Impulsive Differential Equations

by using the Euler Method

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Abstract

The theory of impulsive differential equations is emerging as an important area of investigation since such equations appear to represent a natural framework for mathematical modeling of several real phenomena. There have been intensive studies on the qualitative behavior of solutions of the impulsive differential equations. However, many impulsive differential equations cannot be solved analytically or their solving is complicated. In this paper, we represent the algorithm which follows the theory of impulsive differential equations to solve the impulsive differential equations by using the Euler methods. It is clearly shown the impulsive operators $I_k$ that acts at the moments $t_k$ influence the error. Finally, the better convergence result of the numerical solution is given by solving the numerical examples.

Keywords: Differential Equations, Impulsive Differential Equations, Fixed impulse, Impulsive jump, Euler Method

1. Introduction

Many evolution processes are characterized by the fact that at certain moments of time, they experience a change of state abruptly. This is due to short term perturbations whose duration is negligible in comparison with the duration of the
process. It is assumed naturally that those perturbations act instantaneously, in the form of impulses. Thus impulsive differential equations, by means, differential equations involving impulse effects, are seen as a natural description of observed evolution phenomenon of several real world problems. For example, mechanical system with impact, biological phenomenon involving thresholds, bursting rhythm models in medicine and biology, optimal control models in economics, pharmacokinetics and industrial robotics, and many more, do exhibit impulsive effects. Therefore, it is beneficial to study the theory of impulsive differential equations as a well deserved discipline, due to the increase applications of impulsive differential equations in various fields in the future. The pioneer papers in this theory are written by A. D. Myshkis and V. D. Mil'man in 1960's [9].

In spite of its importance, many solutions regarded to impulsive differential equations are done analytically. Some of the famous researchers who presented significance results are V. Lakshmikantham, D. Bainov, P. Simeonov and many others [1, 3, 4, 5, 6, 7, 8, 10]. However, many impulsive differential equations cannot be solved analytically or if done, their solving is very much complicated [11]. Therefore, numerical solutions of impulsive differential equations has to be studied and the results has to be improved. In this paper, the numerical solutions of impulsive differential equations are sought by using the Euler method. The algorithm proposed is interpreted according to the theory of impulsive differential equations written by V. Lakshmikantham et. al [8]. Based on the theory, the better numerical solution of the problem is illustrated in the examples.

2. Impulsive Differential Equations

Basically, impulsive differential equations consist of three components. A continuous-time differential equation, which governs the state of the system between impulses, an impulse equation, which models an impulsive jump, defined by a jump function at the instant an impulse occurs, and a jump criterion, which defines a set of jump events. Mathematically, the equation takes the form,

\[
x'(t) = f(t, x), \quad t \neq t_k, \quad t \in Z \\
\Delta x(t_k) = I_k(x(t_k^-)), \quad k = 1, 2, \ldots, m
\]

where \( Z \) is any real interval, \( f : Z \times R^n \rightarrow R^n \) is a given function, \( I_k : R^n \rightarrow R^n, k = 1, 2, \ldots, m \) and \( \Delta x(t_k) = x(t_k^+ - x(t_k^-)), k = 1, 2, \ldots, m \). The numbers \( t_k \) are called instants (or moments) of impulse, \( I_k \) represents the jump of state at each \( t_k \), whereas \( x(t_k^+) \) and \( x(t_k^-) \) represent the right limit and the left limit, respectively, of the state at \( t = t_k \). The moments of impulse maybe fixed or depended
on the state of the system. In this paper, we will be concerned with fixed moments only.

Moreover, impulsive differential equations can be classified according to these three components.

1. Systems with impulse at fixed moments. The equations have the following form

\[ x'(t) = f(t, x), \quad t \neq t_k, \]
\[ \Delta x = I_k(x), \quad t = t_k \]  \hspace{1cm} (2.2)

where \( t_0 < t_1 < \ldots < t_k < t_{k+1} < \ldots, \quad k \in \mathbb{Z} \) and for \( t = t_k \), \( \Delta x(t_k) = x(t_k^+) - x(t_k^-) \)

where \( x(t_k^+) = \lim_{h \to 0^+} x(t_k + h) \). We surely see that any solution, \( x(t) \) of (2.2) satisfies

(i) \( x'(t) = f(t, x(t)), \quad t \in (t_k, t_{k+1}) \) and

(ii) \( \Delta x(t_k) = I_k(x(t_k)), \quad t = t_k, \quad k = 1, 2, \ldots \)

2. Systems with impulse at variable times. The equations have the following form

\[ x'(t) = f(t, x(t)), \quad t \neq \tau_k(x), \]
\[ \Delta x = I_k(x(t_k)), \quad t = \tau_k(x), \quad k = 1, 2, \ldots \]  \hspace{1cm} (2.3)

where \( \tau_k = \Omega \to R, \quad \Omega \) is the phase space, and \( \tau_k(x) < \tau_{k+1}(x), \quad k \in \mathbb{Z}, \quad x \in \Omega \).

Systems with variable moments of impulsive effect involve more difficult problems than systems with fixed moments of impulsive effect. This is due to the fact that the moments of impulsive effect of (2.3) depend on the solution, i.e. \( t = \tau_k(x) \), for each \( k \). Therefore, solutions at different starting points will have different points of discontinuity.

