Applied Mathematical Sciences, Vol. 4, 2010, no. 73, 3625 - 3642

Coordinates Transformation and Stabilization for Switching Models of Actuators in Servoelastic Framework

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Abstract

Consideration of the effects of the mounting structure stiffness on electrohydraulic servomechanisms actuating primary flight controls yields a switched control system of seven nonlinear differential equations. The stabilizing control synthesis has to cope with the critical case of a zero eigenvalue of the Jacobian matrices calculated in equilibria. By local coordinates transformations, the switching components are decoupled into a common linear part and nonlinear equations without linear terms. Existence of a Common Lyapunov Function and arguments from Lyapunov-Malkin Theorem prove the stability of all equilibria for the switched system and a desired asymptotic behavior.

Mathematics Subject Classification: 34D20, 93B27, 93B52

Keywords: single-input single-output control system, local coordinates transformation, common Lyapunov function, switched system, stability

1 Introduction

The electrohydraulic servomechanisms (EHS) are widely used in various control technologies. Features as large processing force and stiffness, high payload capabilities, good positioning and power to weight ratio make this type of actuation systems appropriate for positioning aircraft control surfaces (flight control), high power industrial machinery, position control of military gun turrets and antennas or material handling. To control hydraulic systems is always a challenge due to the presence of nonlinearities that arise from fundamental properties such as fluid compressibility, complex flow properties of hydraulic valves and friction characteristic. Control synthesis is supposed to cope with all this and to ensure the desired behavior of EHSs.

The EHS is in fact a tracking system. For such a system a qualitative statement of the tracking (or servomechanism) problem (see [2]) is the following.

Let us suppose that the desired output of the system (e.g., herein, $x_1(t)$) is required to track some desired function (e.g. $x_{1d}(t)$). Provide then a control law $u(t)$ that will cause, eventually in some optimal sense, this tracking ($\lim\limits_{t\to\infty}(x_1(t)-x_{1d}(t))=0$) (see also [38]).

The modern approach in synthesis of control laws for EHSs considered as tracking systems is the development of strategies directly applicable to large classes of nonlinear models. So, in [26], [27], succesive Galerkin approximations of the solution of the Hamilton-Jacobi-Bellman equation, respectively of the Hamilton-Jacobi-Isaacs equation, are used to solve optimal control problems. A nonlinear robust C^∞ control law is obtained in [27]. Energy criteria in the synthesis of EHSs are considered in [21]. Nonlinear adaptive robust strategies are presented in [6], [41], [42] using backstepping and discontinuous projections introduced in [32]. In [13] and [38] the standard theory of backstepping is used to produce controllers for EHSs. Feedback stabilization by Jurdjevic-Quinn methods is developed in [31].

A special case of the tracking problem is the so called regulator problem (see [2]) for which the desired output is an equilibrium of the system. It has the following statement.

Let us suppose that the initial conditions for some solution are different from any equilibrium point. Provide a control law $u(t)$ to bring some components of the solution (e.g. $x_1(t)$) close to the same components of an equilibrium point.

This kind of problems, involving stabilization of equilibria and asymptotic stability with respect to some state variables is solved, for different mathematical models of servomechanisms, in recent works of the authors [11], [12], [14], [39] combining classical and modern tools: the Lyapunov-Malkin approach to critical cases in stability theory ([24]) and geometric methods in control synthesis ([19]). One practical application refers to altitude-hold autopilot synthesis, aiming to maintain a desired altitude of an aircraft allowing the pilot to perform other tasks.

The influence of mounting structure's elasticity on system's stability has been neglected in most models aiming at the control design of EHSs for flight control systems (see [22], [34]).

One original feature of the present paper is to develop control synthesis in a particular servoelastic framework defined by the finite mounting structure stiffness of an EHS. Another characteristic of the approach, used for the first time on this model, is the stabilization of the switched system equilibria (see [23]). Thus, as a component of the aeroservoelastic system, the EHS will contribute to general stability and performance of flight control system.

The paper is organised as follows. In §2, the mathematical model is introduced. In §3 control synthesis is achieved through the use of coordinate transformations as in [19] and of the proof of a Malkin-type theorem for the resulting switched system. A final section is devoted to concluding remarks.

2 The mathematical model

There is a rich literature dedicated to mathematical modelling of servoactuators, both mecanohydraulic and electrohydraulic: from classical books [4], [8], [9], [28] to modern approaches [1], [16], [22], [25].

Figure 1 shows a typical physical model of an EHS in the servoelastic framework given by the consideration of the mounting structure stiffness E. The EHS is a combination of an electrohydraulic servovalve (EHSV) and a hydrocylinder with piston. The paradigm of an EHSV with ideal spool valve ([1], [28]) and having rectangular ports is assumed. An algebraic sign convention is chosen,i.e.the positive incoming in the cylinder flow and negative outgoing from the cylinder flow. The flow into and out of the cylinder is described by two components, one due to the movement of the piston and the other one to compressibility effects.

