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Relativistic Equations' Contrast

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Abstract

This paper postulates a metric that solves differential equations contained in the Einstein's contracted tensor or Riemann-Christoffel's tensor, its solution provides differential equations represented for geodesics, whose solution in turn, is reduced to a relativistic equation able to describe planetary motion in the continuum space-time.

Given the results, it is concluded that the postulated metric as well as the proposed equation describing an arbitrary starting anomaly in elliptical orbit of a gigantic mass, such as a planet, provide evidence of a close link between Classical Newtonian Mechanics and Einstein's Relativity General. Even more, the relativistic equation obtained here, provides a mean to measure the movement of the semi-parameter of a conic section with respect to the true anomaly associated with an elliptical orbit of a gigantic mass. The slow rate of rotation of the semi-major axis is calculated, it is noted to be approximately the predicted value in Einstein's theory of relativity, this corroborates that a gigantic mass drags space and time around itself when it rotates.

Mathematics Subject Classification: 58E10

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Nomenclature

a	Semi-major axes of an elliptical orbit, cm. For Mercury planet: $a = 5.78\left(10^{12}\right)\ cm$
c	Light velocity, $c = 3 \times 10^{10}\ \ cm/s$
$\_C1$, $c5$, $c6$, k E	Constants to be determined
G	Universal gravitation constant, $G = 6.67\left(10^{-8}\right)\ dyne\ \ cm^2/gr^2$
G_{11}, G_{22}, G_{33}, G_{44}	Components of Einstein's contracted tensor or Riemann-Christoffel's tensor
$G_{ij} = R^i_{jkl}$	Riemann-Christoffel's tensor or Einstein's contracted tensor
h	Constant for a body system, cm^2/s. For Mercury planet : $h \approx \sqrt{GMa\left(1-\varepsilon^2\right)} = 2.71\left(10^{19}\right)\ cm^2/s$
M	Planet Mercury mass, $M = 1.99\left(10^{33}\right)\ gr$
p	Semi-parameter of an elliptic conic section, cm
$\overline{FP}$, $\overline{PD}$	The straight line segment in elliptic conic section, cm
q	Quotient of the semi-parameter of the conic section and eccentricity
s	Length of arc, cm
t	Temporal coordinate for a material particle of astronomical dimensions, s
$u = u(\rho)$, $v = v(\rho)$	Unknown functions of ρ to be determined
x^k, x^l	Spatial and temporal coordinates in Relativity, cm and s, respectively

Greek letters

ε	Eccentricity of a conic, cm. For an ellipse: $0 \leq \varepsilon \leq 1$. For the planet Mercury: $\varepsilon = 0.206$
Γ^i_{jk}	Christoffel symbol
ϕ	True anomaly associated with the orbit (spatial coordinate), rad or degrees
ϕ_0	Arbitrary starting anomaly, rad or degrees
ρ	Semi-parameter of the conic section of a gigantic mass or spatial coordinate, cm
θ	Angular coordinate for a relativistic particle (spatial coordinate), rad or degrees
$\_\xi$, $\_\eta$	Infinitesimals of a one-parameter Lie group

1. Introduction

To predict planetary motion in continuum space-time has been one of the most fascinating physical phenomena in Newton's classical mechanics as well as Einstein’s relativity theory. The most important contribution of Newton and Einstein was to observe the world in a different way, this derived in a better understanding of gravity and the motion of the planets as in Heinbockel (1996).
By using an elliptic conic section, Newton described his multiple bodies’ problem, demonstrating that the motion of the gigantic mass can be predicted as a conic section (Stein (1996), Valluri et al. (1997) and Weinstock (1984)). In this paper, Newton’s theory is used to determine both, the arbitrary starting anomaly associated with an elliptic orbit of a gigantic mass, and the rotation’s slow rate of Mercury’s semi-major axis. The solution of the Newton’s equation from classic mechanics is introduced in a relativistic equation obtained from a metric postulated here. The mentioned metric provides nonzero second order Christoffel symbols useful to calculate Einstein’s contracted tensor, these solutions combined with a second metric are used to expand geodesics equations toward a four dimensional space. The solution of the geodesics is a relativistic equation that predicts the behavior of gigantic masses in a continuum space-time.

The obtained results provide evidence of a close relationship between theories as in Tane (2004), also it is confirmed that a gigantic mass drags space and time around itself as it rotates. Also the behavior between the arbitrary starting anomaly and true anomaly associated with an orbit through the slow rate of rotation of the semi-major axis in a planet, can be predicted as in Einstein’s relativity theory, as well as the behavior of the variations observed in the continuum space-time of the semi parameter of the conic section with respect to the true anomaly when a gigantic mass rotates in an elliptic orbit.

Given the results obtained by mean of relativistic equations, it’s concluded that the interval of continuum space-time in a gigantic mass is moving in the same way that the motion of planets predicted by Newton's classic mechanics when the proposed differential equation obtained of the metric here postulated to measure the velocity of change of the arbitrary of the starting anomaly with respect to the true anomaly. This conclusion is ratified by relativistic predictions from other authors such as Heinbockel (1996), Eisenstaedt (1989), Martel and Poisson (2001, 2005) and Martel (2004) and from a previous work by Corral et al. (2008).

2. Modelling

This paper compares Newtonian equations with relativistic equations when describing planetary motion through a metric proposed to solve differential equations in General Relativity, such a metric is different from other authors such as Heinbockel (1996), Eisenstaedt (1989), Martel and Poisson (2001, 2005) and Martel (2004).

Newton's planetary motion can be described with two bodies, ej. the sun and one planet, with motion of both masses taking place in a plane. From Newton's second law, it is shown that the motion of the planet's mass can be described as a conic section.

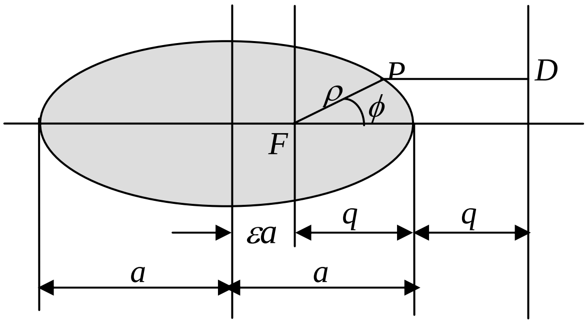

Fig. 1 Elliptic conic section.

