Sensitivity Analysis of Partial Derivatives of 

an European Option Pricing Model

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Abstract. In this paper an attempt has been made to study the European call and put options. The mathematical model has been developed using Black-Scholes formula incorporating parameters like time, strike price, volatility, stock price and interest rate. Appropriate boundary conditions have been framed using physical conditions of the market. Also appropriate transformation has been employed to simplify the model in the form of heat equation. Green functions approach has been employed to solve the boundary value problem. The solution is computed using MATLAB and the numerical results obtained have been used to perform sensitivity analysis of options like Delta, Gamma, Theta, Vega and Rho.

Mathematics Subject Classification: 00A71, 35K05, 35K20

Keywords: green function, diffusion equation, European option, partial derivatives. Put-call parity, transformation, sensitivity, black-scholes equation

Introduction:- Thousands of traders and investors now use Black-Scholes formula every day to value stock options in markets throughout the world [16]. This formula was devised by Black-Scholes & published in 1973. Robert Merton (1973) devised another method to derive the same formula that turned out to have very wide applicability; he also generalized the formula in many directions. Black, Merton and Scholes thus laid the foundation for the rapid growth of markets for derivatives in the last years [13]. Their methodology has paved the way for economic valuations in many areas. It has also generated new types of financial instruments and facilitated more efficient risk management in society [12].
Mathematically it is a final value problem for a second order parabolic equation [1]. An option is a contract that admits the owner the right (not the duty) to buy (‘call option’) or to sell (‘put option’) an asset (typically a stock or a parcel of shares of a company) for a prespecified price \( E \) (‘strike price’) by the date \( T \) to receive some payoffs. The basic problem here is to specify a fair price to charge for permitting these rights. A closely related question is how to hedge the risks that arises when selling these options. ‘European’ options can only be exercised at the expiration date \( T \) [3, 11]. For ‘American’ options exercise is permitted at any time until the expiry date. The notion European or American are not meant geographically, they just declare the type of option. For European options the Black–Scholes equation results after a standard transformation in a boundary value problem (that can be solved explicitly for cases with constant coefficients and simple payoffs), and also for American options it results in a free boundary problem for the heat equation [9].

The standard approach for solving the Black–Scholes equation for European options consists in transforming the original equation to a heat equation. In this paper we consider a European call and put option [8,10]. The treatment of a European option is analogous. The Black Scholes equation is a backward–in–time parabolic equation and posed on a time dependent domain.

The price of options depends on the price of the underlying asset and because options are a wasting asset due to their limited lifetimes, option premiums vary quickly with the price and volatility of the underlying asset and time to expiration of the options contract [7, 2, 4]. Several ratios have been developed to measure this change in price with respect to the price or volatility of the underlying, and the effect of time decay. Most of these ratios are represented by Greek letters like as delta, gamma, theta, and rho. These ratios are sometimes used to determine or measure portfolio strategy [5, 17].

**European Call Option:-**

A call option gives the holder the right to exercise his option at maturity time \( T \). To buy the underlying asset at maturity time \( T \) makes sense if the asset price is higher than the exercise price \( S > E \). One can buy the asset for \( E \) and sell it immediately on the market for \( S \). If this is not the case, then the option is worthless [18]. The value of the option is thus known at maturity time, namely it is either zero or \( S - E \), which is the net amount of profit. The value of a European call \( C(S, t) \), satisfying following PDE and final condition is as given below [12]

\[
\frac{\partial C}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + rS \frac{\partial C}{\partial S} - rC = 0
\]  

(1)

\[
C(S, T) = \max(S - E, 0)
\]  

(2)
Therefore the solution of the above equation is.

\[ C(S, t) = SN(d_1) - Ee^{-r(T-t)}N(d_2) \]  

(3)

Where \( N(d) \) is the cumulative normal distribution function defined by

\[ N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}y^2} dy \]

And

\[ d_1 = \frac{\log\left(\frac{S}{E}\right) + \left(r + \frac{1}{2} \sigma^2\right)(T-t)}{\sigma \sqrt{T-t}} \]  

(4)

\[ d_2 = \frac{\log\left(\frac{S}{E}\right) + \left(r - \frac{1}{2} \sigma^2\right)(T-t)}{\sigma \sqrt{T-t}} \]  

(5)

Now we apply following transformation in equation (1) and (2)

\[ S = S' e^{-r(T-t)}, \quad C = C' e^{-r(T-t)}, \quad C' = C'(S', t) \]

