A Class of Univalent Analytic Functions with Varying Argument of Coefficients Involving Convolution

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Abstract: The object of this paper is to introduce a new class of univalent analytic functions with varying argument of its coefficients, which satisfy a subordinate condition involving convolutions. Coefficient inequality, results on growth and distortion theorems, extreme points, integral means inequality and partial sums of functions belonging to this class are obtained. Some consequences are also discussed.

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1. Introduction

Let $S$ denotes the class of functions of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k,$$

which are analytic and univalent in the open unit disk $\Delta = \{z \in \mathbb{C} : |z| < 1\}$. Denote by $S^\epsilon$ ($\epsilon \in \{0, 1\}$), a subclass of $S$ consisting of functions of the form (1) for which for $a_k \neq 0$, $\arg a_k = \epsilon \pi$, and is known as a class of functions with varying argument of its coefficients. For two functions $f$ and $g$ analytic in $\Delta$, the function $f$ is said to be subordinate to $g$ in $\Delta$, and written as $f(z) \prec g(z)$, if there exists a Schwartz function $w$, which is analytic in $\Delta$, with $w(0) = 0$ and $|w(z)| < 1$ for all $z \in \Delta$, such that $f(z) = g(w(z))$, $z \in \Delta$. Let $P(\alpha)$ denotes the class of analytic functions $p(z) = 1 + p_1 z + p_2 z^2 + \ldots$ satisfying the condition:

$$\Re p(z) > \alpha \ (z \in \Delta, \ 0 \leq \alpha < 1).$$
A convolution (Hadamard product) "∗" of \( f \in S \) of the form (1) and \( g \in S \) of the form:

\[
g(z) = z + \sum_{k=2}^{\infty} b_k z^k
\]

is defined as:

\[
(f \ast g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k = (g \ast f)(z).
\]

Let \( h \in S \) be of the form:

\[
h(z) = z + \sum_{k=2}^{\infty} c_k z^k.
\]

**Definition 1.1.** A functions \( f \in S \) is said to be in class \( P_{g,h;\lambda;A,B} \) if for \(-1 \leq A < B \leq 1\), \(0 \leq \lambda \leq 1\) and for \( g, h \in S \) with \( (f \ast h)(z) \neq 0, z \in \Delta \),

\[
(1 - \lambda) \frac{(f \ast g)(z)}{(f \ast h)(z)} + \lambda \frac{z(f \ast g)'(z)}{(f \ast h)(z)} < \frac{1 + Az}{1 + Bz}.
\]

We denote

\[
P^\epsilon_{g,h;\lambda;A,B} \equiv S^\epsilon \cap P_{g,h;\lambda;A,B}.
\]

Motivated with the work of Silverman [5] and Dziok [2], in this paper, we consider functions with varying argument of its coefficients and define a class of univalent functions satisfying a subordinate condition involving convolutions. Coefficient inequality, results on growth and distortion theorems, extreme points, integral means inequality and partial sums of functions belonging to this class are obtained. Some consequences are also discussed.

## 2. Coefficient Inequality

In this section, we obtain a necessary and sufficient coefficient condition for a function belonging to the class \( P_{g,h;\lambda;A,B} \).

**Theorem 2.1.** Let \( g \in S^{\epsilon+1}, h \in S^\epsilon \) of form (2), (4) respectively with \( B |b_k| [1 - \lambda + k\lambda] + A |c_k| > 0, k \geq 2 \) and \(-1 \leq A < B \leq 1\). If a function \( f \in S \) of the form (1) satisfies the condition

\[
\sum_{k=2}^{\infty} d_k |a_k| \leq B - A,
\]

where \( d_k := |b_k| [1 - \lambda + k\lambda] (1 + B) + |c_k| (1 + A), k \geq 2 \), then \( f \in P_{g,h;\lambda;A,B} \).