In Fig. 1, the motion of the planet's mass is shown; it can be described as an elliptic conic section Heinbockel (1996). In this figure

$\overline{FP}/\overline{PD} = \varepsilon$ $\qquad \overline{FP} = \rho$ $\qquad \overline{PD} = 2q - \rho\cos\phi$

$\rho = p/(1+\varepsilon\cos\phi)$ $\qquad p = 2q\varepsilon$ $\qquad \rho = p/(1+\varepsilon\cos(\phi-\phi_0))$

This way the deduced solution useful to solve Newton and Einstein equations is

$$u = \frac{1}{\rho} = A(1+\varepsilon\cos(\phi-\phi_0)) \qquad \textbf{(1)}$$

where $A = 1/p(\rho,\theta,\phi)$, and $p(\rho,\theta,\phi)$ is called the semi-parameter of an elliptic conic section.

In this work, a metric that can be compared with some from other authors such as Heinbockel and Schwarzschild (Heinbockel (1996), Eisenstaedt (1989), Martel and Poisson (2001, 2005) and Martel (2004)). is postulated. The coordinates cuasi-spherical, this is (ρ,θ,ϕ,t), the metric shows the following form

$$g = -(1/1-2m/v)dt\otimes dt + (1-2m/u)d\rho\otimes d\rho + \rho^2\left(d\theta\otimes d\theta + \sin^2\theta d\phi\otimes d\phi\right) \qquad \textbf{(2)}$$

The computing of General Relativity's curvature tensors in a coordinate basis, requires the following parameters: rank-2 symmetric tensor type of the covariant metric, rank-2 symmetric tensor type of contravariant metric, determinant of the covariant metric component matrix (determinant of components of metric tensor), Christoffel, contracted tensor of Einstein.

For this metric, the covariant tensor metric's components are defined as

$$compts = \begin{bmatrix} 1-\frac{2\,m}{u} & 0 & 0 & 0 \\ 0 & -\rho^2 & 0 & 0 \\ 0 & 0 & -\rho^2\sin(\theta)^2 & 0 \\ 0 & 0 & 0 & -\frac{1}{1-\frac{2\,m}{v}} \end{bmatrix} \qquad \textbf{(3)}$$

where $m = \frac{GM}{c^2}$ is the mass of a relativistic object.

Next, the contravariant components of the metric tensor are determined, the first partials of the covariant components of the metric tensor, and the Christoffel symbols of the first kind are determine. Finally, the nonzero second kind Christoffel symbols is obtained

$$(1,\,1,\,1) = \frac{m\left(\frac{\partial}{\partial \rho}\,\mathrm{u}(\rho)\right)}{\mathrm{u}(\rho)^2\left(1-\frac{2\,m}{\mathrm{u}(\rho)}\right)}$$

$$(1,2,2) = -\frac{u\rho}{2m-u} \qquad (2,3,3) = -\sin(\theta)\cos(\theta)$$

$$(1,3,3) = \frac{u\rho\left(\cos^2(\theta)-1\right)}{2m-u} \qquad (2,2,1) = (2,1,2) = \frac{1}{\rho}$$

$$(1,\,4,\,4) = -\frac{m\left(\frac{\partial}{\partial \rho}\,\mathrm{v}(\rho)\right)}{\mathrm{v}(\rho)^2\left(1-\frac{2\,m}{\mathrm{u}(\rho)}\right)\left(1-\frac{2\,m}{\mathrm{v}(\rho)}\right)^2}$$

$$(4,\,1,\,4) = -\frac{m\left(\frac{\partial}{\partial \rho}\,\mathrm{v}(\rho)\right)}{\left(1-\frac{2\,m}{\mathrm{v}(\rho)}\right)\mathrm{v}(\rho)^2}$$

$$(3,1,3) = \frac{1}{\rho}$$

$$(3,2,3) = (3,3,2) = \frac{\cos(\theta)}{\sin(\theta)} \qquad (4,\,4,\,1) = -\frac{m\left(\frac{\partial}{\partial \rho}\,\mathrm{v}(\rho)\right)}{\left(1-\frac{2\,m}{\mathrm{v}(\rho)}\right)\mathrm{v}(\rho)^2}$$

(4)

The contracted tensor of Einstein or Riemann-Christoffel tensor[1], is

$$G11 = \frac{1}{\rho^2} - \frac{2\,m\left(\frac{\partial}{\partial \rho}\,\mathrm{u}(\rho)\right)}{\rho\,\mathrm{u}(\rho)^2\left(1-\frac{2\,m}{\mathrm{u}(\rho)}\right)} + \frac{m^2\left(\frac{\partial}{\partial \rho}\,\mathrm{v}(\rho)\right)^2}{\left(1-\frac{2\,m}{\mathrm{v}(\rho)}\right)^2\mathrm{v}(\rho)^4}$$

$$G22 = 2\,\frac{u}{-u+2\,m} + \frac{u\,\rho\,m\left(\frac{\partial}{\partial \rho}\,\mathrm{u}(\rho)\right)}{(-u+2\,m)\,\mathrm{u}(\rho)^2\left(1-\frac{2\,m}{\mathrm{u}(\rho)}\right)} - 1 - \frac{u\,\rho\,m\left(\frac{\partial}{\partial \rho}\,\mathrm{v}(\rho)\right)}{(-u+2\,m)\left(1-\frac{2\,m}{\mathrm{v}(\rho)}\right)\mathrm{v}(\rho)^2}$$

---

[1] $G_{ij} = R^{i}_{jkl} \equiv \frac{\partial \Gamma^{i}_{jl}}{\partial x^{k}} - \frac{\partial \Gamma^{i}_{jk}}{\partial x^{l}} + \Gamma^{r}_{jl}\Gamma^{i}_{rk} - \Gamma^{r}_{jk}\Gamma^{i}_{rl}$

$$G33 = -\frac{u\,(-1+\cos(\theta)^2)}{-u+2\,m} - \frac{u\,\rho\,m\left(\frac{\partial}{\partial\rho}\mathrm{v}(\rho)\right)}{(-u+2\,m)\left(1-\frac{2\,m}{\mathrm{v}(\rho)}\right)\mathrm{v}(\rho)^2} - \sin(\theta)^2 + \frac{u\,\rho\,(-1+\cos(\theta)^2)\,m\left(\frac{\partial}{\partial\rho}\mathrm{v}(\rho)\right)}{(-u+2\,m)\left(1-\frac{2\,m}{\mathrm{v}(\rho)}\right)\mathrm{v}(\rho)^2}$$

