DEA-Discriminant Analysis for Three Groups

F. Rezai Balf a, M. Saghaei b and F. Hosseinzadeh Lotfi c*

a Department of Mathematics, Islamic Azad University, Qaemshahr, Iran

b Department of Mathematics, Islamic Azad University, Qaemshahr, Iran

c Department of Mathematics, Science and Research Branch
Islamic Azad University, Tehran, Iran

Abstract

We know DEA-Discriminant Analysis (DEA-DA) is designed to identify the existence of an overlap between two groups, by separating hyperplane, then determining the group membership of a newly sampled observation. The separating hyperplane is identified by minimizing the sum of deviations, so the hyperplane locates between the two groups of observations. In this paper we extended DEA-Discriminant Analysis for three groups $G_1$, $G_2$ and $G_3$ which have $n_1$, $n_2$ and $n_3$ observations, respectively.

Keywords: Discriminant Analysis; Data Envelopment Analysis

1. Introduction

The DEA (Data Envelopment Analysis that is a management science technique) and DA (Discriminant Analysis that is a statistical technique) methods, first proposed by Charnes et al. (1978)[1] and Freed and Glover (1981)[2], respectively. DEA-Discriminant Analysis (new type of DA) is a decisional tool that can predict group membership of a newly sampled data. A new sample is classified into a group by comparing its estimated discriminant score with an evaluation score derived from an estimated discriminant function. For example, we can predict the conditional state of students with use of DEA-DA method for conditional students (expulsion and non-expulsion students) and unconditional students.

* Corresponding Author: farhad@hosseinzadeh.ir
Sueyoshi has been proposed a article referred to as "DEA-Discriminant Analysis in the view of goal programming" (European Journal of Operational Research 115 (1999), 564-582) for classification two groups, but Jahanshahloo et al. in "Discriminant analysis of interval data using Monte Carlo method in assessment of overlap" (Applied Mathematics and Computation 191 (2007), 521-532) showed that Sueyoshi model always cannot specify an overlap between two sets. Now, we claim that Sueyoshi modified model (introduced by Jahanshahloo et al. 2007) has a drawback, too. In this paper, we revise Sueyoshi model and Sueyoshi modified model. The rest of the paper is organized as follows:

In section 2 we give two models, mentioned above. In addition we show our claim with an example. Then we introduce revised Sueyoshi modified model along with an example. In section 3 a model is introduced for three groups. Section 4 examines one numerical example using the proposed method. Finally in section 5 conclusions are presented.

2. Background

Definition

Let $S_1$ and $S_2$ be nonempty sets in $E_n$. A hyperplane $H = \{x: p^T x = \alpha\}$ is said to separate $S_1$ and $S_2$ if $p^T x \geq \alpha$ for each $x \in S_1$ and $p^T x \leq \alpha$ for each $x \in S_2$.

2.1. Sueyoshi model

Suppose that there are $n$ observations $z_j = (z_{1j}, ..., z_{kj}), (j = 1, ..., n)$ such that each observation has $k$ independent factors. Suppose that all observations ($G$) are classified into two groups, Group 1 ($G_1$) and Group 2 ($G_2$), which have $n_1$ and $n_2$ observations, respectively. Also it is assumed that $n_1 + n_2 = n, G_1 \cup G_2 = G, G_1 \neq \emptyset$ and $G_2 \neq \emptyset$. Then we research a hyperplanes to form $\alpha z = d$, such that

\[ \forall z_j \ (z_j \in G_1 \rightarrow \alpha z \geq d) \]
\[ \forall z_j \ (z_j \in G_2 \rightarrow \alpha z \leq d - \varepsilon) \]