**Proof.** To prove \( f \in P_{g,h;\lambda;A,B} \), we have to show

\[
(1 - \lambda) \frac{(f \ast g)(z)}{(f \ast h)(z)} + \lambda \frac{z(f \ast g)'(z)}{(f \ast h)(z)} = \frac{1 + Aw(z)}{1 + Bw(z)}.
\]
or, using corresponding series expansions and \(|w(z)| < 1\ (z \in \Delta)\), we have to show
\[
\left| \sum_{k=2}^{\infty} e^{i \pi} a_k \{ |b_k| (1 - \lambda + k\lambda) + |c_k| \} z^{k-1} \right| < 1
\]
\[
(B - A) - \sum_{k=2}^{\infty} e^{i \pi} a_k \{ B |b_k| (1 - \lambda + k\lambda) + A |c_k| \} z^{k-1}
\]
or, it is sufficient to show
\[
\left| \sum_{k=2}^{\infty} e^{i \pi} a_k \{ |b_k| (1 - \lambda + k\lambda) + |c_k| \} z^{k-1} \right| = \sum_{k=2}^{\infty} e^{i \pi} a_k \{ B |b_k| (1 - \lambda + k\lambda) + A |c_k| \} z^{k-1} < 0.
\]
Since the left hand side of (7)
\[
\leq \sum_{k=2}^{\infty} |a_k| \{ |b_k| (1 - \lambda + k\lambda) + |c_k| \} - (B - A) +
\]
\[
\sum_{k=2}^{\infty} |a_k| \{ B |b_k| (1 - \lambda + k\lambda) + A |c_k| \} = \sum_{k=2}^{\infty} d_k |a_k| - (B - A) < 0,
\]
if (6) holds. This proves the Theorem 2.1.

**Theorem 2.2.** Let \( g \in S^{\varepsilon+1}, \ h \in S^\varepsilon \) of form (2), (4) respectively with \( B |b_k| [1 - \lambda + k\lambda] + A |c_k| > 0, \ k \geq 2 \) and \(-1 \leq A < B \leq 1\). A function \( f \in S^\varepsilon \) of the form (1) belongs to the class \( P^\varepsilon (g, h; \lambda; A, B) \) if and only if the condition (6) holds true.

**Proof.** In view of Theorem 2.1, we only need to show that each function \( f \) of the class \( P^\varepsilon (g, h; \lambda; A, B) \) satisfies the coefficient inequality (6). Let \( f \in P^\varepsilon (g, h; \lambda; A, B) \), then from the definition of the class \( P^\varepsilon (g, h; \lambda; A, B) \), we get
\[
\left| \sum_{k=2}^{\infty} |a_k| \{ |b_k| (1 - \lambda + k\lambda) + |c_k| \} z^{k-1} \right| < 1 \ (z \in \Delta)
\]
or, on putting \( z = re^{i \theta} \ (0 \leq r < 1) \), we get
\[
\sum_{k=2}^{\infty} |a_k| \{ |b_k| (1 - \lambda + k\lambda) + |c_k| \} r^{k-1}
\]
\[
(B - A) - \sum_{k=2}^{\infty} |a_k| \{ B |b_k| (1 - \lambda + k\lambda) + A |c_k| \} r^{k-1}
\]
which, upon letting \( r \to 1^- \), readily yields the assertion (6). □

With the help of Theorem 2.2, following corollaries are obtained:
Corollary 2.3. Let \( g \in S^{\epsilon+1}, h \in S^\epsilon \) of form (2), (4) respectively with \( B |b_k| [1 - \lambda + k\lambda] + A |c_k| > 0, \forall \ k \geq 2, -1 \leq A < B \leq 1 \). If a function \( f \) of the form (1) belongs to the class \( P^\epsilon (g, h; \lambda; A, B) \), then
\[
|a_k| \leq \frac{B - A}{d_k}, \ k \geq 2
\]  
(8)
where \( d_k := |b_k| [1 - \lambda + k\lambda] (1 + B) + |c_k| (1 + A) > 0, \forall \ k \geq 2 \). The result is sharp. The functions \( f_k \) of the form:
\[
f_k = z + \frac{(B - A)e^{i\epsilon\pi}}{d_k} z^k (z \in \Delta, \ k \geq 2)
\]  
(9)
are the extremal functions for \( P^\epsilon (g, h; \lambda; A, B) \) class.