For $\sigma \geq 0$

$$\begin{aligned} c_d w \sigma \sqrt{\frac{2(p_s - p_1)}{\rho}} &= S(\dot z - \dot z_e) + \frac{V_0 + S(z - z_e)}{B}\dot p_1 \\ -c_d w \sigma \sqrt{\frac{2p_2}{\rho}} &= -S(\dot z - \dot z_e) + \frac{V_0 - S(z - z_e)}{B}\dot p_2 \end{aligned} \tag{2.1}$$

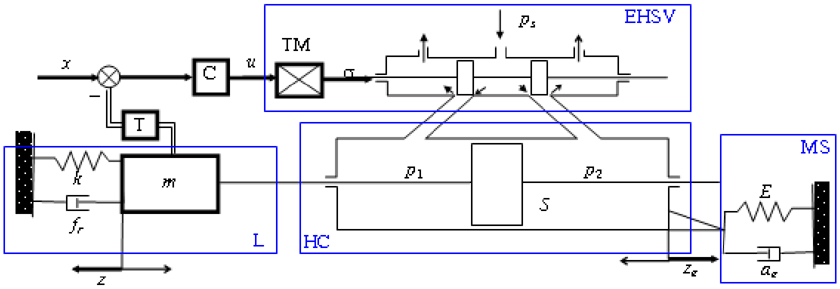

Fig. 1 - Physical model of EHS including finite mounting structure stiffness E C - controller; T - transducer; TM - torque motor; EHSV - electrohydraulic servovalve; L - load; HC - hydraulic cylinder; MS - mounting structure

and for $\sigma < 0$

$$\begin{aligned} c_d w \sigma \sqrt{\frac{2p_1}{\rho}} &= S(\dot{z} - \dot{z}_e) + \frac{V_0 + S(z - z_e)}{B}\dot{p}_1 \\ -c_d w \sigma \sqrt{\frac{2(p_s - p_2)}{\rho}} &= -S(\dot{z} - \dot{z}_e) + \frac{V_0 - S(z - z_e)}{B}\dot{p}_2 \end{aligned} \tag{2.1$'$}$$

(see the Nomenclature).

The equation of motion of the hydraulic cylinder piston assembly is defined as a force balance equation related to the displacement of the load (L) generated by the hydraulic pressures p_1 and p_2

$$m\ddot{z} + f_r z + kz = S(p_1 - p_2). \tag{2.2}$$

Also the equation of motion of mass m_e, related to the hydraulic servo body and to the attached neighbour structure, involves the mounting structure stiffness E and the structural damping a_e

$$m_e\ddot{z}_e + a_e\dot{z}_e + Ez_e = -S(p_1 - p_2). \tag{2.3}$$

Usual values of structural damping a_e are given in the literature [25] as corresponding to 2% of critical damping ratio, so $a_e = 0,04\sqrt{Em_e}$.

The linear displacement induced to spool valve by the torque motor (TM) of EHSV is described as the first order equation

$$\tau\dot{\sigma} + \sigma = k_v u. \tag{2.4}$$

The relative displacement spool-sleeve σ is induced in the presence of the control signal u.

For the system, the aim of the control synthesis is to have a good tracking by the load displacement z – the systems's output – of the specified desired position references signals r – the system's input. The state variables are

$$x_1 := z;\ x_2 := \dot{z};\ x_3 := z_e;\ x_4 := \dot{z}_e;\ x_5 := p_1;\ x_6 := p_2;\ x_7 := \sigma. \tag{2.5}$$

Using the notation

$$C := c_d w \sqrt{\frac{2}{\rho}}\,, \tag{2.6}$$

the canonical first order system derived from equations (2.1)–(2.4) for $x_7 \geq 0$ is

$$\mathcal{S}_1 : \begin{aligned} &\dot{x}_1 = x_2 \quad \dot{x}_2 = -\frac{k}{m}x_1 - \frac{f_r}{m} + \frac{S}{m}(x_5 - x_6) \\ &\dot{x}_3 = x_4, \quad \dot{x}_4 = -\frac{E}{m_e}x_3 - \frac{a_e}{m_e}x_4 - \frac{S}{m_e}(x_5 - x_6) \\ &\dot{x}_5 = \frac{BS}{V_0 + S(x_1 - x_3)}\left(\frac{C}{S}x_7\sqrt{p_s - x_5} - x_2 + x_4\right) \\ &\dot{x}_6 = -\frac{BS}{V_0 - S(x_1 - x_3)}\left(\frac{C}{S}x_7\sqrt{x_6} - x_2 + x_4\right), \\ &\dot{x}_7 = -\frac{1}{\tau}x_7 + \frac{k_v}{\tau}u_1 \end{aligned} \tag{2.7}$$

and, for $x_7 < 0$,

$$\mathcal{S}_2 : \begin{aligned} &\dot{x}_1 = x_2, \quad \dot{x}_2 = -\frac{k}{m}x_1 - \frac{f_r}{m}x_2 + \frac{S}{m}(x_5 - x_6) \\ &\dot{x}_3 = x_4, \quad \dot{x}_4 = -\frac{E}{m_e}x_3 - \frac{a_e}{m_e}x_4 - \frac{S}{m_e}(x_5 - x_6) \\ &\dot{x}_5 = \frac{BS}{V_0 + S(x_1 - x_3)}\left(\frac{C}{S}x_7\sqrt{x_5} - x_2 + x_4\right) \\ &\dot{x}_6 = -\frac{BS}{V_0 - S(x_1 - x_3)}\left(\frac{C}{S}x_7\sqrt{p_s - x_6} - x_2 + x_4\right), \\ &\dot{x}_7 = -\frac{1}{\tau}x_7 + \frac{k_v}{\tau}u_2 \end{aligned} \tag{2.8}$$