$$\begin{aligned} G44 = {} & -2\,\frac{m\left(\frac{\partial}{\partial\rho}\mathrm{v}(\rho)\right)^2}{\mathrm{v}(\rho)^3\left(1-\frac{2\,m}{\mathrm{u}(\rho)}\right)\left(1-\frac{2\,m}{\mathrm{v}(\rho)}\right)^2} - \frac{m^2\left(\frac{\partial}{\partial\rho}\mathrm{v}(\rho)\right)\left(\frac{\partial}{\partial\rho}\mathrm{u}(\rho)\right)}{\mathrm{v}(\rho)^2\left(1-\frac{2\,m}{\mathrm{u}(\rho)}\right)^2\left(1-\frac{2\,m}{\mathrm{v}(\rho)}\right)^2\mathrm{u}(\rho)^2} \\ & - \frac{4\,m^2\left(\frac{\partial}{\partial\rho}\mathrm{v}(\rho)\right)^2}{\mathrm{v}(\rho)^4\left(1-\frac{2\,m}{\mathrm{u}(\rho)}\right)\left(1-\frac{2\,m}{\mathrm{v}(\rho)}\right)^3} + \frac{m\left(\frac{\partial^2}{\partial\rho^2}\mathrm{v}(\rho)\right)}{\mathrm{v}(\rho)^2\left(1-\frac{2\,m}{\mathrm{u}(\rho)}\right)\left(1-\frac{2\,m}{\mathrm{v}(\rho)}\right)^2} \\ & + \frac{2\,m\left(\frac{\partial}{\partial\rho}\mathrm{v}(\rho)\right)}{\rho\,\mathrm{v}(\rho)^2\left(1-\frac{2\,m}{\mathrm{u}(\rho)}\right)\left(1-\frac{2\,m}{\mathrm{v}(\rho)}\right)^2} \\ & + \frac{m^2\left(\frac{\partial}{\partial\rho}\mathrm{u}(\rho)\right)\left(\frac{\partial}{\partial\rho}\mathrm{v}(\rho)\right)}{\mathrm{u}(\rho)^2\left(1-\frac{2\,m}{\mathrm{u}(\rho)}\right)\left(1-\frac{2\,m}{\mathrm{v}(\rho)}\right)\mathrm{v}(\rho)^2} \end{aligned} \qquad \textbf{(5)}$$

In this case, the equations contained in the contracted tensor G_{ij} was set equal to zero as one of many conditions that can be assumed in order to arrive to a metric for the continuum space-time.

The solutions for Einstein's contracted tensor or Riemann-Christoffel's tensor, are

$$sol := \{v = m + m\tanh\left(\_C1 - \frac{1}{2}\right),\ u = 2\,m\},\ \{u = 2\,m,\ v = m - m\tanh\left(\_C1 + \frac{1}{2}\right)\} \qquad \textbf{(6)}$$

This produced the following metrices

$$g_{11} = k \qquad g_{22} = -\rho^2 \qquad g_{33} = -\rho^2\sin^2(\theta) \qquad g_{44} = -\frac{1}{1-2m/\left(m - m\tanh\left(\_ - C1 + 1/2\right)\right)} \qquad \textbf{(7)}$$

Euler-Lagrange equations for geodesic curves are generated in accordance with metrics in equations (7). Here, Christoffel symbols of second kind non-zero components are

$$\begin{array}{ll} \{1,22\} = \frac{\rho}{k} & \{3,13\} = \frac{1}{\rho} \\ \{1,33\} = \frac{\rho \sin(\theta)^2}{k} & \{3,23\} = \frac{\cos(\theta)}{\sin(\theta)} \\ \{2,12\} = \frac{1}{\rho} & \\ \{2,33\} = -\sin(\theta)\cos(\theta) & \end{array} \qquad \textbf{(8)}$$

Now the geodesic equations are generated:

$$\left(\frac{\partial^2}{\partial s^2}\rho(s)\right) + \frac{\rho\left(\frac{\partial}{\partial s}\theta(s)\right)^2}{k} + \frac{\rho\sin(\theta)^2\left(\frac{\partial}{\partial s}\phi(s)\right)^2}{k} = 0 \qquad \textbf{(9)}$$

$$\left(\frac{\partial^2}{\partial s^2}\theta(s)\right) + \frac{2\left(\frac{\partial}{\partial s}\rho(s)\right)\left(\frac{\partial}{\partial s}\theta(s)\right)}{\rho} - \sin(\theta)\cos(\theta)\left(\frac{\partial}{\partial s}\phi(s)\right)^2 = 0 \qquad \textbf{(10)}$$

$$\left(\frac{\partial^2}{\partial s^2}\phi(s)\right) + \frac{2\left(\frac{\partial}{\partial s}\rho(s)\right)\left(\frac{\partial}{\partial s}\phi(s)\right)}{\rho} + \frac{2\cos(\theta)\left(\frac{\partial}{\partial s}\theta(s)\right)\left(\frac{\partial}{\partial s}\phi(s)\right)}{\sin(\theta)} = 0 \qquad \textbf{(11)}$$

$$\frac{\partial^2}{\partial s^2}t(s) = 0 \qquad \textbf{(12)}$$

Equation (10) is identically satisfied if planar orbits are taken in consideration, where $\theta = \pi/2$ is a constant. This value of θ also simplifies equations (9) and (11). equation (11) becomes an exact differential equation

$$\left(\frac{\partial^2}{\partial s^2}\phi(s)\right) + \frac{2\left(\frac{\partial}{\partial s}\rho(s)\right)\left(\frac{\partial}{\partial s}\phi(s)\right)}{\rho} \quad \text{or} \quad \frac{\partial}{\partial s}\phi(s) = \frac{c4}{\rho^2} \qquad \textbf{(13)}$$

Equation (12) becomes an exact differential too

$$\frac{\partial^2}{\partial s^2}t(s) = 0 \quad \text{or} \quad \frac{\partial}{\partial s}t(s) = c5 \qquad \textbf{(14)}$$

where c4 and c5 are integration constants. This allows equation (9) to determine ρ. Substituting results from equations (13) and (14), in conjunction with the length of arc of the proposed metric, the following result is obtained

$$\left(\frac{\partial}{\partial s}s(s)\right)^2 = \left(1 - \frac{2m}{u}\right)\left(\frac{\partial}{\partial s}\rho(s)\right)^2 - \rho^2\left(\frac{\partial}{\partial s}\theta(s)\right)^2 - \rho^2\sin(\theta)^2\left(\frac{\partial}{\partial s}\phi(s)\right)^2 - \frac{\left(\frac{\partial}{\partial s}t(s)\right)^2}{1 - \frac{2m}{v}} \qquad \textbf{(15)}$$

