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A New Analytical Solution for the 2D Advection–Dispersion Equation in Semi-Infinite and Laterally Bounded Domain

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Abstract

In this work the parallel plate geometry has been considered to analytically solve the advection – dispersion equation in porous media. The general solution was obtained by the application of Cosine Fourier Series for the transversal domain, by the application of the Laplace Transform in regard to the temporal dimension and the introduction of a Theta Function. A brief comparison with a commonly used solution for infinite domain has been performed in order to qualitatively evaluate the influence of the presence of the boundaries on the plume profiles behavior.

Keywords: advection, dispersion, 2-D analytical solution, bounded domain

1 Introduction

The transport of contaminants in natural or artificial porous media is an argument of growing interest. The contamination of water by substances of various kind and the study of the behavior of compounds into fixed – bed reactors are objects of several works. A correct simulation of Advection – Dispersion based processes is fundamental to predict, for instance, the dispersion of contaminants in groundwater or to simulate start-ups and shut-downs in fixed bed reactors in order to reduce accident risks.

The transport of solutes in saturated porous media is commonly described by the Advection- Dispersion- Equation (ADE, Bear, 1972). The solution of this equation in real domains often requires the application of discretized numerical methods. In some cases, where an analytical approach is possible, the solutions often deal with one dimensional or two-dimensional flow and constant velocities.

Many authors proposed analytical solutions of two-dimensional ADEs in different domains even with decay or sorption kinetics. Some of them are well explained in Lee (1999). Batu (1989, 1993) evaluated a two dimensional analytical solution considering solute transport for a bounded aquifer by adopting Fourier analysis and Laplace Transform. Aral (1996) furnished a general analytical solution of the two- dimensional solute transport equation with time-dependent dispersion coefficient for an infinite domain aquifer. Tartakowsky et al. (1997) obtained a simplified analytical solution for the 2D steady state convection-dispersion equation adopting velocity and dispersion coefficient varying in space: the solution was obtained by the application of the Dupuit-Forchheimer approximation. Massabò et al. (2006) derived some analytical solutions for the advection dispersion equation in cylindrical coordinates by using a Bessel function expansion.

In this work we propose an exact analytical solution for a two dimensional semi-infinite domain by evolving a common approach based on Fourier series and Laplace transforms introducing the application of a Jacobi – Theta Function. This domain represents a typical experimental device adopted to study solute transport through porous media.

2 Problem formulation

Mass conservation of conservative solutes transported through porous media is described by a partial differential equation known as advection-dispersion equation. Here we consider the transport of solute through a thin chamber homogeneously filled with a homogeneous porous medium. Figure 1 shows a graphical representation of the problem, the water flow is along the x coordinate, the length of the chamber is L and the chamber's width is 2l. Fresh water is fed from the inlet of the chamber while a pulse injection of solute mass is initially provided from a source placed at a distance q from the inlet. For thin chambers, where the thickness is much smaller than the other two dimensions in which the transport phenomenon occurs, the governing equation of solute concentration can be expressed by the two dimensional advection – dispersion equation as follows (Bear (1972), Grove (1976) and Bear et al. (1978)):

$$\frac{\partial C}{\partial t} + u\frac{\partial C}{\partial x} = D_L \frac{\partial^2 C}{\partial x^2} + D_T \frac{\partial^2 C}{\partial y^2} \qquad 1$$

where u is the pore scale velocity, $C(x, y, t)$ ([M/L^3]) is the resident fluid solute concentration, D_L and D_T are the longitudinal and transverse dispersion coefficients respectively. The porous medium is homogeneous, hence the velocity and dispersion coefficients do not vary in space; moreover, the flow is considered to be stationary, thus u , D_L and D_T do not change with time and are considered as constant in the following derivation.
Initial and boundary conditions are to be set in order to solve Equation (1). No solute flux across the lateral boundaries at $y = \pm l$ has to be imposed; the condition can be expressed as a Neumann boundary condition as follows:

$$\left.\frac{\partial C}{\partial y}\right|_{y=\pm l} = 0 \qquad 2$$

As regard the finite longitudinal domain here considered, two boundary conditions must be imposed. The former represents a mass balance equation across the inlet section at x=0, thus the solute flux across the left size of the section equals the flux from the right. The concentration of the inlet stream is zero in our case since the chamber is fed with pure water. The resulting boundary condition is then expressible as a 3rd type (mixed) boundary condition:

$$\left.\left[uC - D_L \frac{\partial C}{\partial x}\right]\right|_{x=0^+} = 0 \qquad 3$$