The small positive number $\varepsilon$ is use to avoid an obvious solution (i.e., all weight are zero). Sueyoshi model process is achieved by the following two steps: (a) the classification and overlap identification and (b) the overlap handling [4].
Classification and Overlap Identification (COI): The first step of DEA-DA is mathematically formulated as follows:

\[ \begin{align*}
\text{Min} \quad & \varphi = \sum_{j \in J_1} s_{1j}^+ + \sum_{j \in J_2} s_{2j}^- \\
\text{s.t.} \quad & \sum_{i=1}^{k} \alpha_i z_{ij} + s_{1j}^+ - s_{1j}^- = d, \quad j \in J_1, \\
& \sum_{i=1}^{k} \beta_i z_{ij} + s_{2j}^+ - s_{2j}^- = d - \eta, \quad j \in J_2, \\
& \sum_{i=1}^{k} \alpha_i = 1, \quad \sum_{i=1}^{k} \beta_i = 1, \\
& s_{1j}^+, s_{1j}^- \geq 0, \quad j \in J_1, \quad s_{2j}^+, s_{2j}^- \geq 0, \quad j \in J_2, \\
& \alpha_i \geq 0, \quad \beta_i \geq 0, \quad i = 1, \ldots, k.
\end{align*} \]  \tag{1}

where \( \eta \) is a very small positive number, also we have:

\[ J_1 = \{ j \mid z_j = (x_j, y_j) \in G_1 \} \text{ and } J_2 = \{ j \mid z_j = (x_j, y_j) \in G_2 \}. \]

Overlap Handling (OH): The second stage for handling overlap is formulated as follows:

\[ \begin{align*}
\text{Min} \quad & \varphi = \sum_{j \in J_1} s_{1j}^+ + \sum_{j \in J_2} s_{2j}^- \\
\text{s.t.} \quad & \sum_{i=1}^{k} \gamma_i z_{ij} + s_{1j}^+ - s_{1j}^- = d, \quad j \in J_1, \\
& \sum_{i=1}^{k} \gamma_i z_{ij} + s_{2j}^+ - s_{2j}^- = d - \eta, \quad j \in J_2, \\
& \sum_{i=1}^{k} \gamma_i = 1, \\
& s_{1j}^+, s_{1j}^- \geq 0, \quad j \in J_1, \quad s_{2j}^+, s_{2j}^- \geq 0, \quad j \in J_2, \\
& \gamma_i \geq 0, \quad i = 1, \ldots, k.
\end{align*} \]  \tag{2}

2.2. Sueyoshi modified model

Jahanshahloo et al. showed with an example that Sueyoshi model (first stage) should be modified [3]. They supposed \( z_j = (x_j, y_j) = (x_{1j}, \ldots, x_{mj}, y_{1j}, \ldots, y_{sj}), (j = 1, \ldots, n) \) that \( m + s = k \). Also \( x_j \) and \( y_j \) are of type of cost and profit indicator, respectively. In what follows, they presented sueyoshi modified model.
2.3. Revise Sueyoshi modified model

Now, we claim that Sueyoshi modified model has a drawback when exist alternative optimal solutions. Then we show our claim with an example and introduce our model for remove it.

2.4. A numerical example

Consider \( G_1 \) and \( G_2 \) to be the groups of data with one input and one output as follows (Table 1):

<table>
<thead>
<tr>
<th>Groups</th>
<th>( Z_j )</th>
<th>x</th>
<th>y</th>
</tr>
</thead>
<tbody>
<tr>
<td>Group 1 (( G_1 ))</td>
<td>P_1</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>P_2</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td></td>
<td>P_3</td>
<td>1</td>
<td>4</td>
</tr>
<tr>
<td></td>
<td>P_4</td>
<td>2</td>
<td>5</td>
</tr>
<tr>
<td>Group 2 (( G_2 ))</td>
<td>Q_1</td>
<td>5</td>
<td>6</td>
</tr>
<tr>
<td></td>
<td>Q_2</td>
<td>6</td>
<td>6</td>
</tr>
<tr>
<td></td>
<td>Q_3</td>
<td>6</td>
<td>8</td>
</tr>
<tr>
<td></td>
<td>Q_4</td>
<td>7</td>
<td>9</td>
</tr>
<tr>
<td></td>
<td>Q_5</td>
<td>8</td>
<td>7</td>
</tr>
</tbody>
</table>

Using model (1) for classification of such observations, the optimal solution is obtained as follows:

\[
\begin{align*}
\alpha_1^+ &= 0, \quad \alpha_2^+ = 1, \quad \beta_1^+ = 1, \quad \beta_2^+ = 0, \quad d^* = 6, \quad \varphi^* = 13,
\end{align*}
\]

\[
\begin{align*}
\alpha_1^- &= (4,3,2,1), \quad \alpha_2^- = (0,0,0,0,0), \quad \beta_1^- = (1,0,0,0,0), \quad \beta_2^- = (0,0,0,1,2).
\end{align*}
\]
Since the optimal objective value of model (1) is equal 13, so there is an overlap between two groups (Fig. 1).