Corollary 2.4. Let \( g \in S^{\epsilon+1}, h \in S^\epsilon \) of form (2), (4) respectively with \( B |b_k| [1 - \lambda + k\lambda] + A |c_k| > 0, -1 \leq A < B \leq 1 \) and \( |b_k|, |c_k| \) are increasing functions of \( k, k \geq 2 \). If \( f \in P^\epsilon (g, h; \lambda; A, B) \), then
\[
\sum_{k=2}^\infty |a_k| \leq \frac{(B - A)}{d_2},
\]
where \( d_2 = |b_2| [1 - \lambda + 2\lambda] (1 + B) + |c_2| (1 + A) \).

Proof. It is verified by the hypothesis that \( d_k := |b_k| [1 - \lambda + k\lambda] (1 + B) + |c_k| (1 + A) \) is an increasing function of \( k, k \geq 2 \). The result is directly obtained by Theorem 2.2. \( \square \)

Corollary 2.5. Let \( g \in S^{\epsilon+1}, h \in S^\epsilon \) of form (2), (4) respectively with \( B |b_k| [1 - \lambda + k\lambda] + A |c_k| > 0 \) and \( -1 \leq A < B \leq 1 \). If \( f \in P^\epsilon (g, h; \lambda; A, B) \) and for \( d_k := |b_k| [1 - \lambda + k\lambda] (1 + B) + |c_k| (1 + A) > 0, 0 < \zeta \leq \frac{\Delta}{k}, \forall \ k \geq 2 \), then
\[
\sum_{k=2}^\infty k |a_k| \leq \frac{(B - A)}{\zeta}.
\]

3. Growth and Distortion Bounds

Theorem 3.1. Let \( g \in S^{\epsilon+1}, h \in S^\epsilon \) of form (2), (4) respectively with \( B |b_k| [1 - \lambda + k\lambda] + A |c_k| > 0, -1 \leq A < B \leq 1 \) and \( |b_k|, |c_k| \) are increasing function of \( k, k \geq 2 \), if \( f \) belongs to the class \( P^\epsilon (g, h; \lambda; A, B) \), then
\[
r - \frac{(B - A)}{d_2} r^2 \leq |f(z)| \leq r + \frac{(B - A)}{d_2} r^2 \quad (|z| = r < 1).
\]  
(10)
where \( d_k := |b_k| [1 - \lambda + k\lambda] (1 + B) + |c_k| (1 + A) \). Also if \( 0 < \zeta \leq \frac{\Delta}{k} \) \( (k \geq 2) \), then
\[
1 - \frac{(B - A)}{\zeta} r \leq |f'(z)| \leq 1 + \frac{(B - A)}{\zeta} r \quad (|z| = r < 1).
\]  
(11)
The result is sharp, with the extremal function given by (9).
Proof. Let \( f \in S^\varepsilon \) be of the form (1). Using Corollary 2.4, we get

\[
|f(z)| \leq |z| + \sum_{k=2}^{\infty} |a_k| |z|^k = r + \sum_{k=2}^{\infty} |a_k| r^k \leq r + \frac{(B - A)}{d_2} r^2
\]

and

\[
|f(z)| \geq |z| - \sum_{k=2}^{\infty} |a_k| |z|^k = r - \sum_{k=2}^{\infty} |a_k| r^k \geq r - \frac{(B - A)}{d_2} r^2,
\]

which prove assertion (10). Again, for \( f \in S^\varepsilon \) be of the form (1) and using Corollary 2.5, we get

\[
|f'(z)| \leq 1 + \sum_{k=2}^{\infty} k |a_k| |z|^{k-1} = 1 + \sum_{k=2}^{\infty} k |a_k| r^{k-1} \leq 1 + \frac{(B - A)}{\zeta} r
\]

and

\[
|f'(z)| \geq 1 - \sum_{k=2}^{\infty} k |a_k| |z|^{k-1} = 1 - \sum_{k=2}^{\infty} k |a_k| r^{k-1} \geq 1 - \frac{(B - A)}{\zeta} r,
\]

which prove assertion (11). \( \Box \)