Two different ways exist in order to set boundary conditions at chamber outlet. The most common in chemical engineering is to assume that concentrations are continuous across the outlet section, namely $C|_{x=L^-} = C|_{x=L^+}$ resulting in:

$$\left.\frac{\partial C}{\partial x}\right|_{x=L^-} = 0 \qquad (4)$$

The other possibility, the one we adopted, was to assume an infinite chamber in the longitudinal direction (x axis for the problem considered), thus in this case we get a Dirichlet boundary condition expressed as:

$$\lim_{x\to+\infty} C(x,y,t) = 0 \qquad (5)$$

Finally, the initial condition for the point pulse injection at y=0 and x=q, yields to:

$$C(x,y,t=0) = M\delta(y)\delta(x-q) \qquad (6)$$

3 The Analytical Solution in Bounded Domain

The derivation of an analytical solution of Equation (1) subject to boundary conditions (2), (3) and (5) and initial condition (6) is here illustrated. In order to solve the equations system proposed, a set of new variables are introduced. The transformed variables are:

$$\breve{C} = \frac{C}{\sqrt{D_L D_T}}, \quad \breve{x} = \frac{x}{\sqrt{D_L}}, \quad \breve{y} = \frac{y}{\sqrt{D_T}}$$
$$\breve{u} = \frac{u}{\sqrt{D_L}}, \quad \breve{l} = \frac{l}{\sqrt{D_T}}, \quad \breve{q} = \frac{q}{\sqrt{D_L}} \qquad (7)$$

By substituting the new variables in the governing equations (1, 2, 3, 5, 6), they become:

$$\frac{\partial \breve{C}}{\partial t} = \frac{\partial^2 \breve{C}}{\partial \breve{x}^2} + \frac{\partial^2 \breve{C}}{\partial \breve{y}^2} - \breve{u}\frac{\partial \breve{C}}{\partial \breve{x}} \qquad (8)$$

$$\left.\frac{\partial \breve{C}}{\partial \breve{y}}\right|_{\breve{y}=\pm\breve{l}} = 0 \qquad (9)$$

$$\left.\left[\breve{u}\breve{C} - \frac{\partial \breve{C}}{\partial \breve{x}}\right]\right|_{\breve{x}=0^+} = 0 \qquad (10)$$

$$\lim_{\breve{x}\to+\infty} \breve{C}(\breve{x},\breve{y},t) = 0 \qquad (11)$$

$$\breve{C}(\breve{x},\breve{y},t=0) = M\delta(\breve{y})\delta(\breve{x}-\breve{q}) \qquad (12)$$

Due to the symmetry of both the transversal boundaries and the injection position with respect to the longitudinal axis, the solution of the PDE will be symmetric as well, resulting to $\breve{C}(\breve{x},\breve{y},t)=\breve{C}(\breve{x},-\breve{y},t)$. Starting from this consideration, it's possible to use a cosine Fourier series expansion for the solution as:

$$\breve{C}(\breve{x},\breve{y},t)=\sum_{n=0}^{+\infty} a_n(\breve{x},t)\cos\left(\frac{n\,\pi\,\breve{y}}{\breve{l}}\right) \qquad 13$$

It is important to notice that Equation (13) satisfies the no flux boundary condition (9). By substituting Equation (13) into (8) we get a partial differential equation for the series coefficients:

$$\frac{\partial a_n}{\partial t}=\frac{\partial^2 a_n}{\partial \breve{x}^2}-\frac{n^2\pi^2}{\breve{l}^2}a_n-\breve{u}\frac{\partial a_n}{\partial \breve{x}}, \qquad n=0,1\ldots,+\infty \qquad 14$$

Similarly, boundaries and initial conditions (10), (11), (12) may be written as:

$$\left[\breve{u}a_n(\breve{x},t)-\frac{\partial a_n}{\partial \breve{x}}\right]\Bigg|_{\breve{x}=0^+}=0, \qquad n=0,1\ldots,+\infty \qquad 15$$

$$\lim_{\breve{x}\to+\infty} a_n(\breve{x},t)=0\,, \qquad n=0,1\ldots,+\infty \qquad 16$$

$$\begin{aligned} a_0(\breve{x},t=0)&=\frac{M}{\breve{l}}\delta(\breve{x}-\breve{q}),\\ a_n(\breve{x},t=0)&=\frac{M}{2\breve{l}}\delta(\breve{x}-\breve{q}), \qquad n=1\ldots,+\infty \end{aligned} \qquad 17$$