Now let us use model (3) for the classification of two groups $G_1$ and $G_2$. Note that if change $u(= 0$ or $1)$ and $v(= 0$ or $1)$ then we get alternative optimal solutions. For $u = v = 0$ and $u = v = 1$ we have, respectively:

\[
\begin{align*}
\alpha_1^* &= -5.67, \quad \alpha_2^* = 6.67, \quad \beta_1^* = 0.67, \quad \beta_2^* = 0.33, \quad d^* = 7.67, \quad \varphi^* = 0, \\
 s_1^+ &= (0,0,0,0), \quad s_1^- = (0,1,13.34,14.34), \quad s_2^+ = (2.33,1.67,1,0,0), \quad s_2^- = (0,0,0,0,0).
\end{align*}
\]

and

\[
\begin{align*}
\alpha_1^* &= -1, \quad \alpha_2^* = 0, \quad \beta_1^* = -2.95, \quad \beta_2^* = 1.95, \quad d^* = -2, \quad \varphi^* = 0, \\
 s_1^+ &= (0,0,0,0), \quad s_1^- = (1,0,1,0), \quad s_2^+ = (0.95,3.9,0,1,7.9), \quad s_2^- = (0,0,0,0,0).
\end{align*}
\]

With regards to the optimal objective value of model (3), it conclude that there is not overlap between the two groups, but hyperplanes of obtained of alternative optimal solutions always is not such (see Fig. 2a, b).

With regards to the optimal objective value of model (3), it conclude that there is not overlap between the two groups, but hyperplanes of obtained of alternative optimal solutions always is not such (see Fig. 2a, b).
The example 2.4 (Table 1) shows that the Sueyoshi model and modified model of it should be revise. The first stage of Sueyoshi modified model reformulated as follows:

\[
\begin{align*}
\text{Min} & \quad \varphi = \sum_{j \in J_1} (s^+_1 + s^+_2) + \sum_{j \in J_2} (s^-_2 + s^-_4), \\
\text{s.t.} & \quad \sum_{i=1}^{k} \alpha_i z_{ij} + s^+_1 - s^-_1 = d, \ j \in J_1, \\
& \quad \sum_{i=1}^{k} \beta_i z_{ij} + s^+_2 - s^-_2 = d - \eta, \ j \in J_2, \\
& \quad \sum_{i=1}^{k} \beta_i z_{ij} + s^+_3 - s^-_3 = d, \ j \in J_1, \\
& \quad \sum_{i=1}^{k} \alpha_i z_{ij} + s^+_4 - s^-_4 = d - \eta, \ j \in J_2, \\
& \quad \sum_{i=1}^{k} \alpha_i = 1 - 2u, \quad \sum_{i=1}^{k} \beta_i = 1 - 2v, \\
& \quad u, v \in \{0,1\}, \\
& \quad s^+_1, s^-_1, s^+_2, s^-_2 \geq 0, \ j \in J_1, \\
& \quad s^+_2, s^-_2, s^+_4, s^-_4 \geq 0, \ j \in J_2.
\end{align*}
\]

where \( \eta \) is a very small positive number. (\( \eta = 0.001 \) is used in this study.)

Here, \( s^+_1, s^-_1 (j \in J_1) \) are positive and negative deviations of the linear discriminant function (\( \sum_{i=1}^{k} \alpha_i z_{ij} \)) and \( s^+_2, s^-_2 (j \in J_1) \) are positive and negative deviations of the linear discriminant function (\( \sum_{i=1}^{k} \beta_i z_{ij} \)) from a discriminant score (\( d \)) of the first group (\( G_1 \)). Similarly, \( s^+_2, s^-_2 (j \in J_2) \) and \( s^+_4, s^-_4 (j \in J_2) \) are positive and negative deviations of the linear discriminant functions \( \sum_{i=1}^{k} \beta_i z_{ij} \) and \( \sum_{i=1}^{k} \alpha_i z_{ij} \) from a discriminant score \( (d - 1)\eta \) of the second group (\( G_2 \)). Both \( s^+_1, s^-_3 (j \in J_1) \) and \( s^-_2, s^-_4 (j \in J_2) \) show the incorrect classification that should minimize.