Equation (9) becomes

$$
\begin{aligned}
&\frac{1}{2}\Bigg(\bigg(2\,\rho(s)\left(\frac{\partial}{\partial s}\rho(s)\right)+2\,\rho(s)\tanh\!\left(\_C1+\frac{1}{2}\right)\left(\frac{\partial}{\partial s}\rho(s)\right)-2\,c5^2\,\rho(s)\left(\frac{\partial}{\partial s}\rho(s)\right)\\
&\quad+2\,c5^2\,\rho(s)\tanh\!\left(\_C1+\frac{1}{2}\right)\left(\frac{\partial}{\partial s}\rho(s)\right)\bigg)\bigg/\left(k\,\rho(s)^2\left(1+\tanh\!\left(\_C1+\frac{1}{2}\right)\right)\right)-2\,\bigg(\\
&\quad\rho(s)^2+\rho(s)^2\tanh\!\left(\_C1+\frac{1}{2}\right)-c5^2\,\rho(s)^2+c5^2\,\rho(s)^2\tanh\!\left(\_C1+\frac{1}{2}\right)+c4^2\\
&\quad+c4^2\tanh\!\left(\_C1+\frac{1}{2}\right)\bigg)\left(\frac{\partial}{\partial s}\rho(s)\right)\bigg/\left(k\,\rho(s)^3\left(1+\tanh\!\left(\_C1+\frac{1}{2}\right)\right)\right)\Bigg)\Bigg/\Bigg(\bigg(\rho(s)^2\\
&\quad+\rho(s)^2\tanh\!\left(\_C1+\frac{1}{2}\right)-c5^2\,\rho(s)^2+c5^2\,\rho(s)^2\tanh\!\left(\_C1+\frac{1}{2}\right)+c4^2\\
&\quad+c4^2\tanh\!\left(\_C1+\frac{1}{2}\right)\bigg)\bigg/\left(k\,\rho(s)^2\left(1+\tanh\!\left(\_C1+\frac{1}{2}\right)\right)\right)\Bigg)^{(1/2)}+\frac{c4^2}{k\,\rho(s)^3}=0
\end{aligned}
$$

(16a)

or

$$
\frac{1}{2}\,\frac{\sqrt{2}\left(2\,\dfrac{\frac{\partial}{\partial s}\rho(s)}{\rho(s)\,k}-\dfrac{(2\,\rho(s)^2+2\,c4^2)\left(\frac{\partial}{\partial s}\rho(s)\right)}{k\,\rho(s)^3}\right)}{\sqrt{\dfrac{2\,\rho(s)^2+2\,c4^2}{k\,\rho(s)^2}}}+\frac{c4^2}{k\,\rho(s)^3}=0 \qquad \textbf{(16b)}
$$

or

$$
\frac{\partial}{\partial\phi}\rho(\phi)=\frac{\sqrt{\rho(\phi)^2+c4^2}\,\sqrt{\frac{1}{k}}\,\operatorname{csgn}\!\left(\overline{(\rho(\phi))}\right)\rho(\phi)}{c4}
$$

(16c)

Using chain rule

$$
\frac{\partial}{\partial s}\rho(s)=\left(\frac{\partial}{\partial\phi}\rho(\phi)\right)\left(\frac{\partial}{\partial s}\phi(s)\right)
$$

Taking into account equation (12) and equation (15) the following form can be written

$$
\frac{1}{2}\,\frac{\sqrt{2}\left(2\,\dfrac{\left(\frac{\partial}{\partial\phi}\rho(\phi)\right)c4}{\rho(\phi)^3\,k}-\dfrac{(2\,\rho(\phi)^2+2\,c4^2)\left(\frac{\partial}{\partial\phi}\rho(\phi)\right)c4}{k\,\rho(\phi)^5}\right)}{\sqrt{\dfrac{2\,\rho(\phi)^2+2\,c4^2}{k\,\rho(\phi)^2}}}+\frac{c4^2}{k\,\rho(\phi)^3}=0 \qquad \textbf{(17)}
$$

The substitution $\rho(\phi)=1/u(\phi)$ in equation (17) becomes

$$\frac{1}{2}\,\frac{\sqrt{2}\left(-2\,\frac{u(\phi)\left(\frac{\partial}{\partial\phi}u(\phi)\right)c4}{k}+\frac{\left(2\,\frac{1}{u(\phi)^2}+2\,c4^2\right)u(\phi)^3\left(\frac{\partial}{\partial\phi}u(\phi)\right)c4}{k}\right)}{\sqrt{\frac{\left(2\,\frac{1}{u(\phi)^2}+2\,c4^2\right)u(\phi)^2}{k}}}+\frac{u(\phi)^3\,c4^2}{k}=0 \qquad \textbf{(18)}$$

Multiplying equation (18) by $du/d\phi$ and integrating it with respect to ϕ the following is obtained

$$\frac{1}{2}\,\frac{\sqrt{2}\left(-2\,\frac{u(\phi)^2\left(\frac{\partial}{\partial\phi}u(\phi)\right)c4}{k}+\frac{\left(2\,\frac{1}{u(\phi)^2}+2\,c4^2\right)u(\phi)^4\left(\frac{\partial}{\partial\phi}u(\phi)\right)c4}{k}\right)}{\sqrt{\frac{\left(2\,\frac{1}{u(\phi)^2}+2\,c4^2\right)u(\phi)^2}{k}}}+\frac{u(\phi)^4\,c4^2}{k}=c6 \qquad \textbf{(19)}$$

Where c6 is an integration’s constant. In order to determine constant c6, equation (15) is written taken the special case, $\theta=\pi/2$, using substitutions from equations (13) and (14) the following is obtained

$$\frac{\left(\frac{\partial}{\partial\phi}\rho(\phi)\right)^2 c4^2}{\rho^4}=\frac{\rho^2+\rho^2\tanh\left(\_C1+\frac{1}{2}\right)-c5^2\,\rho^2+c5^2\,\rho^2\tanh\left(\_C1+\frac{1}{2}\right)+c4^2+c4^2\tanh\left(\_C1+\frac{1}{2}\right)}{k\,\rho^2\left(1+\tanh\left(\_C1+\frac{1}{2}\right)\right)} \qquad \textbf{(20)}$$

