Decomposition Theorems on Vague Sets

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Abstract

In this paper the decomposition theorems on vague sets are established and the relationships among vague sets, fuzzy sets and classical sets are also revealed. At the same time, their properties are obtained. Finally, an example is given to illustrate their applications.

Keywords: Vague Set, Fuzzy Set, Classical Set, Decomposition Theorem

1. Introduction

In 1965, L.A. Zadeh proposed the concept of fuzzy sets in [16] by extending the range of eigenfunction of classical sets from \(\{0, 1\}\) to closed interval \([0, 1]\), it has been

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applied in various fields [see, e.g., 2, 3, 8-11, 13], but the membership function of a fuzzy set is a single valued function, which can't express either the evidence for \( u \in U \) (\( U \) is a universe of discourse) or the evidence against \( u \in U \). For overcoming the shortage, W.L. Gau and D.J. Buehrer proposed the concept of vague set in [4], which is a generalization of the concept of fuzzy set.

Let \( U \) be a universe of discourse, Gau and Buehrer [4] used a truth-membership \( t_A(u) \) which is a lower bound on the grade of membership of \( u \) derived from the evidence for \( u \in U \), and a false-membership \( f_A(u) \) which is a lower bound on the negation of \( u \) derived from the evidence against \( u \in U \) to characterize the lower bounds on \( U \). For each element in \( U \), the two membership functions correspond two numbers in \([0, 1]\) respectively. Namely

\[
t_A : U \rightarrow [0, 1], \quad f_A : U \rightarrow [0, 1].
\]

The lower bounds are used to create a subinterval on \([0, 1]\), namely

\[
[t_A(u), 1 - f_A(u)]
\]

where

\[
t_A(u) + f_A(u) \leq 1.
\]

A vague sets \( A \) on finite (or infinite) universe of discourse \( U \) is denoted as follows

\[
A = \sum_{i=0}^{n} [t_A(u_i), 1 - f_A(u_i)]/u_i \quad (A = \int_U [t_A(u), 1 - f_A(u)]/u)
\]

If a vague value \([t_A(x), 1 - f_A(x)] = [0.3, 0.7]\), then \( t_A(u) = 0.3, f_A(u) = 0.3\). In model of vote, it can be interpreted as "the number of supporter is three, the number of objector also is three and the number of abstainer is four". This shows that vague sets have stronger capability to describe uncertainty than fuzzy sets. Many authors have focused on vague sets and obtained great deal of achievements in theory and application [see, e.g., 1, 5-7, 12, 14, 15, 17].

Decomposition theorems of fuzzy sets are some of the most important theorems in fuzzy mathematics, which play keys role in fuzzy set theory, since they can convert fuzzy set into classical set. This naturally brings us the following question:

Can we convert a vague set into classical set?

Inspired and motivated by decomposition theorems on fuzzy sets, in this paper we propose and established decomposition theorems on vague sets, which show that vague sets can also be expressed by classical sets. Finally, an example is given to illustrate its application.
2. Cut-set of vague set

Firstly, we introduce the following concepts for obtaining decomposition theorems on vague sets.

**Definition 2.1.** Let $U$ be a universe of discourse, where $V(U)$ is the set of all vague sets on $U$, if $A \in V(U)$, $\alpha \in [0, 1]$, $\beta \in [0, 1]$, and $\alpha \leq \beta$, denote $1 - f_A(u) = \gamma_A(u)$. Then

$$A_{\alpha, \beta} = \{ u \in U | t_A(u) \geq \alpha, \quad \gamma_A(u) \geq \beta \}$$ is said to be a $\alpha$-$\beta$-weakly cut-set of $A$.

$$A^{\alpha, \beta} = \{ u \in U | t_A(u) > \alpha, \quad \gamma_A(u) > \beta \}$$ is said to be a $\alpha$-$\beta$-strongly cut-set of $A$.

**Definition 2.2.** Let $U$ be a universe of discourse, where $V(U)$ consists of all vague sets on $U$, if $A \in V(U)$, $\alpha \in [0, 1]$, $\beta \in [0, 1]$.

$A_{\alpha} = \{ u \in U | t_A(u) \geq \alpha \}$ is said to be a $\alpha$-weakly cut-set of truth-membership of $A$.

$A^{\alpha} = \{ u \in U | t_A(u) > \alpha \}$ is said to be a $\alpha$-strongly cut-set of truth-membership of $A$.

$A_{\beta} = \{ u \in U | \gamma_A(u) \geq \beta \}$ is said to be a $\beta$-weakly cut-set of false-membership of $A$.

$A^{\beta} = \{ u \in U | \gamma_A(u) > \beta \}$ is said to be a $\beta$-strongly cut-set of false-membership of $A$.

From the definitions above the following results are easily proved.