The solution of Equation (14) subject to (15), (16) and (17) is obtained by applying the Laplace transform. The Laplace transform of a_n is defined as

$$\hat{a}_n(\breve{x},s)=\int_0^{+\infty} e^{-st}a_n(\breve{x},t)dt \qquad 18$$

where s is sometime called the frequency variable of the Laplace transform. By applying the Laplace transform to Equation (14) and rearranging, we get:

$$\frac{\partial^2\hat{a}_n(\breve{x},s)}{\partial\breve{x}^2}-\breve{u}\frac{\partial\hat{a}_n(\breve{x},s)}{\partial\breve{x}}-\left(s+\frac{n^2\pi^2}{\breve{l}^2}\right)\hat{a}_n(x,s)=-a_n(\breve{x},t=0) \qquad 19$$
$$\forall\; n=0,1,\ldots,+\infty$$

here $a_n(\breve{x},t=0)$ is expressed by Equation (17). Initial and boundary condition than become:

$$\left[\breve{u}\hat{a}_n(\breve{x},s) - \frac{\partial \hat{a}_n}{\partial \breve{x}}\right]\Bigg|_{\breve{x}=0^+} = 0, \qquad \forall\ n = 0,1\ldots,+\infty \tag{20}$$

$$\lim_{\breve{x}\to+\infty} \hat{a}_n(\breve{x},s) = 0 \qquad \forall\ n = 0,1\ldots,+\infty \tag{21}$$

Equation (19) is a non-homogeneous ordinary differential equation that can be solved by the application of classical methods. The general solution is composed by sum of the general integral of the associated homogeneous equation and the particular solution: for every n= 0, 1,…. we have:

$$\begin{aligned}\hat{a}_n(\breve{x},s) = {} & \exp\left[\frac{\breve{x}\breve{u}}{2}\right]\left\{Q_n(s)\exp\left[\breve{x}\frac{\sqrt{W^2+4s}}{2}\right]+\right.\\ & \left.+P_n(s)\exp\left[-\breve{x}\frac{\sqrt{W^2+4s}}{2}\right]\right\}+\\ & +\int_{-\infty}^{+\infty}\left[-a_n(\breve{x},t=0)\right]2\frac{\exp\left[\frac{\breve{u}(\varepsilon-\breve{x})}{2}\right]}{\breve{l}\sqrt{W^2+4s}}\sinh\left[\frac{(\breve{x}-\varepsilon)\sqrt{W^2+4s}}{2}\right]H(\breve{x}-\varepsilon)\,d\varepsilon\end{aligned} \tag{22}$$

where the second term represents the particular solution. Since the initial conditions $a_n(\breve{x},t=0)$ are different for $n=0$ and $n \neq 0$, two different particular solutions are found by substituting $a_n(\breve{x},t=0)$ of Equation (17) in (19):

$$\begin{aligned}\hat{a}_0(\breve{x},s) = {} & \exp\left[\frac{\breve{x}\breve{u}}{2}\right]\cdot\\ & \left\{Q_0(s)\exp\left[\breve{x}\frac{\sqrt{\breve{u}^2+4s}}{2}\right]+P_0(s)\exp\left[-\breve{x}\frac{\sqrt{\breve{u}^2+4s}}{2}\right]\right\}+\\ & -\frac{\exp\left[\frac{\breve{u}(\breve{x}-\breve{q})}{2}\right]}{\breve{l}\sqrt{\breve{u}^2+4s}}\sinh\left[\frac{(\breve{x}-\breve{q})\sqrt{\breve{u}^2+4s}}{2}\right]H(\breve{x}-\breve{q})\end{aligned} \tag{23}$$