We find a new set of equation which is given below.

\[ \frac{\partial C'}{\partial t} + \frac{1}{2} \sigma^2 S'^2 \frac{\partial^2 C'}{\partial S'^2} = 0 \]  

(6)

\[ C'(S', T) = \max(S' - E, 0) \]  

(7)

Above equations (6) along with boundary condition (7) is the European call price for the case that interest rate vanishes i.e \( r = 0 \). We notice that \( C' \) and \( S' \) are risk-neutral expected values of option and asset, given their value at \( t \) [19]. Equation (6) is a backward heat equation with variable coefficient. We consider another transformation.

\[ S' = E^{\frac{x+\frac{1}{2} \sigma^2 \tau}{\sigma^2}}, \quad \tau = T-t, \quad C'(S', t) = EV(x, \tau) \]

On applying this transformation in equations (6) we find a forward heat equation with constant coefficient [19].

\[ \frac{\partial V}{\partial \tau} = \frac{1}{2} \sigma^2 \frac{\partial^2 V}{\partial x^2} \]  

(8)

\[ V(x,0) = \max(e^x - 1, 0) \]  

(9)

It is well-known that the Green’s function of heat equation, satisfying

\[ \frac{\partial G}{\partial \tau} = \frac{1}{2} \sigma^2 \frac{\partial^2 G}{\partial x^2} \]

\[ G(x,0) = \delta(x-x_0) \]
is given by
\[ G(x, \tau; x_0) = \frac{1}{\sqrt{2\sigma^2 \pi \tau}} e^{-\frac{(x-x_0)^2}{2\sigma^2 \tau}} \]

Therefore in order to find the solution of equation (8) and (9) using Green’s formula we get.

\[ V(x, \tau) = \int_{-\infty}^{\infty} \max(e^{x_0} - 1, 0) G(x, \tau; x_0) \, dx_0 \]

\[ = \int_{-\infty}^{\infty} (e^{x_0} - 1) G(x, \tau; x_0) \, dx_0 \]

\[ = \int_{-\infty}^{\infty} (e^{x_0} - 1) \frac{1}{\sqrt{2\sigma^2 \pi \tau}} e^{-\frac{(x-x_0)^2}{2\sigma^2 \tau}} \, dx_0 \]

\[ = \int_{-\infty}^{\infty} e^{x_0} \frac{1}{\sqrt{2\sigma^2 \pi \tau}} e^{-\frac{(x-x_0)^2}{2\sigma^2 \tau}} \, dx_0 - \int_{0}^{\infty} e^{x_0} \frac{1}{\sqrt{2\sigma^2 \pi \tau}} e^{-\frac{(x-x_0)^2}{2\sigma^2 \tau}} \, dx_0 \]

(10)

The second term of equation (10) is-

\[ \int_{0}^{\infty} \frac{1}{\sqrt{2\sigma^2 \pi \tau}} e^{-\frac{(x-x_0)^2}{2\sigma^2 \tau}} \, dx_0 = \int_{-\infty}^{0} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2\sigma^2 \tau}} \, dy = N\left(\frac{x}{\sigma \sqrt{\tau}}\right) \]

(11)

Where

\[ y = \frac{x - x_0}{\sigma \sqrt{\tau}} \]

(12)

Above equation is used at the first equal sign, and the second equal sign is due to the definition of cumulative normal distribution function [15,14]. The first term of (10) can be calculated with a little bit more algebra by the same approach as given below.

\[ \int_{0}^{\infty} e^{x_0} \frac{1}{\sqrt{2\sigma^2 \pi \tau}} e^{-\frac{(x-x_0)^2}{2\sigma^2 \tau}} \, dx_0 = \int_{0}^{\infty} e^{x_0} \frac{1}{\sqrt{2\sigma^2 \pi \tau}} e^{-\frac{(x-x_0)^2}{2\sigma^2 \tau}} \, dx_0 \]

\[ = \int_{0}^{\infty} \frac{1}{\sqrt{2\sigma^2 \pi \tau}} e^{-\frac{x_0^2 - 2x_0(x+\sigma^2 \tau)+x^2}{2\sigma^2 \tau}} \, dx_0 \]