**Theorem 2.1.** Let $A$ be a vague set, for any $\alpha, \beta \in [0, 1]$, the following relationships hold

$$A_{\alpha, \beta} \subseteq A_{\alpha} \subseteq A, \quad A_{\alpha, \beta} \subseteq A_{\beta} \subseteq A, \quad A_{\alpha, \beta} = A_{\alpha} \cap A_{\beta}.$$ where "\subseteq" means "contain".

**Property 2.1.** Let $A \in V(U)$, $\alpha_1, \alpha_2, \beta_1, \beta_2 \in [0, 1]$, if $\alpha_1 > \alpha_2$ and $\beta_1 > \beta_2$, then

$$A_{\alpha_1, \beta_1} \subseteq A_{\alpha_2, \beta_2}, \quad A^{\alpha_1, \beta_1} \subseteq A^{\alpha_2, \beta_2}.$$ 

**Proof.** For any $u \in A_{\alpha_1, \beta_1}$, since

$$t_A(u) \geq \alpha_1 > \alpha_2, \quad \gamma_A(u) \geq \beta_1 > \beta_2,$$

we have $u \in A_{\alpha_2, \beta_2}$. So $A_{\alpha_1, \beta_1} \subseteq A_{\alpha_2, \beta_2}$.

Similarly, we can prove that $A^{\alpha_1, \beta_1} \subseteq A^{\alpha_2, \beta_2}$ also holds. The proof is completed.

**Example 2.1.** A company conduct an annual assessment for its seven units. Seven units are $u_1, u_2, u_3, u_4, u_5, u_6, u_7$, respectively. Let their grades be seven vague values, where $u_1 = [0.9, 1]$, $u_2 = [0.45, 0.75]$, $u_3 = [0.8, 0.95]$, $u_4 = [0.65, 0.8]$, $u_5 = [0.75, 0.95]$, $u_6 = [0.5, 0.6]$, $u_7 = [0.85, 0.9]$, then seven vague values can consist
of a vague sets $A$ as follows: $A = \frac{0.9.1}{u_1} + \frac{0.45.0.75}{u_2} + \frac{0.8.0.95}{u_3} + \frac{0.65.0.8}{u_4} + \frac{0.75.0.95}{u_5} + \frac{0.5.0.6}{u_6} + \frac{0.85.0.9}{u_7}$. Let $U$ be the set of assessment units, then $A \in V(U)$.

Now we need know which ones are excellent in the seven units at the levels $t_A(u) \geq 0.8$ and $\gamma_A(u) \geq 0.8$, and which ones are passed in the seven units at the levels $t_A(u) \geq 0.6$, and $\gamma_A(u) \geq 0.7$. Method of assessment is as follows:

When $t_A(u) \geq 0.8, \gamma_A(u) \geq 0.8$, namely $\alpha = 0.8, \beta = 0.8$, then the set of excellent units is $A_{0.8,0.8} = \{u_1, u_3, u_7\}$ which is a classic sets.

When $t_A(u) \geq 0.6, \gamma_A(u) \geq 0.7$, namely $\alpha = 0.6, \beta = 0.7$, then the set of passed units is $A_{0.6,0.7} = \{u_1, u_3, u_4, u_5, u_7\}$.

In this example, we converted vague sets into classic sets by using cut-sets, and obtain different results at different confidence levels.

### 3. Decomposition theorems on vague sets

Let $A$ be a vague set on universe of discourse $U$, and $A = \int_U \left[ t_A(u), 1 - f_A(u) \right]/u$, then we can obtain two fuzzy sets on $U$ as follows:

$$A_T = \int_U t_A(u)/u, \quad A_\gamma = \int_U \gamma_A(u)/u$$

where $\gamma_A(u) = 1 - f_A(u)$.

**Theorem 3.1 (Decomposition Theorem I)**. Let $U$ be a universe of discourse, where $V(U)$ consists of all vague sets on $U$, if $A \in V(U)$, $\alpha \in [0,1]$, $\beta \in [0,1]$, then

$$A = \{[A_T(u), A_\gamma(u)] | u \in U\}$$

where $A_T = \bigcup_{\alpha \in [0,1]} \alpha(A_T)_\alpha$, $A_\gamma = \bigcup_{\beta \in [0,1]} \beta(A_\gamma)_\beta$, $(\alpha A)(u) = \alpha \land A(u)$, $(\beta A)(u) = \beta \land A(u)$.