In equation (20), the fact just below was taken in consideration

$$e^{-C1}=c^2$$

Where c is light’s velocity, $c=3\left(10^{10}\right)\ cm/s$. Therefore

$$\tanh\left(\_C1+\frac{1}{2}\right)=1$$

Then, the substitution

$$\rho(\phi)=1/u(\phi)$$

reduces equation (20) to the form

$$\left(\frac{\partial}{\partial\phi}u(\phi)\right)^2=\frac{1}{2}\,\frac{u(\phi)^2\left(2\,\frac{1}{u(\phi)^2}+2\,c4^2\right)}{c4^2\,k} \qquad \textbf{(21)}$$

Comparison of equations (21) and (19) suggests the selection

$$c6 := \frac{1}{2}\sqrt{2}\left(-\frac{\mathrm{u}(\phi)^2\sqrt{2}\sqrt{\dfrac{\mathrm{u}(\phi)^2\left(2\dfrac{1}{\mathrm{u}(\phi)^2}+2\,c4^2\right)}{c4^2\,k}}\;c4}{k} + \frac{\dfrac{1}{2}\left(2\dfrac{1}{\mathrm{u}(\phi)^2}+2\,c4^2\right)\mathrm{u}(\phi)^4\sqrt{2}\sqrt{\dfrac{\mathrm{u}(\phi)^2\left(2\dfrac{1}{\mathrm{u}(\phi)^2}+2\,c4^2\right)}{c4^2\,k}}\;c4}{k}\right) \Bigg/ \sqrt{\frac{\left(2\dfrac{1}{\mathrm{u}(\phi)^2}+2\,c4^2\right)\mathrm{u}(\phi)^2}{k}+\frac{\mathrm{u}(\phi)^4\,c4^2}{k}}$$

So, equation (19) takes the following form

$$\left(\frac{\partial}{\partial\phi}\mathrm{u}(\phi)\right)-\sqrt{\frac{1+c4^2\,\mathrm{u}(\phi)^2}{c4^2\,k}}=0 \tag{22}$$

Now the relativistic equation (22) is compared to the Newtonian equation, that is

$$\left(\frac{\partial}{\partial\phi}\mathrm{u}(\phi)\right)^2+\mathrm{u}(\phi)^2-\frac{2\,G\,M\,\mathrm{u}(\phi)}{h^2}+\frac{E}{h^2}=0 \tag{23}$$

That led to the calculation of constant $c4$

$$c4 := -\frac{h}{\sqrt{-k\,\mathrm{u}(\phi)^2\,h^2+2\,k\,G\,M\,\mathrm{u}(\phi)-k\,E-\mathrm{u}(\phi)^2\,h^2}} \tag{24}$$

Substituting equations (13), (14) and (24) into equation (21) the following differential equation is obtained

$$\left(\frac{\partial}{\partial\phi}\mathrm{u}(\phi)\right)^2=\frac{1}{2}\,\mathrm{u}(\phi)^2\,(-k\,\mathrm{u}(\phi)^2\,h^2+2\,k\,G\,M\,\mathrm{u}(\phi)-k\,E-\mathrm{u}(\phi)^2\,h^2)\left(2\frac{1}{\mathrm{u}(\phi)^2}+\frac{2\,h^2}{-k\,\mathrm{u}(\phi)^2\,h^2+2\,k\,G\,M\,\mathrm{u}(\phi)-k\,E-\mathrm{u}(\phi)^2\,h^2}\right)\Big/(h^2\,k) \tag{25}$$

Now, let equation (24) and $\rho(\phi) := \dfrac{1}{\mathrm{u}(\phi)}$ then the differential equation (18) can be written as

$$\frac{1}{2}\,\frac{\sqrt{2}\left(-2\,\frac{\mathrm{u}(\phi)\left(\frac{\partial}{\partial\phi}\mathrm{u}(\phi)\right)c4}{k}+\frac{\left(2\,\frac{1}{\mathrm{u}(\phi)^{2}}+2\,c4^{2}\right)\mathrm{u}(\phi)^{3}\left(\frac{\partial}{\partial\phi}\mathrm{u}(\phi)\right)c4}{k}\right)}{\sqrt{\frac{\left(2\,\frac{1}{\mathrm{u}(\phi)^{2}}+2\,c4^{2}\right)\mathrm{u}(\phi)^{2}}{k}}}+\frac{\mathrm{u}(\phi)^{3}\,c4^{2}}{k}=0 \qquad \textbf{(26)}$$

The solution for equation (26) is given in terms of the solution that Newton gave for equation (23), which is equation (1).

It is known that A is small, so it’s feasible to do the assumption that the solution of equation (26) given for equation (1) is such that ϕ_0 is approximately constant and varies slowly as a function of $A\phi$. Observe that if $\phi_0=\phi_0(A\phi)$, then $d\phi_0/d\phi=\phi_0'A$, where prime denote differentiation with respect to the argument of the function. So

$$\frac{\partial}{\partial\phi}\mathrm{u}(\phi)=-A\,\varepsilon\,\sin(\phi-\phi_0)\left(1-\left(\frac{\partial}{\partial\phi}\phi_0\right)A\right)$$

In this case, the following equation is proposed

$$-A\,\varepsilon\,\sin(\phi-\phi_0)\left(3\,\frac{G^2\,M^2}{c^2\,h^2}-\left(\frac{\partial}{\partial\phi}\phi_0\right)A\right) \qquad \textbf{(27)}$$

Substituting equation (1) and equation (27) into equation (26) and considering that A is small in such a way that terms $O(A^3)...O(A^n)$ can be neglected, also is needed to compute constant terms and coefficient of trigonometrical functions, all that led to the next differential equation

$$ode := \frac{\partial}{\partial\phi}\phi_0(\phi)=3\,\frac{G^2\,M^2}{c^2\,h^2} \qquad \textbf{(28)}$$

whose solution is

$$ans_1 := \{\phi_0(\phi)=3\,\frac{G^2\,M^2\,\phi}{c^2\,h^2}+\_C1\} \qquad \textbf{(29)}$$

Now, the rotation slow rate of Mercury's semi-major axis can be calculated to be approximately

$$\frac{d\phi_0}{dt}=\frac{d\phi_0}{d\phi}\frac{d\phi}{dt}\approx 3\left(\frac{GM}{ch}\right)^2\left(\frac{GM}{a^3}\right)^{1/2}=6.628\times 10^{-14}\quad rad/s=43.01\quad s_{arc}/century \qquad \textbf{(30)}$$

where the term

$$\frac{d\phi}{dt}\approx\left(\frac{GM}{a^3}\right)^{1/2}=0.8290824031\times 10^{-6}\quad s^{-1} \qquad \textbf{(31)}$$

corresponds to Kepler’s third law.

