated in the discussion section.

Term T_{21}

Open loop vibratory forces are independent of deflections and hence linear.

Term T_{22}

Closed loop servo-aero-viscoelastic forces may be linear or nonlinear depending on the nature of the controllers and/or aerodynamic forces.

Term T_{23}

Whenever the analysis includes a change in plate length in either x_1, x_2 or s directions, the results are inherently geometrically nonlinear for any plate material.

Term T_{24} and T_{25}

These terms embody geometric contributions to strain components associated with large displacements. If included then problem is inherently nonlinear.

Terms T_{26} and T_{27}

Constitutive relations used in the definition of flexible boundary conditions dependent on the nature and material of the plate support. Their linear or nonlinear behavior is independent of plate material characterizations as defined by Terms T_1. Indeed, the plate and its supports may be made of totally dissimilar materials.

Terms T_{31} and T_{33}

These terms are inherently nonlinear and if they are linearized they will simply vanish.

3 Discussion

A detailed parametric analysis is somewhat misleading and impractical since stability boundaries for these self-excited problems are most sensitive to the parametric combinations rather than to their individual values. This is due to the complicated phase interrelations between the various terms of Eq. (4) . The panel is subject to buckling, flutter and material failures in time and the prevailing failure mode is the one that occurs first. Fig. 1 depicts deformations in the various plate modes for a particular set of parametric combinations. The contributions of panel creep buckling and composite fiber delaminations are shown in Fig. 2 . Depending on geometry either phenomena could precede the other and thus establish failure modes and survival times. Finally in Fig. 4 linear and nonlinear analysis results are compared indicating that linearization leads to conservative estimates.

The application of designer material analysis [61] with tailored/engineered material properties would significantly improve flutter and failure performances. This is accomplished by minimizing and delaying, but not eliminating, the latter occurrences through proper material property specifications and management.

4 Conclusions

- Due the extensive complexities of the contributory terms to the governing relations, no analytical order of magnitude sensitivity analyses can be performed

- There appears to be no direct relation between flutter velocities and frequencies induced by the main lift forces and those stemming from aerodynamic noise fluctuations

- Wing flutter velocities and frequencies may differ significantly from the corresponding panel ones.

- Critical instability times and material failure lifetimes are unrelated to each other due to the distinct and separate causes defining each event

- In dynamic problems the presence of control forces and/or mechanical or material damping may have either stabilizing or destabilizing effects depending on the phase relations of the entire system
- A limited number of delamination simulations indicate that linear results are significantly more conservative than their nonlinear counterparts based on failure probabilities and lifetimes
- The use of designer materials presents an effective means of structural control without additional devices by best utilizing mechanical material properties to accomplished prescribed tasks
- The urgent need for efficient analytical and computational protocols for the solution of integral-diffferential equations remain a high priority item

Appendix A – Nonlinear Constitutive Relation Formulations

For materials with nonlinear constitutive relations, Eqs. (1) and (2) may be recast in terms of strain and strain rate invariants defined by

$$I_1 = \frac{1}{3}\epsilon_{ii} \qquad I_2 = \frac{1}{3}\epsilon_{ij}\,\epsilon_{ij} \qquad I_3 = \frac{1}{3}\epsilon_{ij}\,\epsilon_{ik}\,\epsilon_{kj} \tag{45}$$

$$\dot{I}_1 = \frac{1}{3}\dot{\epsilon}_{ii} \qquad \dot{I}_2 = \frac{1}{3}\dot{\epsilon}_{ij}\,\dot{\epsilon}_{ij} \qquad \dot{I}_3 = \frac{1}{3}\dot{\epsilon}_{ij}\,\dot{\epsilon}_{ik}\,\dot{\epsilon}_{kj} \tag{46}$$

where $\dot{I}_k \neq dI_k/dt$. The moduli are now defined as

$$\begin{array}{lcrcl}
\text{linear elastic} & \Longrightarrow & E^E_{ijkl} & = & E^E_{ijkl}(x) \\
\text{nonlinear elastic} & \Longrightarrow & E^E_{ijkl} & = & E^E_{ijkl}\left[x, I(x,t)\right] \\
\text{linear viscoelastic} & \Longrightarrow & E_{ijkl} & = & E_{ijkl}\left[x, t, t', T(x,t')\right] \\
\text{nonlinear viscoelastic} & \Longrightarrow & E_{ijkl} & = & E_{ijkl}\Big[x, t, t', T(x,t'), \\
 & & & & \qquad I(x,t'), \dot{I}(x,t')\Big]
\end{array} \tag{47}$$

The viscoelastic relaxation moduli are expressed in terms of Prony series [32] as

linear isothermal material $\Longrightarrow$

$$E_{ijkl}(x,t) = E_{ijkl\infty}(x) + \sum_{n=1}^{N} E_{\underline{ijkl}n}(x)\,\exp\left(-\frac{t}{\tau_{\underline{ijkl}n}(x)}\right) \tag{48}$$

$$\text{nonlinear material} \qquad \Longrightarrow \qquad E_{ijkl}(x,t) \;=\; E_{ijkl\infty}(x,I)$$

$$+ \sum_{n=1}^{N} E_{\underline{ijkln}}(x,I)\; \exp\left(- \int^{t} \frac{ds}{\tau_{\underline{ijkln}}\left[x,s,T(x,s),I(x,s),I'(x,s)\right]} \right) \tag{49}$$

with $I = \{I_1, I_2, I_3\}$ and $I' = \{I_1', I_2', I_3'\} \neq dI/ds$. The nonlinear constitutive relations (47) and (49) constitute a feed back system as the material parameters E_{ijkln} and τ_{ijkln} must be continuously updated as the stress and strain states change with time and position.