\[ = e^{\frac{-x^2-(x+\sigma^2 \tau)^2}{2\sigma^2 \tau}} \int_{0}^{\infty} \frac{1}{\sqrt{2\sigma^2 \pi \tau}} e^{\frac{[(x+\sigma^2 \tau)-x_0]^2}{2\sigma^2 \tau}} \, dx_0 \]

\[ = e^{\frac{-x^2}{2\sigma^2 \tau}} \int_{0}^{\infty} \frac{1}{\sqrt{2\pi \tau}} e^{\frac{y^2}{2\sigma^2 \tau}} \, dy_1 \]
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where

\[ y_1 = \frac{(x + \sigma^2 \tau)x_0}{\sigma \sqrt{\tau}} \]

\[ = e^{x + \frac{1}{2} \sigma^2 \tau} N \left( \frac{x + \sigma^2 \tau}{\sigma \sqrt{\tau}} \right) \]

By combining the first and second terms of equation (10) we find the solution of equations (8) and (9) as follows.

\[ V(x, \tau) = e^{x + \frac{1}{2} \sigma^2 \tau} N \left( \frac{x + \sigma^2 \tau}{\sigma \sqrt{\tau}} \right) - N \left( \frac{x}{\sigma \sqrt{\tau}} \right) \]

Therefore the solution of equations (6) and (7) is-

\[ C'(S', t) = S' N \left( \frac{\log \left( \frac{S}{E} \right) + \frac{1}{2} \sigma^2 (T - t)}{\sigma \sqrt{T - t}} \right) - E N \left( \frac{\log \left( \frac{S}{E} \right) - \frac{1}{2} \sigma^2 (T - t)}{\sigma \sqrt{T - t}} \right) \]

Finally the solution of equations (1) and (2) is-

\[ C(S, t) = S N \left( \frac{\log \left( \frac{S}{E} \right) + \frac{1}{2} \sigma^2 (T - t)}{\sigma \sqrt{T - t}} \right) - E e^{-r(T-t)} N \left( \frac{\log \left( \frac{S}{E} \right) - \frac{1}{2} \sigma^2 (T - t)}{\sigma \sqrt{T - t}} \right) \]

Put-Call Parity Relation for options:-

Put-call parity defines a relationship between the price of a call option and a put option. Both options have the identical strike price and expiry dates [20]. The assumption is that the options are not exercised before expiration day, which necessarily applies to European options. The Black-Scholes equation is

\[ \frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV = 0 \]

Also, with the additional terminal condition \( V(S, T) \) given, a solution exists and is unique. We observe that it is a linear equation, so the linear combination of any two solutions is again a solution [6]. We know that \( S \) is a solution of the Black-Scholes equation and \( E e^{-r(T-t)} \) is also a solution, so \( S - E \) is a solution. At expiration time \( T \), the solution has value \( S - E \).

Now if \( C(S, t) \) is the value of a call option at stock price \( S \) and time \( t < T \), then \( C(S, t) \) satisfies the Black-Scholes equation, and has terminal value \( \max(S - E, 0) \). If \( P(S, t) \) is the value of a put option at stock price \( S \) and time \( t < T \), then \( P(S, t) \) also satisfies the Black-Scholes equation, and has terminal value \( \max(E - S, 0) \)[13]. Therefore by linearity property- \( C(S, t) - P(S, t) \) is a solution and has terminal value \( C(S, T) - P(S, T) = S - E \). By uniqueness, the solutions must be the same, and

\[ C(S, T) - P(S, T) = S - e^{r(T-t)} E, \]
where T-t=Time until expiration of the option
r= Risk-free rate of interest.
E=Strike Price.
S=Stock Price.
A program has been developed in MATLAB 7.5 and implemented on Intel core 2 Duo.P-IV computer to obtain numerical results and perform sensitivity analysis.

**Numerical Results & Discussion:-**

The value of the parameters [5] used for obtaining numerical results are given in tables I, II, III, IV, V &VI. The sensitivity analysis of the parameters like option Delta, Gamma, Vega, Theta and Rho has been performed as given below.