4. Extreme Points

**Theorem 4.1.** Let \( g \in S^{\varepsilon+1}, h \in S^\varepsilon \) of form (2), (4) respectively with \( B |b_k| [1 - \lambda + k\lambda] + A |c_k| > 0, k \geq 2, -1 \leq A < B \leq 1 \) and \( d_k := |b_k|[1 - \lambda + k\lambda] (1 + B) + |c_k|(1 + A), k \geq 2, \) let \( f \in S^\varepsilon \) and for \( z \in \Delta, \)

\[
f_1(z) = z, f_k(z) = z + \frac{(B - A) e^{i\pi}}{d_k} z^k, k \geq 2.
\]

Then \( f \in P^\varepsilon (g, h; \lambda; A, B) \) if and only if it can be expressed in the form

\[
f(z) = \lambda_1 f_1(z) + \sum_{k=2}^{\infty} \lambda_k f_k(z), z \in \Delta, \text{ where } \lambda_k \geq 0 \text{ and } \lambda_1 = 1 - \sum_{k=2}^{\infty} \lambda_k.
\]

**Proof.** Let

\[
f(z) = \lambda_1 f_1(z) + \sum_{k=2}^{\infty} \lambda_k f_k(z) = \left[1 - \sum_{k=2}^{\infty} \lambda_k\right] z + \sum_{k=2}^{\infty} \lambda_k \left[z + \frac{(B - A) e^{i\pi}}{d_k} z^k\right]
\]

\[
= z + \sum_{k=2}^{\infty} \frac{(B - A) e^{i\pi}}{d_k} \lambda_k z^k.
\]

Then from Theorem 2.2, \( f \in P^\varepsilon (g, h; \lambda; A, B) \), since,

\[
\sum_{k=2}^{\infty} \frac{d_k}{(B - A)} \left|\frac{(B - A) e^{i\pi}}{d_k}\lambda_k\right| = \sum_{k=2}^{\infty} \lambda_k = 1 - \lambda_1 \leq 1.
\]
Conversely, let \( f \in P^\epsilon (g, h; \lambda; A, B) \) of the form (1), and setting

\[
\lambda_k = \frac{d_k}{(B - A)} |a_k|, \ k \geq 2 \text{ and } \lambda_1 = 1 - \sum_{k=2}^\infty \lambda_k,
\]

then

\[
f(z) = z + e^{i\epsilon \pi} \sum_{k=2}^\infty |a_k| z^k = z - \sum_{k=2}^\infty \lambda_k z + \sum_{k=2}^\infty \lambda_k (B - A) e^{i\epsilon \pi} d_k z^k
\]
or

\[
f(z) = \lambda_1 f_1(z) + \sum_{k=2}^\infty \lambda_k f_k(z).
\]

This completes the proof. \( \square \)

5. INTEGRAL MEANS INEQUALITY

Littlewood [4] proved the following subordination theorem. (See also Duren [1])

**Lemma 5.1.** [4] If \( f \) and \( g \) are analytic in \( \Delta \) with \( f \prec g \), then for \( \tau > 0 \) and \( z = re^{i\theta} \) \((0 < r < 1)\),

\[
\int_0^{2\pi} |f(z)|^\tau \, d\theta \leq \int_0^{2\pi} |g(z)|^\tau \, d\theta.
\]