By using model (4) for data of Table 1, we obtain:

\[
\begin{align*}
\alpha^*_1 &= 0.33, \quad \alpha^*_2 = -1.33, \quad \beta^*_1 = 0, \quad \beta^*_2 = -1, \quad d^* = -6, \quad \varphi^* = 0, \\
& \quad s^+_1 = (0,0,0,0), \quad s^-_1 = (3.67,2.67,1,0), \quad s^+_2 = (0,0,2,3,1), \quad s^-_2 = (0,0,0,0,0), \\
& \quad s^+_3 = (4,3,2,1), \quad s^-_3 = (0.33,0,2.67,3.67,0.67), \quad s^+_4 = (0,0,0,0,0).
\end{align*}
\]

In other words we have:

\[
0.33x - 1.33y \geq -6, \quad j \in J_1 \quad \text{and} \quad -y < -6, \quad j \in J_2.
\]

This means that there is not overlap between two groups \( G_1 \) and \( G_2 \) (Fig. 3).
In this example, we had a unique optimal solution, namely, if change $u(=0\ or\ 1)$ and $v(=0\ or\ 1)$ don't produce alternative optimal solutions.

3. Extend DEA-DA for three groups

Suppose that all observations ($G$) are classified into three groups, Group 1 ($G_1$), Group 2 ($G_2$) and Group 3 ($G_3$) which have $n_1, n_2$ and $n_3$ observations, respectively. Also it is assumed that $n_1 + n_2 + n_3 = n$ and $G_1 \cup G_2 \cup G_3 = G$.

Hereafter, we will present a method for discriminant three groups $G_1, G_2$ and $G_3$ It can be defined as the following two steps:

**First step:** Should separate a group from other two groups with the help of three models (5), (6) and (7).

**Second step:** We discriminate between the remaining groups with use of hyper-plane performed by model (4).

In the first step of new method, should choose a group from between three groups given. Choice criterion is, have the least total deviation's value with other two groups. So, for this purpose we solve three models (5), (6) and (7) as parallel. Hence, we will have:

![Fig. 3. Not existence of overlap with model (4)](image-url)
Min  \[ \theta = \sum_{j \in J_3} (t^+_{1j} + t^+_{3j}) + \sum_{j \in J_1 \cup J_2} (t^-_{1j} + t^-_{4j}), \]

s.t. \[ \sum_{i=1}^{k} \lambda_i z_{ij} + t^+_{1j} - t^-_{1j} = c, \quad j \in J_3, \]

\[ \sum_{i=1}^{k} \gamma_i z_{ij} + t^+_{2j} - t^-_{2j} = c - \mu, \quad j \in J_1 \cup J_2, \]

\[ \sum_{i=1}^{k} \gamma_i z_{ij} + t^+_{3j} - t^-_{3j} = c, \quad j \in J_3, \]

\[ \sum_{i=1}^{k} \lambda_i z_{ij} + t^+_{4j} - t^-_{4j} = c - \mu, \quad j \in J_1 \cup J_2, \]

\[ \sum_{i=1}^{k} \lambda_i = 1 - 2p, \quad \sum_{i=1}^{k} \gamma_i = 1 - 2q, \]

\[ p, q \in \{0, 1\}, \]

\[ t^+_{1j}, t^-_{1j}, t^+_{2j}, t^-_{2j}, t^+_{3j}, t^-_{3j} \geq 0, \quad j \in J_3, \]

\[ t^+_{4j}, t^-_{4j}, t^+_{4j}, t^-_{4j} \geq 0, \quad j \in J_1 \cup J_2. \]

Model (5) explain situation group \( G_3 \) with respect to group \((G_2 \cup G_3)\).