Obviously one must have $x_i \in (0, p)$, $i = 5, 6$. Then the fifth and the sixth equations in (2.7), (2.8) are deduced from

$$\begin{aligned}\dot{x}_5 &= \frac{B}{V_0 + S(x_1 - x_3)}(C|x_7|\text{sgn}\,[p_s(1+\text{sgn}\,x_7) - 2x_5]\cdot \\ &\cdot\sqrt{\frac{|p_s|(1+\text{sgn}\,x_7) - 2x_5|}{2}} - Sx_2 + Sx_4) \\ \dot{x}_6 &= \frac{B}{V_0 - S(x_1 - x_3)}(C|x_7|\text{sgn}\,[p_s(1-\text{sgn}\,x_7) - 2x_6]\cdot \\ &\cdot\sqrt{\frac{|p_s|(1-\text{sgn}\,x_7) - 2x_6|}{2}} + Sx_2 - Sx_4)\end{aligned} \tag{2.9}$$

With $(x, p) \in G \subset \mathbf{R}^2$ defined by

$$|x| < \frac{V_0}{S}, \quad p \in (0,1), \quad p_s(1-p) - \frac{kx}{S} > 0, \quad p_s p + \frac{kx}{S} > 0. \tag{2.10}$$

introduce, as in [3], [16], [17], [18],

$$\hat{x}_1 = x,\ \hat{x}_2 = 0,\ \hat{x}_3 = -\frac{k}{E}x,\ \hat{x}_4 = 0,\ \hat{x}_5 = p_s p + \frac{k}{S}x,\ \hat{x}_6 = p_s p,\ \hat{x}_7 = 0. \tag{2.11}$$

If u_1 and u_2 are feedback laws that satisfy

$$u_1(\hat{x}) = 0, \quad u_2(\hat{x}) = 0, \quad \hat{x} = (\hat{x}_1, \ldots, \hat{x}_7) \tag{2.12}$$

then $\hat{x}$ are equilibria for both (2.7) and (2.8).

Synthesis of controllers u_1 and u_2 is aimed to assure stability of all $\hat{x}$ in (2.11) and a good asymptotic behavior

$$\lim_{t\to\infty}[x_1(t) - x] = 0. \tag{2.13}$$

Systems (2.7) and (2.8) will be considered as defined for $x_7 \in \mathbf{R}$ and together they compose a switched system (see [33]).

3 Coordinate transformation and control synthesis

For the synthesis of a stabilizing controller we rely on some geometric constructions from [19]. Recall that a single-input single-output control system is given by

$$\begin{aligned}\dot{\bar{x}} &= f(\bar{x}) + g(\bar{x})u \\ y &= h(\bar{x})\end{aligned} \tag{3.1}$$

where $\bar{x}$ belongs to an open set $U \subset \mathbf{R}^n$, $f, g : U \to \mathbf{R}^n$ and $h : U \to \mathbf{R}$ are smooth functions. In [19] one finds a characterization of the systems (3.1) that admit, through coordinate transformation, a normal form that is linear and controllable. The main ingredient is the notion of relative degree.

Let $L_f h$ denote the Lie derivative of h along f, $L_f h = \sum_{j=1}^{n} \frac{\partial h}{\partial x_j} f_j$.

Definition. System (3.1) has relative degree r at a point $\bar{x}^0 \in U$ if

(1) $(L_g L_f^k h)(\bar{x}) = 0$ for every $\bar{x}$ in a neighbourhood of $\bar{x}^0$, for $k = 0, 1, \ldots, r-2$.

(2) $(L_g L_f^{r-1} h)(\bar{x}^0) \neq 0$.

When the relative degree in $\bar{x}^0$ is n then there exists a coordinate transformation that, locally around $\bar{x}^0$, brings (3.1) to a linear and controllable form. When $r < n$ it is still possible to find a coordinate transformation such that the resulting new system has a simpler form: a linear subsystems of r equations that contains all the information on the input-output behavior and a nonlinear subsystems of $(n-r)$ equations.

For systems (2.7) and (2.8), $g(\bar{x}) = (0, 0, \ldots, 0, 1)^\tau$ (τ means transpose), $\bar{x} = (x_1, \ldots, x_7)$,

$$\begin{aligned} f^{(1)}(\bar{x}) = \Bigg(& x_2, -\frac{k}{m}x_1 - \frac{f_r}{m}x_2 + \frac{S}{m}(x_5 - x_6), x_4, \\ & -\frac{E}{m_e}x_3 - \frac{a_e}{m_e}x_4 - \frac{S}{m_e}(x_5 - x_6), \frac{BCx_7\sqrt{p_s - x_5}}{V_0 + S(x_1 - x_3)} - \frac{BS(x_2 - x_4)}{V_0 + S(x_1 - x_3)}, \\ & -\frac{BCx_7\sqrt{x_6}}{V_0 - S(x_1 - x_3)} + \frac{BS(x_2 - x_4)}{V_0 - S(x_1 - x_3)}, -\frac{1}{\tau}x_7 \Bigg) \end{aligned} \tag{3.2}$$