**Theorem 5.2.** Let \( g \in S^{\epsilon+1}, \ h \in S^\epsilon \) of form (2), (4) respectively with \( B |b_k| [1 - \lambda + k\lambda] + A |c_k| > 0, -1 \leq A < B \leq 1, |b_k|, |c_k| \) are increasing function of \( k, k \geq 2, \) and \( f \in P^\epsilon (g, h; \lambda; A, B) \). Also let the functions \( f_l \) be defined by

\[
f_l(z) = z + \frac{(B - A) e^{i\epsilon \pi}}{d_2} z^l, \ l = 2, 3, \ldots \tag{12}
\]

then, for \( \tau > 0 \) and \( z = re^{i\theta} \) \((0 < r < 1)\),

\[
\int_0^{2\pi} |f| \tau \, d\theta \leq \int_0^{2\pi} |f_l| \tau \, d\theta.
\]

**Proof.** Let

\[
f(z) = z + \sum_{k=2}^\infty e^{i\epsilon \pi} |a_k| z^k = z \left( 1 + \sum_{k=2}^\infty e^{i\epsilon \pi} |a_k| z^{k-1} \right)
\]

From (12), we have

\[
f_l(z) = z + \frac{(B - A) e^{i\epsilon \pi}}{d_2} z^l = z \left( 1 + \frac{(B - A) e^{i\epsilon \pi}}{d_2} z^{l-1} \right).
\]
To prove the theorem, it is necessary to show that for \( \tau > 0 \) and \( z = re^{i\theta} \) (0 < \( r < 1 \)),
\[
\int_0^{2\pi} \left| 1 + \sum_{k=2}^{\infty} e^{i\pi} |a_k| z^{k-1} \right|^\tau \, d\theta \leq \int_0^{2\pi} \left| 1 + \frac{(B-A)e^{i\pi}}{d_2} z^{l-1} \right|^\tau \, d\theta.
\]

Thus, by applying Lemma 5.1, it would suffice to show that
\[
1 + \sum_{k=2}^{\infty} e^{i\pi} |a_k| z^{k-1} < 1 + \frac{(B-A)e^{i\pi}}{d_2} z^{l-1}.
\]

If the subordination (13) holds true, then there exist an analytic function \( w \) with \( w(0) = 0 \) and \( |w(z)| < 1 \) such that
\[
1 + \sum_{k=2}^{\infty} e^{i\pi} |a_k| z^{k-1} = 1 + \frac{(B-A)e^{i\pi}}{d_2} \{w(z)\}^{l-1}.
\]

or
\[
\{w(z)\}^{l-1} = \frac{d_2}{(B-A)} \sum_{k=2}^{\infty} |a_k| z^{k-1}
\]

Since, by Corollary 2.4, we have
\[
|w(z)|^{l-1} \leq \frac{d_2}{(B-A)} \sum_{k=2}^{\infty} |a_k| |z|^{k-1} = |z| \frac{d_2}{(B-A)} \sum_{k=2}^{\infty} |a_k| \leq |z| < 1,
\]
which proves the subordination (13). Thus the theorem is proved. \( \square \)

6. Partial Sums

In this section, inequalities involving partial sums for functions belonging to the class \( P^\epsilon (g, h; \lambda; A, B) \) are obtained. Let the partial sums of \( f \in S^\epsilon \) of the form (1) be as follows:
\[
f_1(z) = z \quad \text{and} \quad f_n(z) = z + \sum_{k=2}^{n} e^{i\pi} |a_k| z^k, \quad n \geq 2.
\]

**Theorem 6.1.** Let \( g \in S^{\epsilon+1}, \ h \in S^\epsilon \) of form (2), (4) respectively with \( B|b_k|[1-\lambda+k\lambda]+A|c_k| > 0 \), \( |b_k|, |c_k| \) are increasing function of \( k \) such that \( |b_k|+|c_k| \geq B-A, \ k \geq 2 \), and for \( -1 \leq A < B \leq 1 \), the function \( f(z) \in S^\epsilon \) of the form (1) satisfies
\[
\sum_{k=2}^{\infty} e_k |a_k| \leq 1
\]

where \( e_k := \frac{|b_k|[1-\lambda+k\lambda](1+B)+|c_k|(1+A)}{B-A} \), then \( f \in P^\epsilon (g, h; \lambda; A, B) \) and
\[
\text{Re} \left\{ \frac{f(z)}{f_n(z)} \right\} > 1 - \frac{1}{e_{n+1}}.
\]
\[ \text{Re} \left\{ \frac{f_n(z)}{f(z)} \right\} > \frac{e_{n+1}}{1+e_{n+1}}. \]  \hspace{1cm} (17)

