Numerical Solution of Algebraic Fuzzy Equations
by Adomian Method

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Abstract
In this paper we introduce an algebraic fuzzy equation of degree $n$ with fuzzy coefficients and crisp variable, and we present an iterative method to find the real roots of such equations, numerically. We present an algorithm to generate a sequence that can be converged to the root of an algebraic fuzzy equation.

Keywords: Fuzzy number, Triangular fuzzy number, Adomian Method

1 Introduction

Since 1965 where Zadeh [7] presented fuzzy logic, up to now, this logic is applicable in many fields of sciences. Sometimes outcome of a system depends on the roots of a fuzzy equation. Some fuzzy equations were checked in [6].

There are some works on fuzzy equations in [5]. A recent work has been done on fuzzy linear systems by Muzzioli and Reynaerts [8]. All these methods compute the roots of an algebraic fuzzy equation analytically, but there are not any analytically solution for algebraic fuzzy equations with degree greater than 3.

In this paper we want to solve the algebraic fuzzy equation of degree $n$, with fuzzy coefficients and crisp variable, numerically.

The structure of this paper is as follows. In Section 2 we introduce an algebraic fuzzy polynomial of degree $n$, with fuzzy coefficients and crisp variables and some needed concepts to solve it. In Section 3 an algorithm is presented to
solve an algebraic equation of degree \( n \), numerically. In Section 4 we introduce an algebraic fuzzy equation and numerically algorithm. There is an examples in this Section.

2 Preliminary Notes

Definition 2.1

A fuzzy number is a fuzzy set like \( u : R \to I = [0, 1] \) which satisfies:
1. \( \tilde{u} \) is upper semi continuous.
2. \( \tilde{u}(x) = 0 \) outside some interval \([c,d]\)
3. there are real number \( a, b : c \leq a \leq b \leq d \) for which:
   1.1 \( \tilde{u}(x) \) is monotonic increasing on \([c,a]\).
   1.2 \( \tilde{u}(x) \) is monotonic decreasing on \([b,d]\).
   1.3 \( \tilde{u}(x) = 1, a \leq x \leq b \)

All of the entire fuzzy number (as given definition 2.1) if denoted by \( \mathbf{F}(R) \). Another definition of fuzzy number defined as follow,

Definition 2.2

A fuzzy number \( \tilde{u} \) is a pair \((u, \overline{u})\) of functions \( u(r), \overline{u}(r); 0 \leq r \leq 1 \) which satisfy the following requirements:
1. \( u(r) \) is monotonically increasing left continuous function.
2. \( \overline{u}(r) \) is monotonically decreasing left continuous function.
3. \( u(r) \leq \overline{u}(r), 0 \leq r \leq 1 \).

A popular fuzzy number is trapezoidal fuzzy number with tolerance interval \([a,b]\), left width \( \alpha \) and right width \( \beta \) if its membership function has the following form and we use the notation:
\[
\tilde{u} = (a, b, \alpha, \beta)
\]
Its parametric form:
\[
\begin{align*}
\underline{u}(r) &= a - (1 - r)\alpha \\
\overline{u}(r) &= b + (1 - r)\beta
\end{align*}
\]
If \( a = b \) then trapezoidal transform to triangular fuzzy number and we denote all of the triangular fuzzy number with \( \mathbf{FT}(R) \).[1]

Definition 2.3

Let \( \tilde{v} = (\underline{v}(r), \overline{v}(r)) \), \( \tilde{u} = (u(r), \overline{u}(r)) \). Some results of applying fuzzy arithmetics on fuzzy numbers \( \tilde{v}, \tilde{u} \) are as follows:[7]

- \( x > 0 : x = (x\underline{v}(r), x\overline{v}(r)) \);
- \( x < 0 : x = (x\overline{v}(r), x\underline{v}(r)) \)
\[\tilde{v} + \tilde{u} = (\bar{v}(r) + \bar{u}(r), \bar{v}(r) + \bar{u}(r))\]
\[\tilde{v} - \tilde{u} = (\bar{v}(r) - \bar{u}(r), \bar{v}(r) - \bar{u}(r))\]

**Definition 2.4**

\[\tilde{P}_n(x)\] is a fuzzy polynomial of degree at most \(n\), if there are some fuzzy numbers \(\tilde{a}_0, ..., \tilde{a}_n\) such that

\[\tilde{P}_n(x) = \sum_{j=0}^{n} \tilde{a}_j x^j\]  \(\text{(1)}\)

### 3 functional iteration

Our aim consists in solving in the set \(R\) the equation

\[a_n x^n + a_{n-1} x^{n-1} + ... + a_0 = 0\]

where \(a_i \in R, 1 \leq i \leq n,\) and \(a_n a_1 a_0 \neq 0\).

It is proved that Adomian method leads to the accurate solution (one of the roots belong to \(R\)).

We want to give some condition on the \(a_i, 1 \leq i \leq n\) which assure the convergence of the decomposition series.