The slow variation in Mercury’s semi-major axis, $d\phi_0/dt$, has been observed and measured, it’s value is $6.628 \times 10^{-14} \quad rad/s = 43.01 \quad s_{arc}/century$.

Newtonian mechanics couldn’t predict the changes observed in Mercury’s semi-major axis, but Einstein’s theory of relativity could.

The prediction of $d\phi_0/dt$ provides evidence of the fact that planet Mercury is dragging space-time as noted in NASA (1998): “An international team of NASA and university researchers has found the first direct evidence of a phenomenon predicted 91 years ago using Einstein’s theory of general relativity- that the Earth is dragging space and time around itself as it rotates”.

On the other hand, equation (16c) is an equation in quadrature format. In this case, the ordinary diffetential equation (ODE), is first order and the right side term depends only on the dependent variable.

Taken in account their symmetries, all ODEs "missing dependent variable" exhibit the symmetry $[\_\xi = 0, \_\eta = 1]$, also all ODEs "missing independent variable" show the symmetry $[\_\xi = 1, \_\eta = 0]$. Equation (16c), in both cases has the solution

$$[\_\xi = 1, \ \_\eta = 0] \tag{32}$$

In order to solve equation (16c) it’s needed to look for a symmetry generator given an ODE, this is done by means of algorithms that determines the coefficients of the symmetry generator (infinitesimals: $\_\xi(\phi,\rho)$ and $\_\eta(\phi,\rho)$ of a one-parameter Lie group, this one makes the given ODE invariant) as well as a possible functional form for the infinitesimals in order to test our own heuristics looking for infinitesimals if none of the algorithms are successful. In all cases, the solution obtained was equation (32) (Cheb-Terrab et al (1997), Cheb-Terrab (2001) and Hydon (2000)).

Solution of equation (16c) by using quadrature format and symmetries of equation (32), is given by

$$\phi + \frac{\operatorname{csgn}(\rho(\phi))\operatorname{csgn}(c4)\left(\ln(2) + \ln\left(\frac{c4\,(c4 + \operatorname{csgn}(c4)\sqrt{\rho(\phi)^2 + c4^2}\,)}{\rho(\phi)}\right)\right)}{\sqrt{\frac{1}{k}}} + \_C1 = 0 \tag{33}$$

Using integrating factor for equation (16c), led to

$$\mu := \frac{1}{\sqrt{\rho(\phi)^2 + c4^2}\ \operatorname{csgn}(\overline{(\rho(\phi))})\ \rho(\phi)} \tag{34}$$

Solution for the related exact ODE is

$$-\left(\sqrt{\frac{1}{k}}\,\phi+\mathrm{csgn}(\rho(\phi))\,\mathrm{csgn}(c4)\ln(2)+\mathrm{csgn}(\rho(\phi))\,\mathrm{csgn}(c4)\ln\left(\frac{c4\,(c4+\mathrm{csgn}(c4)\sqrt{\rho(\phi)^2+c4^2})}{\rho(\phi)}\right)-\_C1\,c4\right)/c4=0 \qquad \textbf{(35)}$$

3. Results and discussions

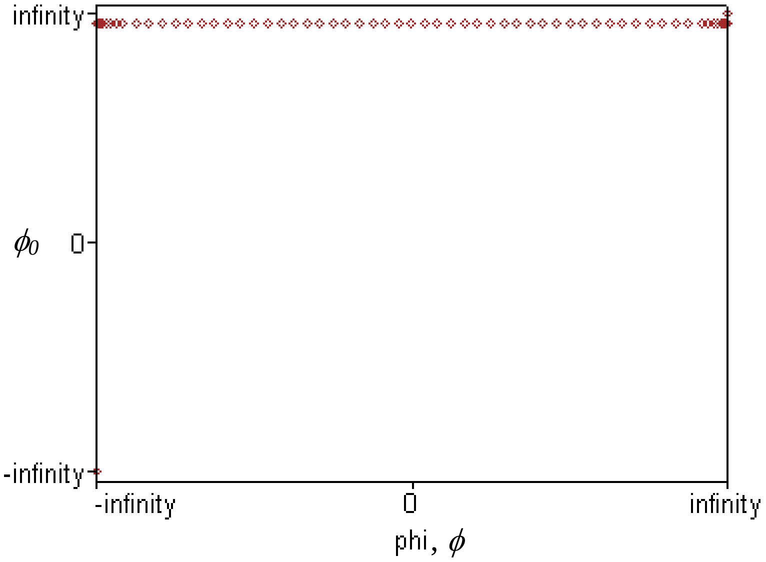

Fig. 2 Arbitrary starting anomaly, ϕ_0, versus true anomaly associated with the orbit, ϕ, of planet Mercury in the space-time by means of the postulated metric.

The mathematical model was simulated by mean of relativistic equations in accordance with the postulated metric as well as differential equation proposed to predict the behavior of gigantic masses in the continuum space-time.

Fig (2) shows the behavior of the arbitrary starting anomaly, ϕ_0, with respect to the true anomaly associated with orbit, ϕ, (equation (29)). In this figure it can be observed a similar behavior to that obtained by the solution of Newton’s equation and the model of Corral et al. (2008).

This response of anomalies obtained through singularity shown in Fig. 2, allows to glimpse the uncertainty of the existence of a mass subject to an infinitely huge gravitational field, as could be a mass that penetrates into a Schwarzschild’s black hole (Eisenstaedt (1989), Martel and Poisson (2001, 2005), Martel (2004), Melia (2003) and Bryner (2008)), or the same black hole predicted by Stephen Hawking in Hawking (1970, 1974), Penrose (1996) and Michael (1983).

The metric postulated here allows to predict with good approximation of the slow variation in Mercury’s semi-major axis, $d\phi_0/dt$, in accordance with the Einstein’s relativity theory. The next terms were included in the differential equation obtained for this metric: universal gravitation constant, G; mass of a planet, M ; velocity of the light, c and the constant for a body system, h. The assignation of these values to the equation, allowed the determination of the value of the arbitrary starting anomaly for the solution that Newton gave to his equation (equation (23)). Results can be shown in Fig. 3. In this figure it can be appreciated that the curve corresponding to Newton's model overlaps the Heinbockel one.

Graph of Fig. 3. a) corresponds to the semi-parameter's behavior of the conic section of a gigantic mass, ρ , with respect to the true anomaly associated with the orbit, ϕ. This figure shows curves of models from other authors previously reported in Corral et al. (2008). This figure shows that the gigantic mass, specifically the one of planet Mercury, moves in the same interval of space-time ($4.6e+12$ to $7e+12$) that is, by using models that propose different metric in Corral et al. (2008), ej. Schwarzchild and Heinbockel's models as well as Newton's model obtained in this paper, but not with Newton's model simulated previously in Corral et al. (2008), where it is seen that the mass acts in an interval of approximately $2.2e+12$ to $3.5e+12$.