Min  \[ \varphi = \sum_{j \in J_1} (s^+_{1j} + s^+_{3j}) + \sum_{j \in J_2 \cup J_3} (s^-_{2j} + s^-_{4j}), \]

s.t. \[ \sum_{i=1}^{k} \alpha_i z_{ij} + s^+_{1j} - s^-_{1j} = d, \quad j \in J_1, \]

\[ \sum_{i=1}^{k} \beta_i z_{ij} + s^+_{2j} - s^-_{2j} = d - \epsilon, \quad j \in J_2 \cup J_3, \]

\[ \sum_{i=1}^{k} \beta_i z_{ij} + s^+_{3j} - s^-_{3j} = d, \quad j \in J_1, \]

\[ \sum_{i=1}^{k} \alpha_i z_{ij} + s^+_{4j} - s^-_{4j} = d - \epsilon, \quad j \in J_2 \cup J_3, \]

\[ \sum_{i=1}^{k} \alpha_i = 1 - 2u, \quad \sum_{i=1}^{k} \beta_i = 1 - 2v, \]

\[ u, v \in \{0, 1\}, \]

\[ s^+_{1j}, s^-_{1j}, s^+_{2j}, s^-_{2j} \geq 0, \quad j \in J_1, \]

\[ s^+_{2j}, s^-_{3j}, s^+_{4j}, s^-_{4j} \geq 0, \quad j \in J_2 \cup J_3. \]
Model (6) identify the existence or non-existence of an overlap between two groups, $G_2$ and $(G_1 \cup G_3)$.

Similarly, model (7) determine overlap measure between $G_3$ and $(G_1 \cup G_2)$.

\[
\begin{align*}
\text{Min} \quad & \psi = \sum_{j \in J_2} (r_{1j}^+ + r_{3j}^+) + \sum_{j \in J_1 \cup J_3} (r_{2j}^- + r_{4j}^-), \\
\text{s.t.} \quad & \sum_{l=1}^{k} \delta_l z_{lj} + r_{1j}^+ - r_{1j}^- = b, \quad j \in J_2, \\
& \sum_{l=1}^{k} \sigma_l z_{lj} + r_{2j}^+ - r_{2j}^- = b - \eta, \quad j \in J_1 \cup J_3, \\
& \sum_{l=1}^{k} \sigma_l z_{lj} + r_{3j}^+ - r_{3j}^- = b, \quad j \in J_2, \\
& \sum_{l=1}^{k} \delta_l z_{lj} + r_{4j}^+ - r_{4j}^- = b - \eta, \quad j \in J_1 \cup J_3, \\
& \sum_{l=1}^{k} \delta_l = 1 - 2l, \quad \sum_{l=1}^{k} \sigma_l = 1 - 2h, \\
& l, h \in \{0,1\}, \\
& r_{1j}^+, r_{1j}^-, r_{2j}^+, r_{2j}^- \geq 0, \quad j \in J_2, \\
& r_{3j}^+, r_{3j}^-, r_{4j}^+, r_{4j}^- \geq 0, \quad j \in J_1 \cup J_3.
\end{align*}
\]

where $J_1 \cup J_2 = \{ j \mid z_j = (x_j, y_j) \in G_1 \cup G_2 \}$, $J_1 \cup J_3 = \{ j \mid z_j = (x_j, y_j) \in G_1 \cup G_3 \}$ and $J_2 \cup J_3 = \{ j \mid z_j = (x_j, y_j) \in G_2 \cup G_3 \}$.

Suppose $\varphi^*$, $\psi^*$ and $\theta^*$ are object function value three models (5), (6) and (7). Also, without loss of generality, it is assumed that $\varphi^* = \text{Min} \{ \varphi^*, \psi^*, \theta^* \}$. Since, model (5) use for discriminant two groups $G_1$ and $G_2 \cup G_3$. If be not $\text{Min} \{ \varphi^*, \psi^*, \theta^* \}$ unique, then we select arbitrary one of them.

Now, we will use under model for discriminant $G_2$ and $G_3$. 