**Table:-I**

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Symbol</th>
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</thead>
<tbody>
<tr>
<td>Stock price</td>
<td>S</td>
<td>100</td>
</tr>
<tr>
<td>Strike /Exercise Price</td>
<td>E</td>
<td>95</td>
</tr>
<tr>
<td>Interest rate</td>
<td>R</td>
<td>10%</td>
</tr>
<tr>
<td>Option Maturity</td>
<td>T-t</td>
<td>12 months =1yr</td>
</tr>
<tr>
<td>Volatility</td>
<td>σ</td>
<td>.25</td>
</tr>
</tbody>
</table>

**Fig: - 1 Graph between Option Price and Strike Price for put-call parity**

(i) **Sensitivity Analysis of the option:-** The sensitivity analysis is the most important factor in the market analysis and the parameters involved in market analysis are known collectively as the ‘Greeks’ delta, gamma, theta, Vega and Rho. Technically speaking these are partial derivatives of the option pricing model [7]. This means that they measure the change in the calculated option value for a given change in one of the inputs, all other inputs remaining constant. In Fig (1) it is observed that the option price of call option (broken line) decreases with increase in strike price. Further the option price of put option(continuous line) increases as the strike price increases. Thus the two options are just opposite to each other in nature.
Delta \( \frac{\partial C}{\partial S} \): The most important of the ‘Greeks’ is the option delta. This measures the sensitivity of the option value to a given small change in the price of the underlying. A bought call has positive delta. This means that the value of the contract increases as the share price rises \(^8\). This is similar to a short or ‘bear’ position in the underlying. However, delta is not simply a sensitivity value. It also tells an option dealer how much of the underlying stock can be traded if they manage or hedge the risks involved in writing options.

**Table:-2**

<table>
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<th>Parameters</th>
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<tr>
<td>Stock price</td>
<td>S</td>
<td>70</td>
</tr>
<tr>
<td>Strike /Exercise Price</td>
<td>E</td>
<td>35</td>
</tr>
<tr>
<td>Interest rate</td>
<td>R</td>
<td>10%</td>
</tr>
<tr>
<td>Option Maturity</td>
<td>T-t</td>
<td>12 months =1yr</td>
</tr>
<tr>
<td>Volatility</td>
<td>( \sigma )</td>
<td>.25</td>
</tr>
</tbody>
</table>

**Fig:-2 Graph between Option Price and Stock Price and time**

In Fig (2) it is observed that the option price at time \( t=0 \) increases rapidly when stock price is 30 units and option price becomes maximum when stock price is 40 units. But at time \( t=12 \) months (1 yr) option price increases gradually at strike price of 10 units and becomes maximum when strike price is 40 units. The major changes in option price are observed for stock price between 30-40 units. This
indicates that delta is most sensitive for stock price between 30-40 units at time t=0 but at t=12 months the sensitivity is observed beginning from stock price of 20 units.

**Gamma:** \[ \left( \frac{\partial^2 C}{\partial S^2} \right) \] - Gamma is the second partial derivative of option price w.r.t. the underlying. It measures the sensitivity of Gamma to changes in the underlying. Gamma grows arbitrarily large as an at-the-money option approaches expiration (it grows like \(1/\sqrt{\text{time to expiration}}\)), call and put option have the same gamma.

**Table:-3**

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Symbol</th>
<th>Numerical value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stock price</td>
<td>S</td>
<td>70</td>
</tr>
<tr>
<td>Strike /Exercise Price</td>
<td>E</td>
<td>35</td>
</tr>
<tr>
<td>Interest rate</td>
<td>R</td>
<td>10%</td>
</tr>
<tr>
<td>Option Maturity</td>
<td>T-t</td>
<td>12 months =1yr</td>
</tr>
<tr>
<td>Volatility</td>
<td>(\sigma)</td>
<td>.25</td>
</tr>
</tbody>
</table>

**Fig:-3 Graph between Option Price and Stock Price and time**

In Fig (3) it is observed that the option price at time t=0 increases when stock price is 30 units and option price becomes maximum when stock price is 40 units. But at time t=12 months (1yr) option price decreases gradually at strike price of above 40 units and becomes linear when strike price goes up to 70 units. The
option price is most sensitive when stock price varies between 25 to 40 units. This indicates the sensitivity of Gamma.

\[ \text{Vega} \left( \frac{\partial C}{\partial \sigma} \right) : \text{ - Vega is options sensitivity to changes in volatility over the life of option. Calls and puts have the same Vega.} \]

**Table:-4**

<table>
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<tr>
<th>Parameters</th>
<th>Symbol</th>
<th>Numerical value</th>
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</thead>
<tbody>
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<td>S</td>
<td>70</td>
</tr>
<tr>
<td>Strike /Exercise Price</td>
<td>E</td>
<td>35</td>
</tr>
<tr>
<td>Interest rate</td>
<td>R</td>
<td>10%</td>
</tr>
<tr>
<td>Option Maturity</td>
<td>T-t</td>
<td>12 months =1yr</td>
</tr>
<tr>
<td>Volatility</td>
<td>( \sigma )</td>
<td>0.25,0.35,0.45,0.55</td>
</tr>
</tbody>
</table>