$$\hat{a}_n(\breve{x},s)=\exp\left[\frac{\breve{x}\breve{u}}{2}\right]\cdot$$
$$\left\{Q_n(s)\exp\left[\breve{x}\frac{\sqrt{W^2+4s}}{2}\right]+P_n(s)\exp\left[-\breve{x}\frac{\sqrt{W^2+4s}}{2}\right]\right\}+$$
$$-2\frac{\exp\left[\frac{\breve{u}(\breve{x}-\breve{q})}{2}\right]}{\breve{l}\sqrt{W^2+4s}}\sinh\left[\frac{(\breve{x}-\breve{q})\sqrt{W^2+4s}}{2}\right]H(\breve{x}-\breve{q}) \qquad 24$$
$$\forall\ n=1,\ldots,+\infty$$

where $\breve{W}^2$ is constituted as follows:

$$\breve{W}^2=\breve{u}^2+4\frac{n^2\pi^2}{\breve{l}^2}\qquad \forall\ n=1,\ldots,+\infty \qquad 25$$

$Q_n(s)$ and $P_n(s)$ have to be determined using the boundary conditions given by Equation (20) and (21). $Q_0(s)$ and $Q_n(s)$ can be computed by using the right boundary condition for the longitudinal domain Equation (21). The limits for $x\to+\infty$ are reported in Equation (26).

$$\lim_{\breve{x}\to+\infty}\hat{a}_0(\breve{x},s)=\lim_{\breve{x}\to+\infty}\exp\left[\frac{\breve{x}\breve{u}}{2}\right]\exp\left[\breve{x}\frac{\sqrt{\breve{u}^2+4s}}{2}\right]\left\{Q_0(s)-\frac{\exp\left[-\frac{\breve{u}\breve{q}}{2}\right]\exp\left[-\frac{\breve{q}\sqrt{\breve{u}^2+4s}}{2}\right]}{2\breve{l}\sqrt{\breve{u}^2+4s}}\right\}$$
$$\lim_{\breve{x}\to+\infty}\hat{a}_n(\breve{x},s)=\lim_{\breve{x}\to+\infty}\exp\left[\frac{\breve{x}\breve{u}}{2}\right]\exp\left[\breve{x}\frac{\sqrt{W^2+4s}}{2}\right]\left\{Q_n(s)-\frac{\exp\left[-\frac{\breve{u}\breve{q}}{2}\right]\exp\left[-\frac{\breve{q}\sqrt{W^2+4s}}{2}\right]}{\breve{l}\sqrt{W^2+4s}}\right\} \qquad 26$$

By imposing in (26) that the limit of $\hat{a}_n$ is zero for x that goes to infinity, $Q_0(s)$

and $Q_n(s)$ can be found to be:

$$Q_0(s) = \frac{\exp\left[-\frac{\breve{u}\,\breve{q}}{2}\right]\exp\left[-\frac{\breve{q}\sqrt{\breve{u}^2+4s}}{2}\right]}{2\,\breve{l}\,\sqrt{\breve{u}^2+4s}}$$
$$Q_n(s) = \frac{\exp\left[-\frac{\breve{u}\,\breve{q}}{2}\right]\exp\left[-\frac{\breve{q}\sqrt{W^2+4s}}{2}\right]}{\breve{l}\,\sqrt{W^2+4s}} \qquad 27$$

$P_0(s)$ and $P_n(s)$ can be computed by using the upstream boundary condition Equation (15). By substituting $\hat{a}_0(\breve{x},s)$ and $\hat{a}_n(\breve{x},s)$ in (20) and rearranging:

$$\left[\breve{u}\hat{a}_0(\breve{x},s)-\frac{\partial\hat{a}_0}{\partial\breve{x}}\right]_{\breve{x}=0^+} =$$
$$=-\frac{1}{2}\frac{Q_0(s)\left(-\breve{u}\sqrt{\breve{u}^2+4s}+\breve{u}^2+4s\right)-P_0(s)\left(\breve{u}\sqrt{\breve{u}^2+4s}+\breve{u}^2+4s\right)}{\sqrt{\breve{u}^2+4s}}$$
$$\left[\breve{u}\hat{a}_n(\breve{x},s)-\frac{\partial\hat{a}_n}{\partial\breve{x}}\right]_{\breve{x}=0^+} = \qquad 28$$
$$=-\frac{1}{2}\frac{Q_n(s)\left(-\breve{u}\sqrt{\breve{W}^2+4s}+\breve{W}^2+4s\right)-P_n(s)\left(\breve{u}\sqrt{\breve{W}^2+4s}+\breve{W}^2+4s\right)}{\sqrt{\breve{W}^2+4s}}$$