With attention to method introduced for the classification three groups, we can use

**Classification of a new sample:**

Now, \((\alpha^*, \beta^*, \gamma^*, \delta^*, \alpha^*, \beta^*), (\lambda^*, \gamma^*, \alpha^*)\) and \((\tau^*, \rho^*, \alpha^*)\) are the optimal solutions of models (5), (6), (7) and (8), respectively. Toward this end, consider \(m^{th}\) observation newly sampled, whose value is denoted by \(Z_{im}\), can be identified by the following criteria.

(a) If

\[
\sum_{i=1}^{k} \alpha_i^* Z_{im} \geq d^* \quad \text{and} \quad \sum_{i=1}^{k} \beta_i^* Z_{im} \geq d^* \quad \text{then} \quad Z_{im} \in G_1.
\]

(b) If

\[
\sum_{i=1}^{k} \alpha_i^* Z_{im} < d^* \quad \text{and} \quad \sum_{i=1}^{k} \beta_i^* Z_{im} < d^* \quad \text{then} \quad Z_{im} \in G_2 \cup G_3.
\]

Now, should determine location \(Z_{im}\). These processes are given in below:

(b. 1) If

\[
\begin{align*}
Min & \quad \chi = \sum_{j=1}^{j_2} (e_{1j}^+ + e_{3j}^+) + \sum_{j=1}^{j_3} (e_{2j}^- + e_{4j}^-), \\
\text{s.t.} & \quad \sum_{i=1}^{k} \tau_i z_{ij} + e_{1j}^+ - e_{1j}^- = a, \quad j \in J_2, \\
& \quad \sum_{i=1}^{k} \rho_i z_{ij} + e_{3j}^+ - e_{3j}^- = a - \xi, \quad j \in J_3, \\
& \quad \sum_{i=1}^{k} \rho_i z_{ij} + e_{2j}^+ - e_{2j}^- = a, \quad j \in J_2, \\
& \quad \sum_{i=1}^{k} \tau_i z_{ij} + e_{4j}^+ - e_{4j}^- = a - \xi, \quad j \in J_3, \\
& \quad \sum_{i=1}^{k} \tau_i = 1 - 2w, \quad \sum_{i=1}^{k} \rho_i = 1 - 2f, \\
& \quad w, \ f \in \{0,1\}, \quad e_{1j}^+, e_{3j}^+, e_{2j}^+, e_{4j}^+ \geq 0, \quad j \in J_2, \\
& \quad e_{2j}^-, e_{3j}^-, e_{4j}^- \geq 0, \quad j \in J_3.
\end{align*}
\]
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\[ \sum_{i=1}^{k} \tau_{im}^* Z_{im} \geq a^* \text{ and } \sum_{i=1}^{k} \rho_{im}^* Z_{im} \geq a^* \text{ then } Z_{im} \in G_2. \]

(b. 2) If
\[ \sum_{i=1}^{k} \tau_{im}^* Z_{im} < a^* \text{ and } \sum_{i=1}^{k} \rho_{im}^* Z_{im} < a^* \text{ then } Z_{im} \in G_3. \]

(b. 3) If
\[ \sum_{i=1}^{k} \rho_{im}^* Z_{im} \leq a^* < \sum_{i=1}^{k} \tau_{im}^* Z_{im} \text{ or } \sum_{i=1}^{k} \tau_{im}^* Z_{im} \leq a^* < \sum_{i=1}^{k} \rho_{im}^* Z_{im} \text{ then } Z_{im} \in G_2 \cap G_3. \]

Hence, we use model (2) for determine situation newly sampled observation.

(c) If
\[ \sum_{i=1}^{k} \beta_{im}^* Z_{im} \leq d^* < \sum_{i=1}^{k} \alpha_{im}^* Z_{im} \text{ or } \sum_{i=1}^{k} \alpha_{im}^* Z_{im} \leq d^* < \sum_{i=1}^{k} \beta_{im}^* Z_{im} \] is identified fo the new-
sampled \( m \)th observation, then there is an overlap and the observation belongs to \( G_1 \cap (G_2 \cup G_3) \), then the group membership of the observation is determined by the model (2). Therefore, happen either (a) or (b).