The elastic moduli can be readily extracted from Eqs. (48) and (49) as

$$E^{E}_{ijkl} \;=\; E_{ijkl\infty} \;+\; \sum_{n=1}^{N} E_{ijkln} \tag{50}$$

The influence of structural damping (friction [56]) can be included through structural damping coefficients $g_{ijkl} \geq 0$ [59] and modifying elastic moduli to $(1 + \imath\, g_{\underline{ijkl}}) E^{E}_{\underline{ijkl}}$ with $\imath = \sqrt{-1}$. Structural damping commonly occurs in bolted, bonded, welded or riveted joints and has a range of accepted values $0 \leq g_{ijkl} \leq .05$.

When structural damping effects are to be included, then Eqs. (48) and (49) must be modified to read

$$\text{linear material} \qquad \Longrightarrow \qquad E_{ijkl}(x,t) \;=\; \Big[1 + \imath\, g_{\underline{ijkl}}(x)\Big]\; E^{E}_{\underline{ijkl}}(x)$$

$$+ \sum_{n=1}^{N} E_{\underline{ijkln}}(x) \left[\exp\left(- \frac{\xi(x,t)}{\tau_{\underline{ijkln}}(x)} \right) \;-\; 1 \right] \tag{51}$$

with the reduced time ξ defined for thermo-rheologically simple materials (TSM) as

$$\xi(x,t) \;=\; \int_{-\infty}^{t} a_T\left[T(x,t')\right]\, dt' \tag{52}$$

and

$$\text{nonlinear material} \qquad \Longrightarrow \qquad E_{ijkl}(x,t) \;=\; \Big[1 + \imath\, g_{\underline{ijkl}}(x)\Big]\; E^{E}_{\underline{ijkl}}(x,I)$$

$$+ \sum_{n=1}^{N} E_{\underline{ijkln}}(x,I) \left[\exp\left(- \int^{t} \frac{ds}{\tau_{\underline{ijkln}}\left[x,s,T(x,s),I(x,s),I'(x,s)\right]} \right) \;-\; 1 \right] \tag{53}$$

These nonlinear constitutive relations, in what ever form they may be exhibited, represent a closed loop system since the material property parameters E_{ijkln} and τ_{ijkln} are functions of strains and their time histories and must constantly be updated as the strains vary in time.

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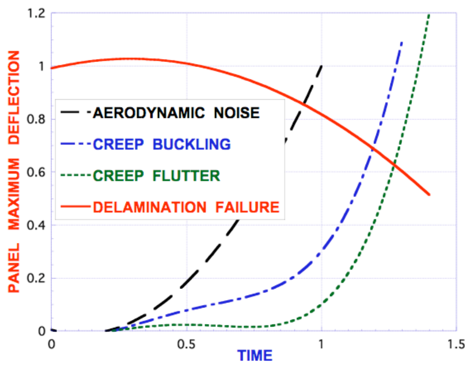

Figure 1: The Various Panel Conditions

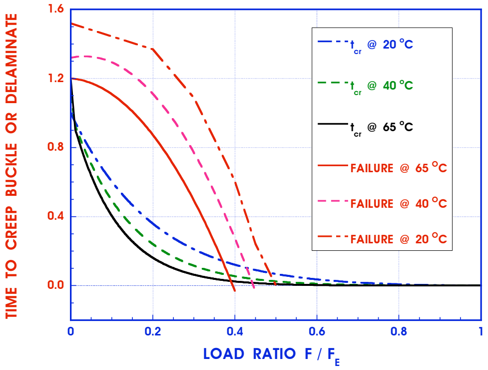

Figure 2: Time to Creep Buckle or Delaminate

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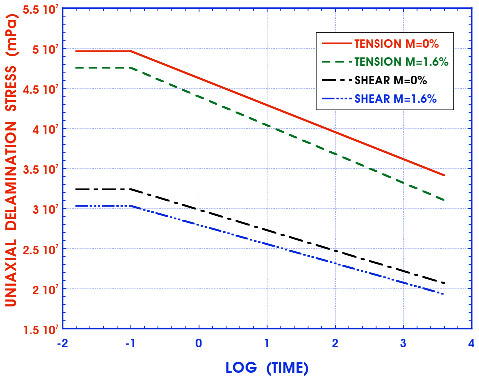

Figure 3: Uniaxial Delamination Stresses [46]

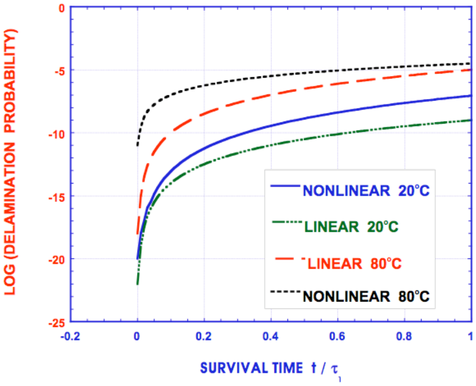

Figure 4: Probability of Delamination