The final expressions for $P_0(s)$ and $P_n(s)$ can be found by imposing Equation (28) to be equal to zero:

$$P_0(s)=\frac{Q_0(s)(-\breve{u}\sqrt{\breve{u}^2+4s}+\breve{u}^2+4s)}{\breve{u}\sqrt{\breve{u}^2+4s}+\breve{u}^2+4s}$$
$$P_n(s)=\frac{Q_n(s)(-\breve{u}\sqrt{\breve{W}^2+4s}+\breve{W}^2+4s)}{\breve{u}\sqrt{\breve{W}^2+4s}+\breve{W}^2+4s} \qquad 29$$

The final expression for $\hat{a}_n(\breve{x},s)$ is so expressed as follows:

$$\hat{a}_0(\breve{x},s)=\frac{\breve{Z}_0(\breve{x})[1-H(\breve{x}-\breve{q})]\exp\left[\frac{(\breve{x}-\breve{q})\sqrt{\breve{u}^2+4s}}{2}\right]}{\sqrt{\breve{u}^2+4s}}+$$
$$-\frac{Z_0(\breve{x})\breve{u}\exp\left[-\frac{(\breve{x}+\breve{q})\sqrt{\breve{u}^2+4s}}{2}\right]}{\breve{u}^2+4s+\breve{u}\sqrt{\breve{u}^2+4s}}+$$
$$+\frac{Z_0(\breve{x})\exp\left[-\frac{(\breve{x}+\breve{q})\sqrt{\breve{u}^2+4s}}{2}\right]}{\breve{u}+\sqrt{\breve{u}^2+4s}}+ \qquad 30$$
$$+\frac{Z_0(\breve{x})H(\breve{x}-\breve{q})\exp\left[-\frac{(\breve{x}-\breve{q})\sqrt{\breve{u}^2+4s}}{2}\right]}{\sqrt{\breve{u}^2+4s}}$$

$$\hat{a}_n(\breve{x},s)=\frac{2\breve{Z}_0(\breve{x})[1-H(\breve{x}-\breve{q})]\exp\left[\frac{(\breve{x}-\breve{q})\sqrt{\breve{W}^2+4s}}{2}\right]}{\sqrt{\breve{W}^2+4s}}+$$
$$-\frac{2\breve{Z}_0(\breve{x})\breve{u}\exp\left[-\frac{(\breve{x}+\breve{q})\sqrt{\breve{W}^2+4s}}{2}\right]}{\breve{W}^2+4s+\breve{u}\sqrt{\breve{W}^2+4s}}+ \qquad 31$$
$$+\frac{2\breve{Z}_0(\breve{x})\exp\left[-\frac{(\breve{x}+\breve{q})\sqrt{\breve{W}^2+4s}}{2}\right]}{\breve{u}+\sqrt{\breve{W}^2+4s}}+\frac{2\breve{Z}_0(\breve{x})H(\breve{x}-\breve{q})\left[-\frac{(\breve{x}-\breve{q})\sqrt{\breve{W}^2+4s}}{2}\right]}{\sqrt{\breve{W}^2+4s}}$$

where

$$\breve{Z}_0(\breve{x})=\frac{1}{2\breve{l}}\exp\left[\frac{\breve{u}(\breve{x}-\breve{q})}{2}\right] \qquad 32$$

It is possible to obtain the final solution by the application of the inverse Laplace Transform to $\hat{a}_0(x,s)$ and $\hat{a}_n(x,s)$, yielding to:

$$a_0(\breve{x},t) = \frac{1}{4}\frac{\exp\left[\frac{\breve{u}(\breve{x}-\breve{q})}{2}\right]\exp\left[-\frac{\breve{u}^2 t}{4}-\frac{(\breve{x}-\breve{q})^2}{4t}\right]}{\breve{l}\sqrt{t\pi}} + \frac{1}{4}\frac{\exp\left[\frac{\breve{u}(\breve{x}-\breve{q})}{2}\right]\exp\left[-\frac{\breve{u}^2 t}{4}-\frac{(\breve{x}+\breve{q})^2}{4t}\right]}{\breve{l}\sqrt{t\pi}} + -\frac{\breve{u}}{4}\frac{\exp(\breve{x}\breve{u})\left(1+erf\left(-\frac{\breve{q}+\breve{x}+\breve{u}t}{2\sqrt{t}}\right)\right)}{\breve{l}} \quad 33$$