for (2.7) and

$$\begin{aligned} f^{(2)}(\bar{x}) = \Bigg(& x_2, -\frac{k}{m}x_1 - \frac{f_r}{m}x_2 + \frac{S}{m}(x_5 - x_6), x_4, \\ & -\frac{E}{m_e}x_3 - \frac{a_e}{m_e}x_4 - \frac{S}{m_e}(x_5 - x_6), \frac{BCx_7\sqrt{x_5}}{V_0 + S(x_1 - x_3)} - \frac{BS(x_2 - x_4)}{V_0 + S(x_1 - x_3)}, \\ & -\frac{BCx_7\sqrt{p_s - x_6}}{V_0 - S(x_1 - x_3)} + \frac{BS(x_2 - x_4)}{V_0 - S(x_1 - x_3)}, -\frac{1}{\tau}x_7 \Bigg) \end{aligned} \tag{3.3}$$

for (2.8). Due to (2.12) it is natural to consider

$$h(\bar{x}) = x_1 - \hat{x}_1 \tag{3.4}$$

as output of (2.7) and (2.8). Then

$$L_{f^{(1)}} h = x_2, \; L_{f^{(1)}}^2 h = -\frac{k}{m}x_1 - \frac{f_r}{m}x_2 + \frac{S}{m}(x_5 - x_6)$$

$$\begin{aligned} L^3_{f^{(1)}}h = {} & \frac{f_r k}{m^2}x_1 + \left(\frac{f_r^2}{m^2} - \frac{k}{m}\right)x_2 - \frac{Sf_r}{m^2}(x_5 - x_6) + \\ & + \frac{S}{m}\left[\frac{BCx_7\sqrt{p_s - x_5}}{V_0 + S(x_1 - x_3)} + \frac{BCx_7\sqrt{x_6}}{V_0 - S(x_1 - x_3)} + \frac{2BSV_0(x_4 - x_2)}{V_0^2 - S^2(x_1 - x_3)^2}\right] \end{aligned} \tag{3.5}$$

It is obvious that $(L_g L^k_{f^{(1)}}h)(x) = 0$, $\forall x \in \mathbf{R}^7$, $k = 0, 1, 2$ and

$$\begin{aligned} L_g L^3_{f^{(1)}}h(\hat{x}) = {} & \frac{\partial}{\partial x_7}(L^3_{f^{(1)}})(\hat{x}) = \\ = {} & \frac{SBC}{m}\left(\frac{\sqrt{p_s - \hat{x}_5}}{V_0 + S(\hat{x}_1 - \hat{x}_3)} + \frac{\sqrt{\hat{x}_6}}{V_0 - S(\hat{x}_1 - \hat{x}_3)}\right) \neq 0 \end{aligned} \tag{3.6}$$

so it follows that (2.7) has relative degree 4 in $\hat{x}$ for every $\hat{x}$ given in (2.10). Also

$$\begin{aligned} & (L_{f^{(2)}}h)(\bar{x}) = x_2,\ (L^2_{f^{(2)}}h)(\bar{x}) = -\frac{k}{m}x_1 - \frac{f_r}{m}x_2 + \frac{S}{m}x_5 - \frac{S}{m}x_6, \\ & (L^3_{f^{(2)}}h)(\bar{x}) = -\frac{k}{m}x_2 - \frac{f_r}{m}\left[-\frac{k}{m}x_1 - \frac{f_r}{m}x_2 + \frac{S}{m}(x_5 - x_6)\right] + \\ & + \frac{S}{m}\left[\frac{BCx_7\sqrt{x_5}}{V_0 + S(x_1 - x_3)} + \frac{BCx_7\sqrt{p_s - x_6}}{V_0 - S(x_1 - x_3)} + \frac{2BSV_0(x_4 - x_2)}{V_0^2 - S^2(x_1 - x_3)^2}\right] \end{aligned}$$

Again $(L_g L^k_{f^{(2)}}h)(\bar{x}) = 0$, $\forall \bar{x} \in \mathbf{R}^7$, $k = 0, 1, 2$ and

$$(L_g L^3_{f^{(2)}}h)(\hat{x}) = \frac{BSC}{m}\left(\frac{\sqrt{\hat{x}_5}}{V_0 + S(\hat{x}_1 - \hat{x}_3)} + \frac{\sqrt{p_s - \hat{x}_6}}{V_0 - S(\hat{x}_1 - \hat{x}_3)}\right) \neq 0 \tag{3.7}$$

Thus both systems (2.7) and (2.8) have relative degree 4 in $\hat{x}$ for every $\hat{x}$ given in (2.10).