Fig. 3 b) shows the curve of Newton's model obtained in this paper and Fig. 3 c) shows the similarity of behavior of Newton's curve obtained in this paper with the curve of Heinbockel which cannot be distinguish so clearly in the Fig. 3 a).

The resulting solution of equation (16a), equation (33), can be caused by the curvature of the continuum space-time in Fig. 3.

Also a graphic showing the variation of the semi-parameter of the conic section, ρ , is presented, with respect to the true anomaly, ϕ, for a gigantic mass as the planet Mercury passing through the space-time (see Figs. 4-6).

Table 1 shows some small variations of rho (ρ) in certain specific points of phi (ϕ) in the space-time of Fig. 4.

In Fig. 5 a) presents the constant behavior of ϕ at right and left of the semi-major axis with regard to ρ in the range 4.6×10^{12} at 7×10^{12} of the interval of space-time as in Corral et al. (2008). In Fig. 5 b) can be appreciated the behavior of ρ with regard to ϕ, on it, taken as reference an elliptic orbit with $0\ rad$ in the semi-major axis, as ϕ increases, ρ diminishes and vice versa, it is the expected behavior.

In Fig. 6 a) and b) is appreciated that for certain constant points of ϕ , the semi-parameter's length of the orbit, ρ, varies slowly as in Fig. 4.

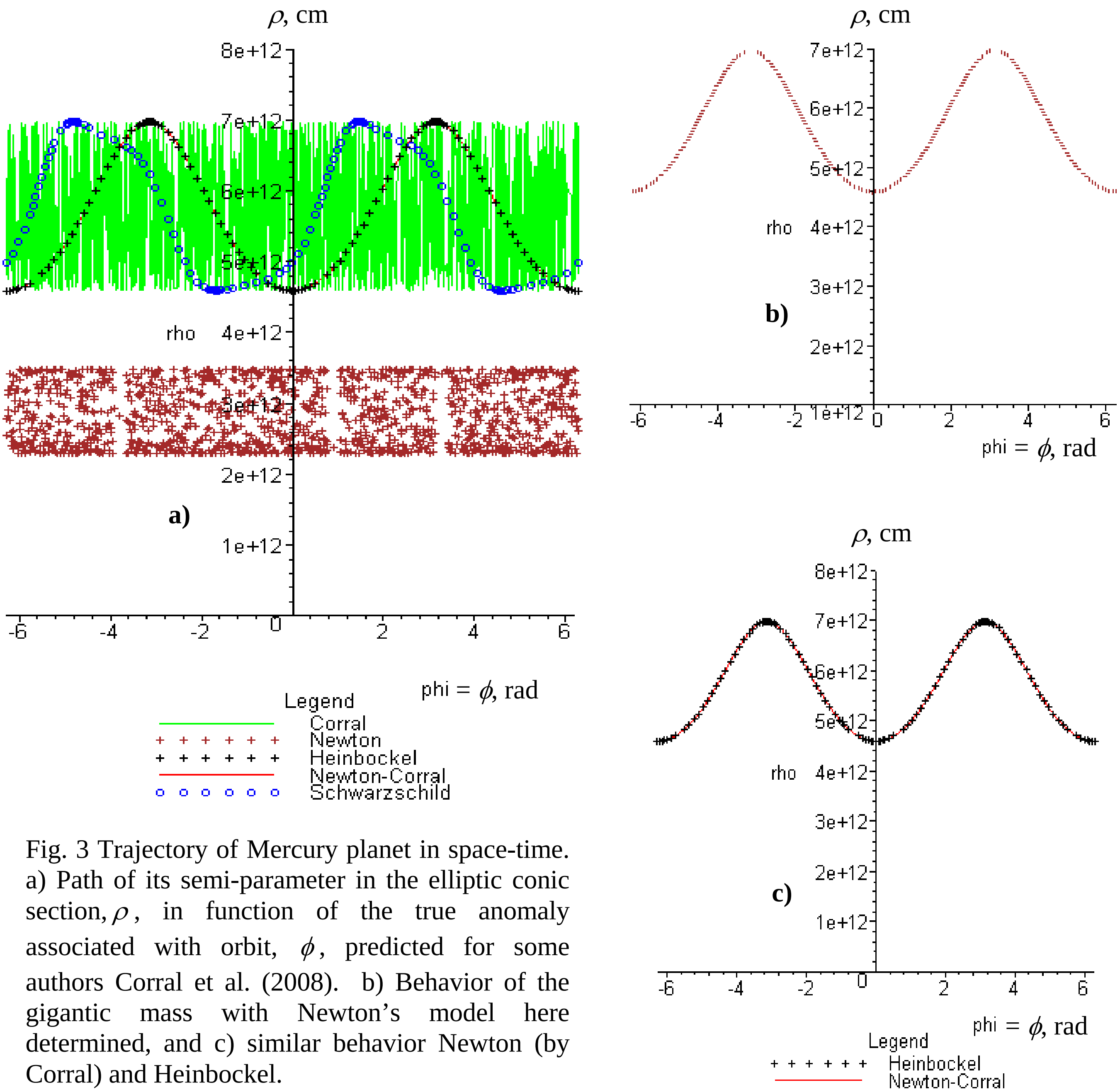

Fig. 3 Trajectory of Mercury planet in space-time. a) Path of its semi-parameter in the elliptic conic section, ρ, in function of the true anomaly associated with orbit, ϕ, predicted for some authors Corral et al. (2008). b) Behavior of the gigantic mass with Newton's model here determined, and c) similar behavior Newton (by Corral) and Heinbockel.