$$a_n(\breve{x},t) = \frac{1}{2}\frac{\exp\left[\frac{\breve{u}(\breve{x}-\breve{q})}{2}\right]\left\{\exp\left[-\frac{\breve{W}^2 t}{4}-\frac{(\breve{x}-\breve{q})^2}{4t}\right]+\exp\left[-\frac{\breve{W}^2 t}{4}-\frac{(\breve{x}+\breve{q})^2}{4t}\right]\right\}}{\breve{l}\sqrt{t\pi}} + -\frac{\breve{u}}{2\breve{l}}\exp(\breve{u}\breve{x})\left(erfc\left(\frac{\breve{x}+\breve{q}}{2\sqrt{t}}+\frac{\breve{u}\sqrt{t}}{2}\right)\right)\exp\left(-\frac{n^2\pi^2}{\breve{l}^2}t\right) \quad 34$$

In relation to the expression of $\breve{W}^2$ presented in (25), it is possible to notice that the following relation between a_0 and a_n is true:

$$a_n(\breve{x},t) = 2\, a_0(\breve{x},t)\, \exp\left[-\frac{n^2\pi^2}{\breve{l}^2}t\right] \quad 35$$

By substituting Equation (35) to (13), we get:

$$\breve{C}(\breve{x},\breve{y},t) = a_0(\breve{x},t)\left[1+2\sum_{n=0}^{+\infty}\cos\left(\frac{n\,\pi\,\breve{y}}{\breve{l}}\right)\exp\left(-\frac{n^2\pi^2}{\breve{l}^2}t\right)\right] \quad 36$$

The series appearing in the solution is the Jacobi Theta $\theta_3(z,q)$ which is defined as follows (Abramovitz & Stegun, chapter 16):

$$\theta_3(z,q) = 1+2\sum_{n=1}^{+\infty} q^{n^2}\cos(2nz) \quad 37$$

The final form of solution C, changing back to the initial variables, is:

$$C(x,y,t)=\frac{M}{4l}\theta_3\left(\frac{\pi}{2}\frac{y}{l},\exp\left(-\frac{D_T t\pi^2}{l^2}\right)\right)\left\{\frac{1}{\sqrt{D_L\pi t}}\left[\exp\left(-\frac{(x-q-ut)^2}{4D_L t}\right)+\right.\right.$$
$$\left.\left.+\exp\left(-\frac{(x+q-ut)^2}{4D_L t}-\frac{uq}{D_L}\right)\right]-\frac{u}{D_L}\exp\left(\frac{ux}{D_L}\right)erfc\left(\frac{x+q+ut}{2\sqrt{D_L t}}\right)\right\} \quad 38$$

4 Discussion and Results

In this section we compare the new solution with the solution of a 2D transport problem in an unbounded domain for a pulse injection point at x=q (Bear (1972)):

$$C(x,y,t)=\frac{M}{4\pi t\sqrt{D_L D_T}}\exp\left(-\frac{(x-q-ut)^2}{4D_L t}-\frac{y^2}{4D_T t}\right) \quad 39$$

If we consider conducting experiments at fixed device dimensions and fixed pore scale velocity, the solute transport is only regulated by the dispersion properties.
The lateral boundary condition has no influence on the phenomenon when the transport of the solute by transversal dispersion is much slower than the transport by advection, in this case during the passage into the chamber the solute does not reach the borders. In Figure 2 we compare the contours obtained with our solution and the Bear's one, for fixed longitudinal dispersion (D_L = 10^{-7}) and different values of the transverse dispersion coefficient (D_T = 10^{-8}, 10^{-9}). The difference between the solutions is, obviously, mainly evident close to the edge of the chamber and it is maximum at $\frac{y}{l}=1$. The upstream boundary condition influences the solution through both advection and longitudinal dispersion at fixed position of the source injection. In principle, for small longitudinal dispersion coefficients, the transport of the solute by advection is faster than the transport by dispersion, and so also by back-dispersion, hence the upstream boundary condition does not influence the solution. Figure 3 shows the contour plots obtained by the two analytical solutions at fixed transverse dispersion coefficient (D_T= 10^{-8}) and different longitudinal ones (D_L=10^{-7}, 10^{-8}). It is possible to notice that, close to the inlet section, the profile of the new solution is different from the Bear's one in both magnitude of the concentration (10% in the case proposed in Figure 3) and position of the center of mass. Both the former and the latter differences are induced by the presence of the upstream boundary condition which rebounds the solute back to the chamber.