**Proof.** As (15) holds, from Theorem 2.2 \( f \in P^\epsilon (g,h; \lambda; A, B) \). Again, as it is easy to verify by the hypothesis that \( e_{k+1} > e_k > 1, k \geq 2 \),

\[ \sum_{k=2}^{n} |a_k| + e_{n+1} \sum_{k=n+1}^{\infty} |a_k| \leq \sum_{k=2}^{\infty} e_k |a_k| \leq 1. \] \hspace{1cm} (18)

Set

\[ w_1(z) = e_{n+1} \left\{ \frac{f(z)}{f_n(z)} - \left( 1 - \frac{1}{e_{n+1}} \right) \right\} = 1 + \frac{e_{n+1} \sum_{k=n+1}^{\infty} e^{i\epsilon \pi} |a_k| z^{k-1}}{1 + \sum_{k=2}^{n} e^{i\epsilon \pi} |a_k| z^{k-1}} \]

on using (18)

\[ \left| \frac{w_1(z) - 1}{w_1(z) + 1} \right| = \left| \frac{\sum_{k=n+1}^{\infty} e^{i\epsilon \pi} |a_k| z^{k-1}}{2 + 2 \sum_{k=2}^{\infty} e^{i\epsilon \pi} |a_k| z^{k-1} + e_{n+1} \sum_{k=n+1}^{\infty} e^{i\epsilon \pi} |a_k| z^{k-1}} \right| \leq \frac{e_{n+1} \sum_{k=n+1}^{\infty} |a_k|}{2 - 2 \sum_{k=2}^{n} |a_k| - e_{n+1} \sum_{k=n+1}^{\infty} |a_k|} \leq 1, \]

which readily yields the assertion (16) of Theorem 6.1.

Similarly, if

\[ w_2(z) = (1 + e_{n+1}) \left\{ \frac{f_n(z)}{f(z)} - e_{n+1} \right\} = 1 - \frac{(1 + e_{n+1}) \sum_{k=n}^{\infty} e^{i\epsilon \pi} |a_k| z^{k-1}}{1 + \sum_{k=2}^{\infty} e^{i\epsilon \pi} |a_k| z^{k-1}}. \]

Again with the use of (18)

\[ \left| \frac{w_2(z) - 1}{w_2(z) + 1} \right| = \left| \frac{-\sum_{k=n+1}^{\infty} e^{i\epsilon \pi} |a_k| z^{k-1}}{2 + 2 \sum_{k=2}^{\infty} e^{i\epsilon \pi} |a_k| z^{k-1} - (e_{n+1} + 1) \sum_{k=n+1}^{\infty} e^{i\epsilon \pi} |a_k| z^{k-1}} \right| \leq \frac{(e_{n+1} + 1) \sum_{k=n+1}^{\infty} |a_k|}{2 - 2 \sum_{k=2}^{n} |a_k| - (e_{n+1} - 1) \sum_{k=n+1}^{\infty} |a_k|} \leq 1, \]
which proves (17) of Theorem 6.1.