Table 1 Variations of the length of ρ in specific points of ϕ

Values in the ϕ axis	ϕ, *rad* (accumulated)	ρ, *cm*
-294961	0	-0.84 at -0.8
-294960	1	-0.76 at -0.6
-294959	2	-0.56 at -0.44
-294958	3	-0.4 at -0.32
-294956.8	4	-0.28 at -0.24
-294956	5	-0.2 at -0.16

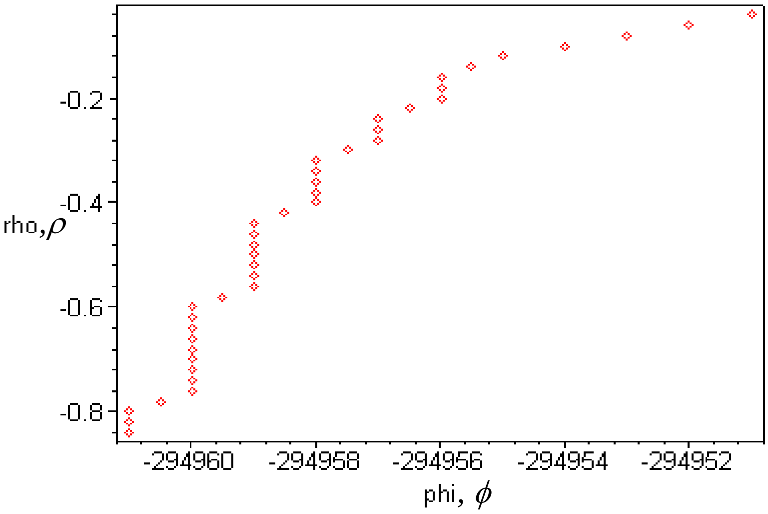

Fig. 4 Variable behavior of semi-parameter's length in its elliptic conic section, ρ, while ϕ remaining constant in certain points of space-time.

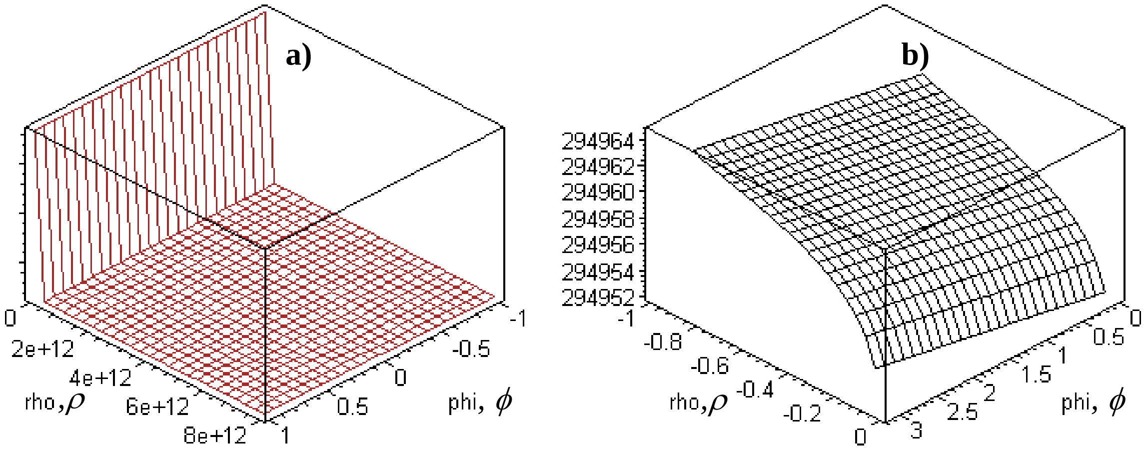

Fig. 5 a) Path of a gigantic mass in the space-time in both sides of the semi-major axis on an elliptic orbit. b) Graph showing the simultaneous falling behavior of ϕ with rising of ρ, putting $\phi = 0 \quad rad$ as reference in the semi-major axis on an elliptic orbit.

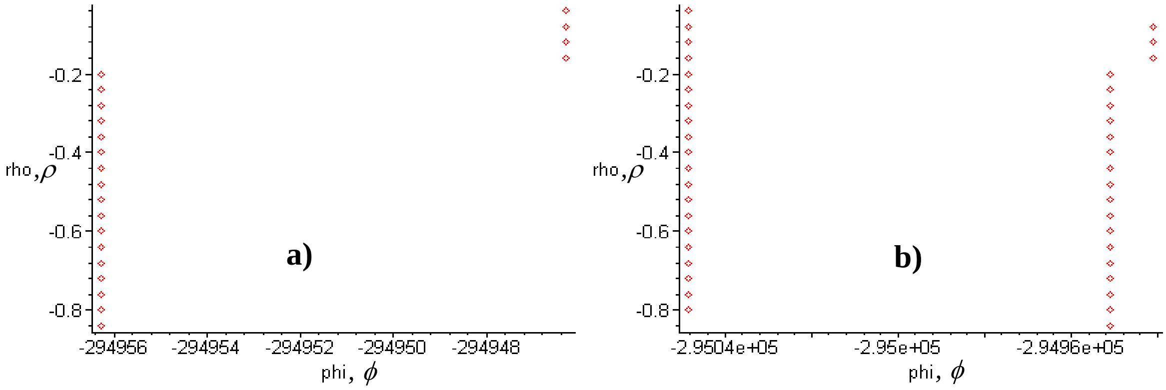

Fig. 6 Slow rate of rotation of the semi-parameter on the elliptic conic section, ρ : a) located at -294956.4 rad and -294946.4 rad of ϕ, and b) located at -2.9504e+05 rad and -2.94950e+5 rad, same as at -2.94950e+5rad and -2.94940e+5rad of ϕ, respectively.

4. Conclusions

The postulated metric in this paper allows the proposal of a differential equation able to predict the arbitrary starting anomaly, with it, the rotation's slow rate of the semi-major axis in seconds of arc per century can be calculated, it agrees with Einstein's prediction. Given the previously mentioned, it was corroborated that in accordance to the postulated metric, a gigantic mass drags space and time around itself as it rotates in the continuum space-time. Also, with this metric, can be predicted the behavior of the spatial coordinates in an elliptic orbit on which a gigantic mass is moving in the continuum space-time. These coordinates to correspond to the semi parameter of the conic section, ρ, and the true anomaly associated with the orbit, ϕ.

Respect to this, it's appreciated the slow variation presented by the longitude of ρ in specific points of ϕ as well as the simultaneous form in which ρ rises when ϕ falls, taking as reference $\phi = 0$ radians in the semi-major axis of an elliptical orbit.

Even more, the proposed differential equation that should calculate the arbitrary starting anomaly, determines in a very precise way the behavior of gigantic masses' movement in space-time, as predicted by Newton for planetary movement in classic mechanics, it is identical to the metric proposed by Heinbockel for planet Mercury with relativistic equations.

That represents evidence of a close link between Classical Mechanics and General Relativity, as in previous work in Corral et al. (2008).

The postulated metric here is able to obtain a differential equation that in fact, locates the planet Mercury's movement as described by Newton's equation for gigantic masses at the same space-time interval than the movement of these masses modeled by means of relativistic equations describing planetary motion.

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