4. Example

4.1. An illustrative example

Consider \( G_1 \), \( G_2 \) and \( G_3 \) to be the groups of data with one input and one output as follows (Table 2):
Using proposed newly method for classification of such observations, the optimal solution of models (5), (6) and (7) is obtained as follows:

\[ \alpha_1^* = 0.6, \quad \alpha_2^* = 0.4, \quad \beta_1^* = -0.25, \quad \beta_2^* = 1.25, \quad d^* = 4, \quad \varphi^* = 0, \]
\[ s_1^+ = (0,0,0), \quad s_2^- = (2.6,0,3.4), \]
\[ s_2^+ = (1.75,3.25,0.75,3.5,1,2.5,3,2,0), \quad s_2^- = (0,0,0,0,0,0,0,0), \]
\[ s_3^+ = (0,0,0), \quad s_3^- = (1.75,4.25,0), \]
\[ s_4^+ = (2.6,2.4,1.6,1.8,1,0.8,3,2,0), \quad s_4^- = (0,0,0,0,0,0,0,0,0). \]

and

\[ \delta_1^* = -0.33, \quad \delta_2^* = -0.67, \quad \sigma_1^* = -0.33, \quad \sigma_2^* = -0.67, \quad b^* = -1.67, \quad \psi^* = 4, \]
\[ r_1^+ = (1,0), \quad r_1^- = (0,0), \]
\[ r_2^+ = (0,0,1.33,1.4,67,3.67,4.67,0,0.33,2.33), \quad r_2^- = (0,0.33,0,0,0,0,0,0.67,0), \]
\[ r_3^+ = (1,0), \quad r_3^- = (0,0), \]
\[ r_4^+ = (0,0,1.33,1.4,67,3.67,4.67,0,0.33,2.33), \quad r_4^- = (0,0.33,0,0,0,0,0.67,0). \]

and

\[ \lambda_1^* = -0.5, \quad \lambda_2^* = -0.5, \quad \gamma_1^* = -0.5, \quad \gamma_2^* = -0.5, \quad c^* = -3, \quad \theta^* = 5, \]
\[ t_1^+ = (0,0,0,0,0,0,1), \quad t_1^- = (2.1,5.1,5,1,0,0), \]
\[ t_2^+ = (0,0,3.5,1.5,4), \quad t_2^- = (0.5,1,0,0), \]
\[ t_3^+ = (0,0,0,0,0,1), \quad t_3^- = (2.1,5.1,5,1,0,0), \]
\[ t_4^+ = (0,0,3.5,1.5,4), \quad t_4^- = (0.5,1,0,0). \]

Because \( \varphi^* = \text{Min} \{ \varphi^*, \psi^*, \theta^* \} \) and \( \varphi^* = 0 \), then there is not overlap between two groups \( G_1 \) and \( (G_2 \cup G_3) \) (Fig. 4).

In other words we have:

\[ 0.6x + 0.4y \geq 4, \quad j \in J_1 \quad \text{and} \quad -0.25x + 1.25y < 4, \quad j \in J_2 \cup J_3. \]
By using model (8) is obtained as follows:

$$
\tau^*_1 = -0.25, \quad \tau^*_2 = -0.75, \quad \rho^*_1 = -0.5, \quad \rho^*_2 = -0.5, \quad a^* = -1.5, \quad \chi^* = 4,
$$

$$
e^+_1 = (1.25,0), \quad e^-_1 = (0,0),
$$

$$
e^+_2 = (0,0,0.5,1.5,1.5,2.5), \quad e^-_2 = (0.5,0,0,0,0,0),
$$

$$
e^+_3 = (1,0.5), \quad e^-_3 = (0,0),
$$

$$
e^+_4 = (0,0.25,0.5,1.5,1.5,2.5), \quad e^-_4 = (0.5,0,0.25,0,0,0).
$$

Then it concludes that there is an overlap (Fig. 5).
5. Conclusions

This article has extended DEA-DA for three groups, it identifies the situation with respect to each other. Before introduce new method, we showed by an example that previous methods for separating two groups have drawback, then repair it. As a future extension of this study, we propose cases as follows:

(a) Change type of data.
(b) Change number of groups.
(c) The use from nonlinear models or nonlinear discriminant functions for classification.
(d) Researchers can choice criterion for separate a group from other two groups as the least overlap score in all of stages that should present other algorithm for it.

References


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