Consider then the following coordinate transformations

$$\begin{aligned} & \Phi^{(1)}_1(\bar{x}) = x_1 - \hat{x}_1, \quad \Phi^{(1)}_2(\bar{x}) = x_2, \quad \Phi^{(1)}_3(\bar{x}) = (L^2_{f^{(1)}}h)(\bar{x}), \\ & \Phi^{(1)}_4(\bar{x}) = (L^3_{f^{(1)}}h)(\bar{x}), \quad \Phi^{(1)}_5(\bar{x}) = x_3 - \hat{x}_3, \quad \Phi^{(1)}_6(\bar{x}) = x_4, \\ & \Phi^{(1)}_7(\bar{x}) = x_5 - \hat{x}_5 \end{aligned} \tag{3.8}$$

and

$$\begin{aligned} & \Phi^{(2)}_1(\bar{x}) = \Phi^{(1)}_1(\bar{x}), \quad \Phi^{(2)}_2(\bar{x}) = \Phi^{(1)}_2(\bar{x}), \quad \Phi^{(2)}_3(\bar{x}) = \Phi^{(1)}_3(\bar{x}), \\ & \Phi^{(2)}_4(\bar{x}) = (L^3_{f^{(2)}}h)(\bar{x}), \quad \Phi^{(2)}_5(\bar{x}) = \Phi^{(1)}_5(\bar{x}), \quad \Phi^{(2)}_6(\bar{x}) = \Phi^{(1)}_6(\bar{x}), \\ & \Phi^{(2)}_7(\bar{x}) = \Phi^{(1)}_7(\bar{x}). \end{aligned} \tag{3.9}$$

The Jacobi matrices of $\Phi^{(1)}$ and $\Phi^{(2)}$ calculated in equilibria are

$$J^{(l)} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -\dfrac{k}{m} & -\dfrac{f_r}{m} & 0 & 0 & \dfrac{S}{m} & -\dfrac{S}{m} & 0 \\ j_{41} & j_{42} & 0 & j_{44} & -\dfrac{Sf_r}{m^2} & \dfrac{Sf_r}{m^2} & j_{47}^{(l)} \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

and $\det J^{(l)} = -\dfrac{S}{m} j_{47}^{(l)} \neq 0$, $l = 1, 2$ so $\Phi^{(l)}$ are local diffeomorphisms around $\hat{x}$. Remark that $\Phi^{(l)}(\hat{x}) = 0$, $l = 1, 2$ for every $\hat{x}$ defined in (2.10).

Define, following [19],

$$u_l(\bar{x}) = \frac{1}{(L_g L^3_{f^{(l)}} h)(\bar{x})} \left[-(L^4_{f^{(l)}} h)(\bar{x}) + \sum_{j=1}^{6} c_j \Phi_j^{(l)}(\bar{x}) \right], \ l = 1, 2 \tag{3.10}$$

where it is useful to remark that in (3.6) and (3.7) $(L_g L^3_{f^{(l)}} h)(\bar{x}) \neq 0$ for every $\bar{x}$ in the domains where $f^{(1)}$, respectively $f^{(2)}$ are defined.

If new coordinates are introduced through

$$z_j = \Phi_j^{(l)}(\bar{x}), \quad j = 1, \ldots, 7, \quad l = 1, 2 \tag{3.11}$$

systems (2.7), (2.8) turn into

$$\Sigma_1 : \begin{array}{l} \dot{\xi} = D\xi \\ \dot{z}_7 = q_7^{(1)}(z) \end{array} \quad ; \qquad \Sigma_2 : \begin{array}{l} \dot{\xi} = D\xi \\ \dot{z}_7 = q_7^{(2)}(z) \end{array} \tag{3.12}$$

with $\xi = (z_1, \ldots, z_6)^\tau$, $z = (z_1, \ldots, z_7) = (\xi^\tau, z_7)$,

$$D = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ c_1 & c_2 & c_3 & c_4 & c_5 & c_6 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -\dfrac{k}{m_e} & -\dfrac{f_r}{m_e} & -\dfrac{m}{m_e} & 0 & -\dfrac{E}{m_e} & -\dfrac{a_e}{m_e} \end{pmatrix}. \tag{3.13}$$

Introduce, for further use

$$x_0 = \hat{x}_1 - \hat{x}_3.$$

The inverses $\Psi^{(l)} = \Phi^{(l)-1}$ of the transformations (3.8), (3.9) are given, for $l = 1, 2$, by

$$x_1 = \Psi_1^{(l)}(z) = z_1 + \hat{x}_1,\ x_2 = \Psi_2^{(l)}(z) = z_2,\ x_3 = \Psi_3^{(l)}(z) = z_5 + \hat{x}_3,\ x_4 = \Psi_4^{(l)}(z) = z_6,$$

$$x_5 = \Psi_5^{(l)}(z) = z_7 + \hat{x}_5,\ x_6 = \Psi_6^{(l)}(z) = -\frac{m}{S}z_3 - \frac{k}{S}z_1 - \frac{fr}{S}z_2 + z_7 - \frac{k}{S}\hat{x}_1 + \hat{x}_5$$

and, for $l = 1$,

$$\begin{aligned} x_7 = \Psi_7^{(1)}(z) = \frac{m}{BCS}\left[\frac{\sqrt{p_s - z_7 - \hat{x}_5}}{V_0 + S(z_1 - z_3 + x_0)} + \frac{\sqrt{\Psi_6^{(1)}(z)}}{V_0 - S(z_1 - z_3 + x_0)}\right]^{-1} \cdot \\ \cdot \left[z_4 + \frac{k}{m}z_2 + \frac{fr}{m}z_3 - \frac{2BCSV_0(z_6 - z_2)}{V_0^2 - S^2(z_1 - z_3 + x_0)^2}\right] \end{aligned} \tag{3.14}$$