5 Conclusions

We presented an analytical solution for the advection – dispersion equation in 2D by taking into account a semi-infinite and laterally bounded domain and a point-like injection. The domain simulated is very often used for experiments in fluid flow. At typical values of the two dispersion coefficients, for some combination of them, the proposed solution that takes into account the physical domain gives results that can highly differ from the classical ones where the domain is considered unbounded.

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7 Figures

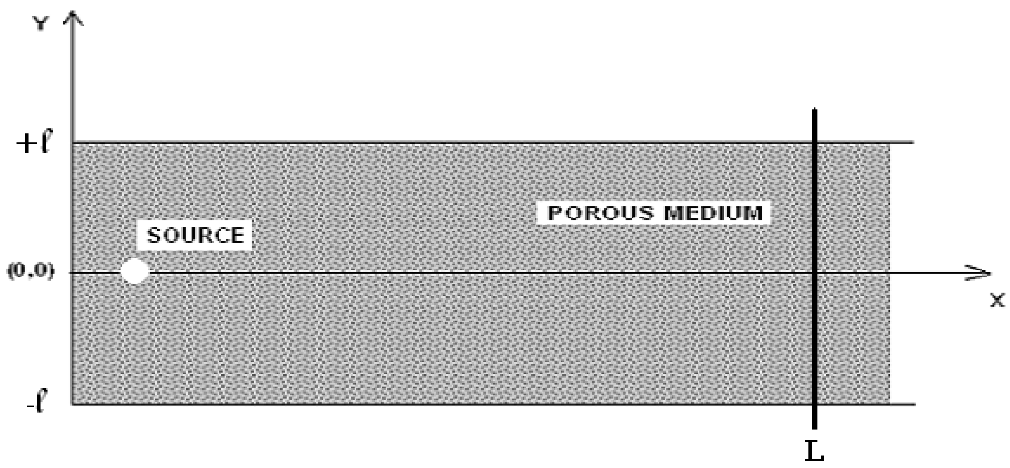
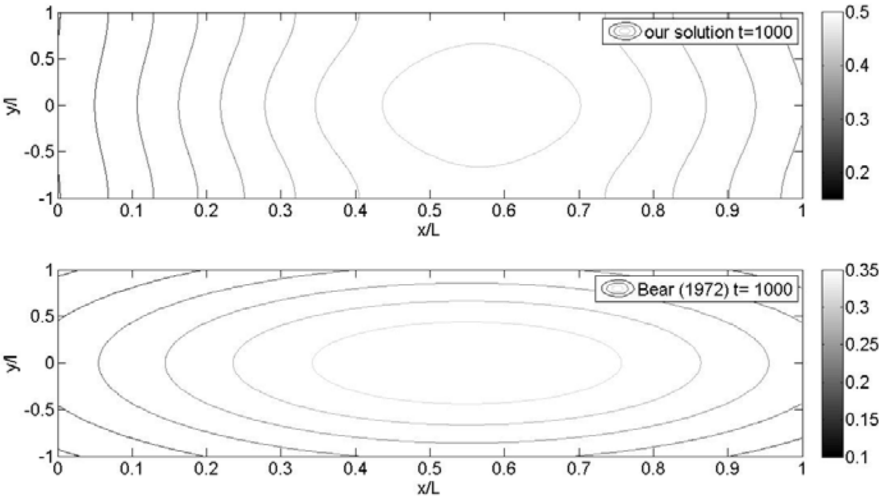
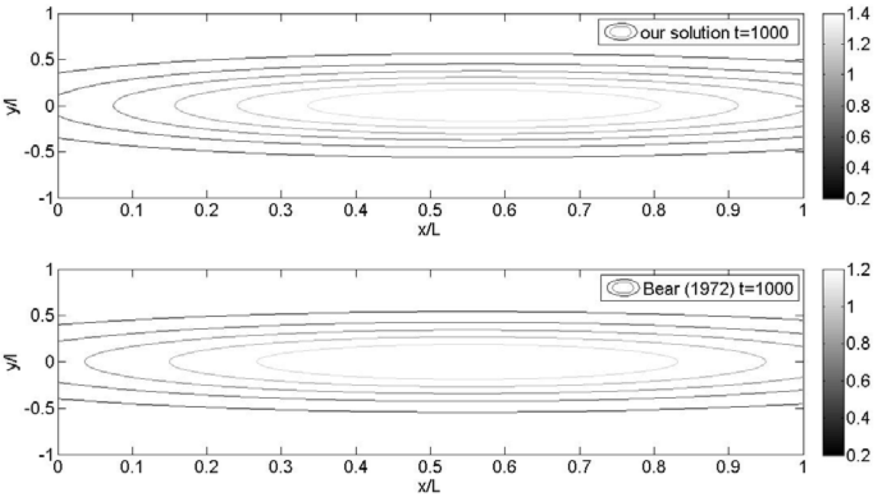