7. SOME CONSEQUENT RESULTS

Corollary 7.1. Let \( g \in S^{\epsilon+1}, h(z) = z, 0 < \lambda \leq 1, -1 < A < 0 \), if a function \( f(z) \in S^\epsilon \) satisfies

\[
(1 - \lambda) \frac{(f * g)(z)}{z} + \lambda (f * g)'(z) < 1 + Az,
\]

then \( \frac{(f * g)(z)}{z} \in P(1 - |A|) \) and \( \frac{zf'(f * g)}{f * g} \in P(\alpha), \alpha = \left( 1 - \frac{(1+|A|)}{\lambda(1-|A|)} \right). \)

Proof. Let \( p(z) := \frac{(f * g)(z)}{z} \), then we get

\[
p(z) + \lambda z f'(z) < 1 + Az.
\]

By a well known Lemma of Hallenbeck and Ruscheweyh [3], we get that \( p(z) \prec 1 + Az \) and hence,

\[
\left| \frac{(f * g)(z)}{z} \right| > 1 - |A|,
\]

which yields

\[
\text{Re} \left\{ \frac{(f * g)(z)}{z} \right\} > 1 - |A|.
\]

Again, with the use of hypothesis, we get

\[
\left| (f * g)'(z) - \frac{(f * g)(z)}{z} \right| < \frac{(1 + |A|)}{\lambda(1 - |A|)} \left| \frac{(f * g)(z)}{z} \right|.
\]

That evidently yields:

\[
\left| \frac{zf'(f * g)}{f * g} - 1 \right| < \frac{(1+|A|)}{\lambda(1-|A|)}.
\]

Therefore \( \text{Re} \left\{ \frac{zf'(f * g)}{f * g} \right\} > \alpha \), where \( \alpha = \left( 1 - \frac{(1+|A|)}{\lambda(1-|A|)} \right) \), which proves the result.

Taking \( \lambda = 0 \) and 1 respectively in Theorem 2.2, we get following results:

Corollary 7.2. Let \( g \in S^{\epsilon+1} \) of form (2), \( h(z) = z \) with \(-1 \leq A < B \leq 1, 0 < B \leq 1\). A function \( f \in S^\epsilon \) satisfies \( \frac{zf'(f * g)}{f * g} \prec \frac{1 + Az}{1 + Bz}, z \in \Delta \), if and only if

\[
\sum_{k=2}^{\infty} (1 + B) |b_k||a_k| \leq B - A.
\]

Corollary 7.3. Let \( g \in S^{\epsilon+1} \) of form (2), \( h(z) = z \) with \(-1 \leq A < B \leq 1, 0 < B \leq 1\). A function \( f \in S^\epsilon \) satisfies \( (f * g)' \prec \frac{1 + Az}{1 + Bz}, z \in \Delta \), if and only if

\[
\sum_{k=2}^{\infty} (1 + B) k |b_k||a_k| \leq B - A.
\]
Taking \( g(z) = \frac{z}{(1-z)^2} \), \( h(z) = z \) in Theorem 2.2, we get following result:

**Corollary 7.4.** Let \(-1 \leq A < B \leq 1\) and \(Bk [1 - \lambda + k\lambda] > 0\). A function \( f \in S^e \) satisfies \( (1 - \lambda) f'(z) + \lambda (zf'(z))' < \frac{1 + A z}{1 + B z} \), \( z \in \Delta \), if and only if
\[
\sum_{k=2}^{\infty} k [1 - \lambda + k\lambda] (1 + B) |a_k| \leq B - A.
\]

Taking \( g(z) = \frac{z}{(1-z)^2} \), \( h(z) = z \) in Theorem 2.2, we get following result:

**Corollary 7.5.** Let \(-1 \leq A < B \leq 1\) and \(B [1 - \lambda + k\lambda] > 0\). A function \( f \in S^e \) satisfies \( (1 - \lambda) \frac{f(z)}{z} + \lambda f'(z) < \frac{1 + A z}{1 + B z} \), \( z \in \Delta \), if and only if
\[
\sum_{k=2}^{\infty} [1 - \lambda + k\lambda] (1 + B) |a_k| \leq B - A.
\]

**Remark 7.6.** Giving some suitable values to \( |b_k| \) in Corollaries 7.1, 7.2 and 7.3, we can get results based on the classes defined by certain well known operators.

**References**


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