#### 3.1 The Adomian method applied to algebraic equations

Let us consider the equation

\[a_n x^n + a_{n-1} x^{n-1} + ... + a_0 = 0\]  \(\text{(2)}\)

where

\[
\begin{cases}
    n \geq 2 \\
    a_i \in R, 1 \leq i \leq n \\
    a_n \neq 0, a_1 \neq 0, a_0 \neq 0
\end{cases}
\]  \(\text{(3)}\)

which can be transformed in the following (canonical) relationship,

\[x = \left(\frac{-a_n}{a_1}\right) x^n + \left(\frac{-a_{n-1}}{a_1}\right) x^{n-1} + ... + \left(\frac{-a_2}{a_1}\right) x^2 + \left(\frac{-a_0}{a_1}\right)\]  \(\text{(4)}\)

Adomian’s method consists calculating the solution in the series form:

\[x = \sum_{0}^{\infty} x_p\]  \(\text{(5)}\)
and each term \((\frac{-a_i}{a_1})x^i\) \((2 \leq i \leq n)\) is identified to a series of special polynomials:

\[
(\frac{-a_i}{a_1})x^i = f_i(x) = x = \sum_{0}^{\infty} A^i_p 
\]

(6)

These polynomials \(A^i_p\) which depend on the \((p - 1)\) first terms of the series (2.4) are given by:

\[
A^i_p = A^i_p(x_0, x_1, ..., x_p) = \frac{1}{p!} \left. \frac{d^p}{d\lambda^p} \left[ f_i \left( \sum_{k=0}^{p} \lambda^k x_k \right) \right] \right|_{\lambda=0} 
\]

(7)

Putting series (2.5) and (2.4) into (2.3) leads to:

\[
\sum_{p=0}^{\infty} x_p = \sum_{p=0}^{\infty} A^n_p + \sum_{p=0}^{\infty} A^{n-1}_p + \ldots + \sum_{p=0}^{\infty} A^2_p + (\frac{-a_0}{a_1}) 
\]

(8)

We can therefore use the identifications:

\[
\begin{cases} 
  x_0 = (\frac{-a_0}{a_1}) \\
  x_1 = A^n_0 + A^{n-1}_0 + \ldots + A^2_0 \\
  \vdots \\
  x_{p+1} = A^n_p + A^{n-1}_p + \ldots + A^2_p 
\end{cases}
\]

(9)

The exact solution of the equation (2.1) is now entirely determined. But in practice, all the terms of the series cannot be determined. We use the approximation,

\[
\phi_k = \sum_{p=0}^{k-1} x_p 
\]

with

\[
\lim_{k \to +\infty} \phi_k = x 
\]

For study of the convergence theorems must be see the[9].

4 fuzzy equation

In this section we define fuzzy equation then solving with Adomian Method.

Definition 4.1

Fuzzy equation defined as follow,

\[
\tilde{P}_n(x) = \tilde{a} 
\]

(10)
that \( \tilde{a} \) have same LR as \( \tilde{a}_j \) in \( \tilde{P}_n(x) \)

If we denote \( \tilde{a}_j = (a_j, \overline{a}_j) \) and \( \tilde{a} = (a, \overline{a}) \) where

\[
a_j = \sum_{i=0}^{n} b_{ij} r^i, \quad \overline{a}_j = \sum_{i=0}^{n} c_{ij} r^i, \quad a = \sum_{i=0}^{n} h_i r^i, \quad \overline{a} = \sum_{i=0}^{n} k_i r^i
\]

and \( x \) if be positive number then we will have,

\[
\left( \sum_{j=0}^{n} a_j x^j, \sum_{j=0}^{n} \overline{a}_j x^j \right) = (a, \overline{a})
\]

then:

\[
\sum_{j=0}^{n} a_j x^j = a, \quad \sum_{j=0}^{n} \overline{a}_j x^j = \overline{a}
\]

where by replacing (11) into (13) then:

\[
\sum_{j=0}^{n} b_{ij} x^j = h_i, \quad \sum_{j=0}^{n} c_{ij} x^j = k_i \quad i = 0, 1, 2, ..., n
\]

You see that fuzzy equation transform to crisp equation and with attention to section(3.1) we can approximate (14) with Adomian method.

5 Main Results

These are the main results of the paper.

Theorem 5.1

If solutions of the equations (14) were be equal together then the equation (10) have solution.

proof: It is obvious.

Example 5.2

In the follow fuzzy equation the exact solution is \( x^* = 1 \) then with Adomian Method will have:

\[
(r, 2 - r)x^2 + 5(1 + r, 4 - 2r)x + (2 + r, 4 - r) = (3 + 7r, 26 - 12r)
\]

\[
x_0 = 1.2, \quad x_1 = -0.288, \quad x_2 = 0.13824
\]

The solution can be approximated by,

\[
x \approx x_0 + x_1 + x_2 = 1.05
\]
References


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