Figure 1: the domain considered. The L- section has been chosen as the outlet one. The source zone is evidenced with a white circle.

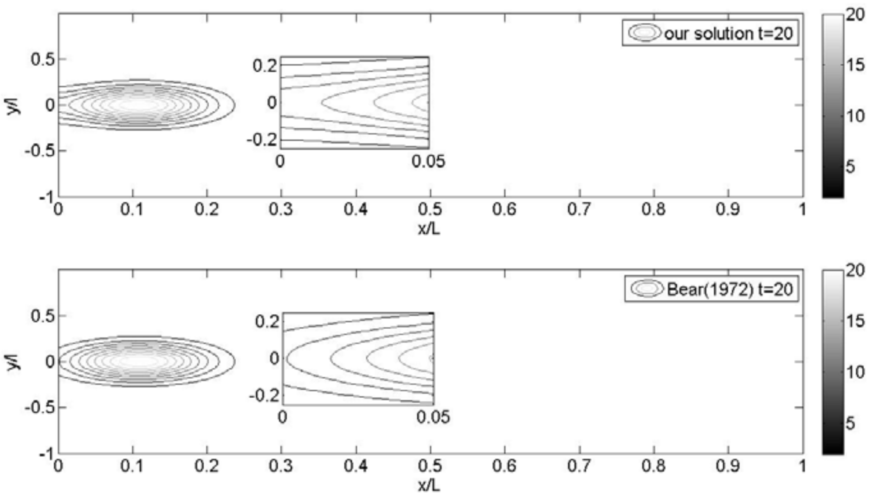


(a)

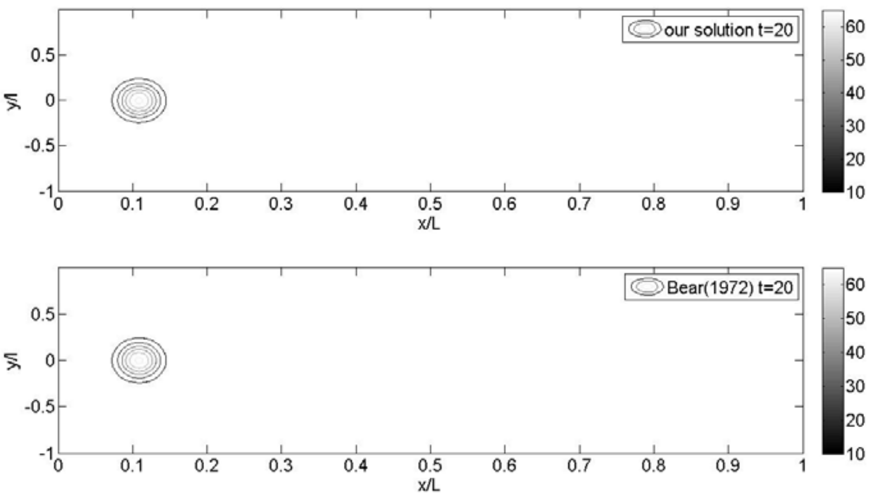


(b)

Figure 2: fixed $D_L=10^{-7}$; a) influence of lateral boundary conditions ($D_T=10^{-8}$); b) no influence of lateral boundary conditions ($D_T=10^{-9}$).



(a)



(b)

Figure 3: fixed $D_T=10^{-8}$; a)influence of the impervious boundary condition ($D_L=10^{-7}$). The small boxes in the subplots contains the enlargements of the contours in proximity of the upper bound; b) no influence of the impervious boundary condition ($D_L=10^{-8}$).

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