**Proof.** Obviously, $(A_T)_\alpha$ is a classical set, and its eigenfunction is

$$\chi(A_T)_\alpha (u) = \begin{cases} 1 & t_A(u) \geq \alpha \\ 0 & t_A(u) < \alpha \end{cases}$$

hence, for any $u \in U$, we have

$$\bigvee_{\alpha \in [0,1]} [\alpha \land \chi(A_T)_\alpha (u)] = \left\{ \bigvee_{0 \leq \alpha \leq t_A(u)} [\alpha \land \chi(A_T)_\alpha (u)] \right\} \bigvee_{t_A(u) < \alpha \leq 1} [\alpha \land \chi(A_T)_\alpha (u)] = \bigvee_{0 \leq \alpha \leq t_A(u)} \alpha = t_A(u),$$

namely

$$A_T = \bigcup_{\alpha \in [0,1]} \alpha(A_T)_\alpha.$$
Similarly, we have

\[ A_\gamma = \bigcup_{\beta \in [0,1]} \beta(A_\gamma)_\beta. \]

The proof is completed.

Decomposition theorem I shows that vague set \( A \) can be divided into two fuzzy sets \( A_T \) and \( A_\gamma \). But \( A_T \) and \( A_\gamma \) can be converted into classic sets by using the Decomposition theorems of fuzzy sets, so vague sets can be divided into classic sets. In other words, vague sets can be expressed by classical sets.

**Example 3.1.** Let \( U = \{u_1, u_2, u_3\}, A \in V(U), A = \{[0.1, 0.6], [0.6, 0.7], [0.2, 0.3]\}, \)

then \( A_T = (0.1, 0.6, 0.2), A_\gamma = (0.6, 0.7, 0.3), (A_T)_{0.1} = \{u_1, u_2, u_3\}, (A_T)_{0.6} = \{u_2\}, \)

\( (A_T)_{0.2} = \{u_2, u_3\} \), hence

\[
A_T = \bigcup_{\alpha \in [0,1]} \alpha(A_T)_\alpha = 0.1(A_T)_{0.1} \bigcup 0.6(A_T)_{0.6} \bigcup 0.2(A_T)_{0.2}
\]

\[
= \left( \frac{0.1}{u_1} + \frac{0.1}{u_2} + \frac{0.1}{u_3} \right) \bigcup \left( \frac{0.6}{u_2} \right) \bigcup \left( \frac{0.2}{u_2} + \frac{0.2}{u_3} \right)
\]

\[
= \frac{0.1}{u_1} + \frac{0.6}{u_2} + \frac{0.2}{u_3}.
\]

Similarly \( A_\gamma = \frac{0.6}{u_1} + \frac{0.7}{u_2} + \frac{0.3}{u_3} \). By using Decomposition Theorem I, we have

\[ A = \{[0.1, 0.6], [0.6, 0.7], [0.2, 0.3]\}. \]

**Corollary 3.1.1:** Let \( A \in V(U) \), for any \( u \in U \), we have

\[ t_A(u) = \vee\{\alpha \in [0,1]|u \in (A_T)_\alpha\}, \]

\[ \gamma_A(u) = \vee\{\beta \in [0,1]|u \in (A_\gamma)_\beta\}. \]

**Theorem 3.2(Decomposition Theorem II).** Let \( U \) be a universe of discourse, where \( V(U) \) consists of all vague sets on \( U \), if \( A \in V(U), \alpha \in [0,1], \beta \in [0,1] \), then

\[ A = \{[A_T(u), A_\gamma(u)]|u \in U}\],

where \( A_T = \bigcup_{\alpha \in [0,1]} \alpha(A_T)^\alpha, A_\gamma = \bigcup_{\beta \in [0,1]} \beta(A_\gamma)^\beta, (\alpha A)(u) = \alpha \land A(u), (\beta A)(u) = \beta \land A(u) \).

**Corollary 3.2.1:** Let \( A \in V(U) \), for any \( u \in U \), we have

\[ t_A(u) = \vee\{\alpha \in [0,1]|u \in (A_T)^\alpha\}, \]
Theorem 3.3 (Decomposition Theorem III). Let $A \in V(U)$, if there exists set-value mappings

$$\gamma \in \{ \beta \in [0, 1] | u \in (A_{\gamma})^{\beta} \}.$$

**Proof:** (1) It follow from Decomposition Theorem 3.1 and 3.2 that

$$\gamma \in A \frac{A_{\gamma}(u)}{\gamma}(A_{\gamma})^\alpha \subseteq H_T(\alpha) \subseteq (A_T)^{\alpha},$$

such that for any $\alpha, \beta \in [0, 1]$, $A_{\alpha, \beta} \subseteq H(\alpha, \beta) \subseteq A_{\alpha, \beta}$, $(A_T)^\alpha \subseteq H_T(\alpha) \subseteq (A_T)^\alpha$, $(A_{\gamma})^\beta \subseteq H_{\gamma}(\beta) \subseteq (A_{\gamma})^\beta$, then

1. $A = \{ [A_T(u), A_{\gamma}(u)] | u \in U \}$, where $A_T = \bigcup_{\alpha \in [0, 1]} \alpha H_T(\alpha), A_{\gamma} = \bigcup_{\beta \in [0, 1]} \beta H_{\gamma}(\beta)$.