while for $l = 2$,

$$\begin{aligned} x_7 = \Psi_7^{(2)}(z) = \frac{m}{BCS}\left[\frac{\sqrt{z_7 + \hat{x}_5}}{V_0 + S(z_1 - z_3 + x_0)} + \frac{\sqrt{p_s - \Psi_6^{(2)}(z)}}{V_0 - S(z_1 - z_3 + x_0)}\right]^{-1} \cdot \\ \cdot \left[z_4 + \frac{k}{m}z_2 + \frac{fr}{m}z_3 - \frac{2BCSV_0(z_6 - z_2)}{V_0^2 - S^2(z_1 - z_3 + x_0)^2}\right] \end{aligned}$$

Since $\Psi^{(l)}(0) = \hat{x}$, $l = 1, 2$ it follows that

$$\Psi_7^{(l)}(0) = 0,\ l = 1, 2. \tag{3.15}$$

Now we compute $q_7^{(1)}$ and $q_7^{(2)}$. For $l = 1$

$$\begin{aligned} \dot{z}_7 = \dot{x}_5 = \frac{BCx_7\sqrt{p_s - x_5}}{V_0 + S(x_1 - x_3)} - \frac{BS(x_2 - x_4)}{V_0 + S(x_1 - x_3)} = \\ = \frac{BC\Psi_7^{(1)}(z)\sqrt{p_s - z_7 - \hat{x}_5}}{V_0 + S(z_1 - z_3 + x_0)} - \frac{BS(z_2 - z_6)}{V_0 + S(z_1 - z_3 + x_0)} = q_7^{(1)}(z) \end{aligned} \tag{3.16}$$

and, for $l = 2$,

$$\begin{aligned} \dot{z}_7 = \dot{x}_5 = \frac{BCx_7\sqrt{x_5}}{V_0 + S(x_1 - x_3)} - \frac{BS(x_2 - x_4)}{V_0 + S(x_1 - x_3)} = \\ = \frac{BC\Psi_7^{(2)}(z)\sqrt{z_7 + \hat{x}_5}}{V_0 + S(z_1 - z_3 + x_0)} - \frac{BS(z_2 - z_6)}{V_0 + S(z_1 - z_3 + x_0)} = q_7^{(2)}(z). \end{aligned} \tag{3.17}$$

It is useful for what follows to introduce also

$$\dot{z}_6 = -\frac{k}{m_e}z_1 - \frac{fr}{m_e}z_2 - \frac{m}{m_e}z_3 - \frac{E}{m_e}z_5 - \frac{a_e}{m_e}z_6 := q_6(z).$$

Remark that, by (3.13) and (3.15), $\frac{\partial q^{(1)}}{\partial z_7}(0) = \frac{BC\sqrt{p_s - \hat{x}_5}}{V_0 + Sx_0}\frac{\partial \psi_7^{(1)}}{\partial z_7}(0) = 0$ and also by (3.14) and (3.15) $\frac{\partial q^{(2)}}{\partial z_7}(0) = \frac{BC\sqrt{\hat{x}_5}}{V_0 + Sx_0}\frac{\partial \psi_7^{(2)}}{\partial z_7}(0) = 0$. Then the Jacobian matrices

$$B^{(l)} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ c_1 & c_2 & c_3 & c_4 & c_5 & c_6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ -\frac{k}{m_e} & -\frac{f_r}{m_e} & -\frac{m}{m_e} & 0 & -\frac{E}{m_e} & -\frac{a_e}{m_e} & 0 \\ b_{71}^{(l)} & b_{72}^{(l)} & b_{73}^{(l)} & b_{74}^{(l)} & b_{75}^{(l)} & b_{76}^{(l)} & 0 \end{pmatrix} \tag{3.18}$$

for systems Σ_l, $l = 1, 2$, in (3.11), calculated in zero, have $\lambda = 0$ as an eigenvalue, thus their characteristic polynomials are

$$P^{(l)}(\lambda) = \lambda P_1^{(l)}(\lambda), \quad l = 1, 2. \tag{3.19}$$

Suppose $P_1^{(l)}$ are Hurwitz for $l = 1, 2$.

Following the approach in the Lyapunov-Malkin Theorem in [24], perform linear transformations on systems Σ_l, $l = 1, 2$ such that in the resulting new systems the last equations have no linear terms.