**Fig:-4(a)** Graph between Option Price and Stock Price and time when volatility is 0.25
Fig: -4(b) Graph between Option Price and Stock Price and time when volatility is 0.35

Fig: -4 (c) Graph between Option Price and Stock Price and time when volatility is 0.45
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In Fig (4(a) it is observed that the option price at time $t=0$ increases when stock price is 30 units and option price becomes maximum when stock price is 40 units. But in Fig 4 b, c, d this rise in option price is seen earlier than stock price of 30 units. This rise in option price becomes more and more earlier than the stock price of 30 units with the increase in volatility. Also the option price at $t=0$ falls down to around 45 units in fig 4 (a) & (b) and around 50 units in figs 4 (c) & (d). Thus it is observed in all fig 4 (a), (b), (c) & (d) that at time $t=0$ & $t=12$ the changes in option price occur in wider range of stock price with the increase in volatility of the option. At time $t=1$ yr we observe in fig 4 (a), (b), (c) & (d) that rise in option price is gradual initially for low volatility of an option and this rise in option price becomes more & more sharp with the increase in volatility of the option at the same time the fall in option price is sharp at low volatility but becomes more and more gradual with increase in volatility rate. Thus the range of stock price for changes of option price widens with the increase in volatility of an option. Therefore we infer that vega is highly sensitive to changes in volatility.

**Theta** - $\Theta = \frac{\partial C}{\partial T}$: Theta measures an option's sensitivity to a decrease in time to expiration. That is, if option value decreases as expiration approaches, theta is negative. Theta is often close to, but not precisely equal to, negative gamma.
Table:-5

<table>
<thead>
<tr>
<th>Parameters</th>
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</tr>
</thead>
<tbody>
<tr>
<td>Stock price</td>
<td>S</td>
<td>70</td>
</tr>
<tr>
<td>Strike /Exercise Price</td>
<td>E</td>
<td>35</td>
</tr>
<tr>
<td>Interest rate</td>
<td>R</td>
<td>10%</td>
</tr>
<tr>
<td>Option Maturity</td>
<td>T-t</td>
<td>6,12,18 months</td>
</tr>
<tr>
<td>Volatility</td>
<td>( \sigma )</td>
<td>.25</td>
</tr>
</tbody>
</table>

Fig:-5(a) Graph between Option Price and Stock Price and time T=6month

Fig:-5 (b) Graph between Option Price and Stock Price and time T=1yr
Fig:-5(c) Graph between Option Price and Stock Price and time T=18months

In figs 5 (a), (b) & (c) we observe that the option price increases with increase in option maturity time. Further the range of stock price showing sensitivity increases with increases in maturity time.

Rho \( \frac{\partial C}{\partial r} \): - Rho measures options sensitivity to changes in the risk-free rate. In a Black-Scholes model, the risk-free rate is both the discount rate applied to future (risky and risk-free) cash flows, and the stocks expected total return [7].

<table>
<thead>
<tr>
<th>Parameters</th>
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<tbody>
<tr>
<td>Stock price</td>
<td>S</td>
<td>70</td>
</tr>
<tr>
<td>Strike /Exercise Price</td>
<td>E</td>
<td>35</td>
</tr>
<tr>
<td>Interest rate</td>
<td>R</td>
<td>5%</td>
</tr>
<tr>
<td>Option Maturity</td>
<td>T-t</td>
<td>12 months =1yr</td>
</tr>
<tr>
<td>Volatility</td>
<td>( \sigma )</td>
<td>.25</td>
</tr>
</tbody>
</table>
Fig:-6 Graph between Option Price and Stock Price and time when r=5%
In Fig (6) it is observed that the option price at time t=0 increases slowly when stock price is below 30 units and option price becomes maximum when stock price is 40 units. But at time t=12 months (1yr) option price takes maximum value at strike price of 70 units.
Thus the model developed here yields very interesting results regarding partial derivatives of the option. Such models can be developed to study sensitivity of Greeks under various market conditions and the information generated may be of great use to financial experts for making predictions of stock market.

References

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