2. $\alpha_1 < \alpha_2$ and $\beta_1 \leq \beta_2 \Rightarrow H(\alpha_1, \beta_1) \supseteq H(\alpha_2, \beta_2)$.

3. $A_{\alpha, \beta} = \bigcap_{a < \alpha, b < \beta} H(a, b)$ (if $a \neq 0, \beta \neq 0$), $A^{\alpha, \beta} = \bigcup_{a > \alpha, b > \beta} H(a, b)$ (if $\alpha \neq 1, \beta \neq 1$), where $H(\alpha, \beta) = H_T(\alpha) \cap H_{\gamma}(\beta)$.
In addition
\[
\bigcap_{a<\alpha,b<\beta} H(\alpha, \beta) \subseteq \bigcap_{a<\alpha,b<\beta} A_{\alpha,\beta} = A(\lor_{a<\alpha,b<\beta} \alpha, \lor_{a<\alpha,b<\beta} \beta) = A_{\alpha,\beta} \quad (\alpha \neq 0, \beta \neq 0),
\]
hence
\[
A_{\alpha,\beta} = \bigcap_{a<\alpha,b<\beta} H(a,b).
\]
Similarly, we can also obtain
\[
A^{\alpha,\beta} = \bigcup_{a>\alpha,b>\beta} H(\alpha, \beta), \quad (\alpha \neq 1, \beta \neq 1).
\]

**Corollary 3.3.1:** Let \( A \in V(U) \), we have
\[
t_A(u) = \lor\{\alpha \in [0,1]|u \in H_T(\alpha)\},
\]
\[
\gamma_A(u) = \lor\{\beta \in [0,1]|u \in H_\gamma(\beta)\}.
\]

**Example 3.2.** Let \( U = \{u_1, u_2, u_3, u_4, u_5\}, \)
\[
H(\alpha, \beta) = \begin{cases}
\{u_1, u_2, u_3, u_4, u_5\} & 0 \leq \alpha \leq 0.2, 0 \leq \beta \leq 0.3 \\
\{u_1, u_2, u_3, u_5\} & 0.2 \leq \alpha \leq 0.5, 0.3 < \beta \leq 0.5 \\
\{u_1, u_3, u_5\} & 0.5 \leq \alpha \leq 0.6, 0.5 < \beta \leq 0.7 \\
\{u_1, u_3\} & 0.6 \leq \alpha \leq 0.7, 0.7 < \beta \leq 0.8 \\
\{u_3\} & 0.7 < \alpha \leq 1, 0.8 < \beta \leq 1
\end{cases}
\]

Try to find vague set \( A \).

Owing to the biggest \( \alpha \) value is 0.7 and the biggest \( \beta \) value is 0.8 in all \( H(\alpha, \beta) \) which include \( u_1 \), so \( t_A(u_1) = 0.7, \gamma_A(u_1) = 0.8 \), similarly, we have \( t_A(u_2) = 0.5, \gamma_A(u_2) = 0.5; t_A(u_3) = 1, \gamma_A(u_3) = 1; t_A(u_4) = 0.2, \gamma_A(u_4) = 0.3; t_A(u_5) = 0.6, \gamma_A(u_5) = 0.7 \), then vague set \( A \) can be denoted as follows
\[
A = \frac{[0.7,0.8]}{u_1} + \frac{[0.5,0.5]}{u_2} + \frac{[1,1]}{u_3} + \frac{[0.2,0.3]}{u_4} + \frac{[0.6,0.7]}{u_5}
\]

**Remark:** Decomposition Theorem III shows that vague set \( A \) can be expressed by \( A_{\alpha,\beta} \) or \( A^{\alpha,\beta} \), as well as by more common set-value mapping \( H(\alpha, \beta) \). Particularly, when \( H(\alpha, \beta) = A_{\alpha,\beta} \) or \( H(\alpha, \beta) = A^{\alpha,\beta} \), Decomposition theorem III reduces to Decomposition theorem I or II respectively. So Decomposition theorem I and II are special cases of Decomposition theorem III. This flexibility of \( H(\alpha, \beta) \) makes it may be widely used in practical problems.
4. Conclusion

Vague sets as a generalized version of fuzzy sets have stronger capability to describe uncertainty than fuzzy sets. As a new tool vague sets have been applied in pattern recognition, decision-making analysis and so on. In this paper, Decomposition theorems on vague sets are proposed to reveal the relationships among vague sets, fuzzy sets and classic sets. This work may be useful for the future research on vague sets.

References


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