If w satisfies $\dot{w} = B^{(l)}w$, $l = 1, 2$, $B^{(l)}$ given in (3.18), $\eta = \sum_{j=1}^{7} w_j$ satisfies $\dot{\eta} = 0$ if

$$d_4^{(l)}c_1 - d_6^{(l)}\frac{k}{m_e} + d_7^{(l)}b_{71}^{(l)} = 0, \quad d_1^{(l)} + d_4^{(l)}c_2 - d_6^{(l)}\frac{fr}{m_e} + d_7^{(l)}b_{72}^{(l)} = 0,$$
$$d_2^{(l)} + d_4^{(l)}c_3 - d_6^{(l)}\frac{m}{m_e} + d_7^{(l)}b_{73}^{(l)} = 0, \quad d_3^{(l)} + d_4^{(l)}c_4 + d_7^{(l)}b_{74}^{(l)} = 0,$$
$$d_4^{(l)}c_5 - d_6^{(l)}\frac{E}{m_e} + d_7^{(l)}b_{75}^{(l)} = 0, \quad d_5^{(l)} + d_4^{(l)}c_6 - d_6^{(l)}\frac{a_e}{m_e} + d_7^{(l)}b_{76}^{(l)} = 0.$$

Since $P_1^{(l)}$ in (3.19) are Hurwitz, one can take $d_7^{(l)} = 1$, $l = 1, 2$ and then the systems with unknowns $d_1^{(l)}, \ldots, d_6^{(l)}$ have unique solutions. Introduce

$$y = z_7 + \sum_{j=1}^{6} d_j^{(l)} z_j. \tag{3.20}$$

Then

$$\dot{y} = q_7^{(l)}\left(z_1, \ldots, z_6, y - \sum_{j=1}^{6} d_j^{(l)} z_j\right) + d_1^{(l)} z_2 + d_2^{(l)} z_3 + d_3^{(l)} z_4 +$$
$$+ d_4^{(l)}\left(\sum_{i=1}^{6} c_i z_i\right) + d_5^{(l)} z_6 + d_6^{(l)} q_6(z) := Y^{(l)}(z_1, \ldots, z_6, y).$$

It follows from (3.14), (3.15), (3.16) and (3.17) that

$$Y^{(l)}(0, \ldots, 0, y) = q_7^{(l)}(0, \ldots, 0, y) = \frac{BC}{V_0 + Sx_0} \Psi_7(0, \ldots, 0, y) = 0$$

(3.12) become

$$\tilde{\Sigma}_l \begin{array}{l} \dot{\xi} = D\xi \\ \dot{y} = Y^{(l)}(\xi^\tau, y) \end{array}, \quad l = 1, 2. \tag{3.21}$$

We can prove now the main result of the paper concerning the stability of all equilibria $\hat{x}$ in (2.10). The stability of an equilibrium point for a switching system is defined similarly to the way stability of equilibria is defined for nonswitching systems (see [23], [25], [33]).

Theorem 3.1. *Suppose u_l in* (3.10) *are constructed such that $P_1^{(l)}$ in* (3.19) *are Hurwitz polynomials for $l = 1, 2$. Then every equilibrium point in* (2.11) *is simple uniformly stable for the switched system* (2.7), (2.8) *and if initial data are closed to such an equilibrium point, $\lim_{t\to\infty}[x_1(t) - \hat{x}_1] = 0$ for $(x_1(t), \ldots, x_7(t))$ a solution.*

Proof. It is immediate from (3.18) that $P_1^{(l)}$ are Hurwitz for $l = 1, 2$ if and only if D in (3.13) is Hurwitz. Actually $P_1^{(1)} = P_1^{(2)}$ = characteristic polynomials of $D = (d_{ij})_{i,j}$.

Since D is Hurwitz there exists a strictly positive definite matrix P, the unique solution of the Lyapunov equation $D^\tau P + PD = -I$. With dot denoting the scalar product, $V\xi = \xi \cdot P\xi$ is a Lyapunov function for $\dot{\xi} = D\xi$ that satisfies

$$\sum_{i=1}^{6} \frac{\partial V}{\partial z_i}\left(\sum_{j=1}^{6} d_{ij} z_j\right) = -\sum_{i=1}^{6} z_i^2$$

(see, for example, [5]). Choose $\alpha > 0$ such that

$$-\sum_{i=1}^{6} z_i^2 + 2\alpha V(\xi) \quad \text{be negative definite.} \tag{3.22}$$

Introduce

$$\eta_j(t) = e^{\alpha t} z_j(t), \quad j = 1, \ldots, 6. \tag{3.23}$$

With I the identity matrix of order 6, (3.20) become

$$\tilde{\Sigma}_l' \begin{array}{l} \dot{\eta} = (D + \alpha I)\eta \\ \dot{y} = Y^{(l)}(e^{-\alpha t}\eta, y) \end{array}, \quad l = 1, 2. \tag{3.24}$$

It follows from (3.22) that, for every solution of (3.24), $\frac{d}{dt}V[\eta(t)] < 0$, $\forall t \geq 0$. Then $V[\eta(t)] < V[\eta(0)]$, $\forall t \geq 0$ and from the properties of Lyapunov functions we infer global existence and $||\eta(t)|| \leq K$ $\forall t \geq 0$ (see, for example [10]) ($||\ ||$ denotes the euclidean norm). If $||\eta(0)||$ is small enough, K can be made as small as we want.

From (3.22) if follows that

$$||\xi(t)|| \leq K\, e^{-\alpha t}\ \forall t \geq 0. \tag{3.25}$$

Since $Y^{(l)}(0, \ldots, 0, y) = 0$, $l = 1, 2$ and $Y^{(l)}$ contains only powers of the variables equal or greater than two, there exists $M > 0$, independent of $K \leq 1$, such that

$$|Y^{(l)}[\xi(t), y(t)]| \leq K\, M\, e^{-\alpha t} \quad \forall t \geq 0,\ l = 1, 2.$$

Since, from (3.21), $y(t) = y(0) + \int_0^t Y^{(l)}[\xi^\tau(s), y(s)]ds$, $l = 1, 2$ it follows that

$$|y(t)| \leq |y(0)| + \frac{KM}{\alpha} \quad \forall t \geq 0. \tag{3.26}$$

Let $\delta > 0$ be given and let $\gamma(\delta) < \frac{\delta}{2}$ be so small that $||\eta(0)|| < \gamma(\delta)$, $|y(0)| < \gamma(\delta)$ imply $\frac{KM}{\alpha} < \frac{\delta}{2}$. (3.25) and (3.26) imply that if $||(\xi(0)^\tau, y(0))|| < \gamma(\delta)$ then $||(\xi(t)^\tau, y(t))|| < \delta$ for every $t \geq 0$. From (3.20) it follows that for every $\omega > 0$ there exists $\gamma(\omega) > 0$ such that $||z(0)|| < \gamma(\omega)$ implies $||z(t)|| < \omega$ $\forall t \geq 0$ for z a solution of (3.12).

Let $\varepsilon > 0$ be given. By the continuity of $\Psi^{(l)}$, $l = 1, 2$ there exists ω_ε such that $||z|| < \omega_\varepsilon \Rightarrow ||\Psi^{(l)}(z) - \hat{x}|| < \varepsilon$, $l = 1, 2$. Then, if $||z(0)|| < \gamma(\omega_\varepsilon)$ it follows that $||\bar{x}(t) - \hat{x}|| < \varepsilon$ $\forall t \geq 0$ for solutions of $\mathcal{S}_1$ or $\mathcal{S}_2$ with u_1, u_2 given in (3.10). The continuity of $\phi^{(l)}$, $l = 1, 2$ implies that there exists $\delta_\varepsilon > 0$ such that $||\bar{x}(0) - \hat{x}|| < \delta_\varepsilon \Rightarrow ||z(0)|| < \gamma(\omega_\varepsilon)$ so $||\bar{x}(t) - \hat{x}|| < \varepsilon$ $\forall t \geq 0$ whenever $\bar{x}$ is a solution of the switching system $\mathcal{S}_1, \mathcal{S}_2$ thus stability of equilibria is proved. The asymptotic behaviour (2.12) follows from (3.25).

4 Concluding remarks

The research described in this paper combines classical Lyapunov-Malkin approach to critical cases in stability theory with geometric control synthesis for a seven dimensional mathematical model of an EHS in a servoelastic framework. The strong nonlinearity due to the directional change of valve opening is adressed to by consideration of a two component switched type system that defines the mathematical model of the EHS in servoelastic framework. For each component of the switched system a controller is provided by a geometric construction using coordinate transformations. The control problem solved in this way is not a tracking problem but a problem of stabilization of equilibria. The domain of initial conditions concernes the load constrained positions and constrained pressures in the actuator cylinder chambers. This problem of control synthesis,which ensures the stability of all admissible equilibria,is meaningful: for an EHS used in an autopilot subsystems it means stability of the airplane evolution for a stationary maneuver such as flight at constant altitude (associated to $k = 0$, thus $x = 0$ (see (2.11)) and $p = 0.5$) or turn with constant curvature radius ($k > 0$ and $p \neq 0.5$). The main step is to prove the existence of a Lyapunov function for a linear system of order 6 which is common to both components of the switched system after coordinates transformation.

This new result in the theory of switched systems can be useful in other situations too.

Nomenclature related to the mathematical model of EHS

Acronymes:

EHS - electrohydraulic servomechanism

Variables:

$z(t)$ EHS load displacement (defined from the center of the actuator cylinder $[m]$)

$z_e(t)$ mounting structure displacement

$p_i(t)$ pressures in actuator cylinder chambers $(i = 1, 2)[N/m^2]$

$\sigma(t)$ valve spool displacement relatively to its sleeve, defined from the valve's neutral position $[m]$

x references input (command) ($[m]$):

Load parameters:

m total mass of piston and load referred to piston $[kg]$

k load spring gradient $[N/m]$

f_r viscous damping coefficient of load $[Ns/m]$

Mounting structure parameters

m_e mass of EHS body and attached neighbour structure $[kg]$

E mounting structure stiffness $[N/m]$

a_e structural damping $[Ns/m]$

EHS parameters:

p_s supply pressure to valve $[N/m^2]$

S actuator piston area $[m^2]$

V_0 bulk modulus of oil $[N/m^2]$

k_v proportional valve displacement/voltage coefficient $[m/V]$
c_d valve discharge coefficient
ρ oil density $[kg/m^3]$
w valve port width $[m]$
σ time constant of servovalve $[s]$

Other notations:
t time variable $[s]$
$\dot{x}$ derivative of function x with respect to time t
$L_f h$ Lie derivative of function h along f.

Acknowledgement. This work is partially supported by CNCSIS Grant 84/2007.

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Received: June, 2008