3. Autonomous systems with impulse. The equations take the form

\[ x'(t) = f(x), \quad x \notin M, \]
\[ \Delta x = I(x), \quad x \in M \]  \hspace{1cm} (2.4)

Let the sets \( M(t) = M, \quad N(t) = N \) and the operator \( A(t) = A \) be independent of \( t \) and let \( A:M \to N \) be defined by \( Ax = x + I(x) \), where \( I: \Omega \to \Omega \). Whenever, any solution
$x(t) = x(t, 0, x_0)$ hits the set $M$ at some time $t$, the operator $A$ instantly transfers the point $x(t) \in M$ into the point $y(t) = x(t) + I(x(t)) \in N$.

Generally, the solutions of the impulsive differential equations are piecewise continuous functions with points of discontinuity at the moments of the impulse effect.

In this paper, we denote $S = \{t_k : k \in \mathbb{Z}\} \subset \mathbb{R}$ where $t_k < t_{k+1}$ for all $k \in \mathbb{Z}$, $t_k \to +\infty$ when $k \to +\infty$ and $t_k \to -\infty$ when $k \to -\infty$. If $\Omega \subset \mathbb{R}$ is any real interval, we suppose that $x(t) = [x_1(t), x_2(t), \ldots, x_n(t)]^T$, is a vector of unknown functions, and

$$f : \Omega \times \mathbb{R}^n \to \mathbb{R}^n,$$

$$f(t, x) = \begin{bmatrix} f_1(t, x_1(t), x_2(t), \ldots, x_n(t)) \\ f_2(t, x_1(t), x_2(t), \ldots, x_n(t)) \\ \vdots \\ f_n(t, x_1(t), x_2(t), \ldots, x_n(t)) \end{bmatrix}$$

is continuous function on every set $[t_k, t_{k+1}] \times \mathbb{R}^n$.

**Definition 2.1**

A system of differential equation of the form

$$\frac{dx}{dt} = f(t, x), \quad t \neq t_k$$

with conditions

$$\Delta x |_{t = t_k} = x(t_k^+) - x(t_k^-) = I_k(x(t_k))$$

where $I_k : \mathbb{R}^n \to \mathbb{R}^n$ are continuous operators, $k = 0, \pm 1, \pm 2, \ldots$ is called impulsive differential equation (IDE) at fixed impulse.
3. Properties of Solutions of IDEs

The problem of existence and uniqueness of the solutions of impulsive differential equations is similar to that of corresponding ordinary differential equations. The continuability of solutions is affected by the nature of the impulsive action.

**Definition 3.1**

A solution of the IDE (2.5) means a piecewise continuous \( x : J \rightarrow R \) with piecewise continuous first derivative such that

1. \( \frac{dx(t)}{dt} = f(t, x(t)), \quad t \neq \tau_k \)
2. \( x(\tau_k^+) - x(\tau_k^-) = I_k(x(\tau_k)), \quad k = 0, \pm 1, \pm 2, ... \)

**Theorem 3.1**

Let the function \( f : R \times \Omega \rightarrow R^n \) be continuous on the sets \((\tau_k, \tau_{k+1}] \times \Omega, \quad k \in \mathbb{Z}\) and for each \( k \in \mathbb{Z} \) and \( x \in \Omega \), suppose there exists the finite limit of \( f(t, x) \) as \((t, y) \rightarrow (\tau_k, x), \quad t > \tau_k \). Then, for each \((t_0, x_0) \in R \times \Omega\), there exists \( T > t_0 \) and a solution \( x : (t_0, T) \rightarrow R^n \) of the problem (2.5) with initial condition \( x(t_0^+) = x_0 \). Furthermore, if the function \( f \) is locally Lipschitz continuous with respect to \( x \) in \( R \times \Omega \), then this solution is unique.

Let \( x(t) \) be the solution of IDE (2.5) with initial condition \( x(t_0^+) = x_0 \), then \( x(t) \) can be represented as

\[
x(t) = \begin{cases} 
x_0 + \int_{t_0}^t f(s, x(s)) \, ds + \sum_{t_l \leq t < t_{l+1}} I_k(x(t_l)) & , \quad t \in \Omega^+
\x_0 + \int_{t_0}^t f(s, x(s)) \, ds - \sum_{t_l \leq t < t_{l+1}} I_k(x(t_l)) & , \quad t \in \Omega^-
\end{cases}
\]

where \( \Omega^+ \) and \( \Omega^- \) are the maximal intervals on which the solution can be continued to the right or to the left of the point \( t = t_0 \) respectively.
**Theorem 3.2**

Assume that \( f \in C[I \times E^n, E^n] \) and satisfies, \( d[f(t,u), f(t,v)] \leq Ld[u,v] \), \( L > 0 \) for \((t,u), (t,v) \in I \times E^n \). Then the initial value problem (2.5) has a unique solution \( u(t) = u(t,t_0,u_0) \) on \( I \).

We also need the following known [8] impulsive differential inequalities result. For this purpose, we let PC denote the class of piecewise continuous functions from \( R \) to \( R \) with discontinuous of the first kind only at \( t = t_k ; k = 1,2,\ldots \) We can now state the needed results.

**Theorem 3.3**

Assume that

(A0) the sequence \( \{t_k\} \) satisfies \( 0 \leq t_0 < t_1 < t_2 < \ldots < t_k < \ldots \) with \( t_k \to \infty \) as \( k \to \infty \);

(A1) \( m \in PC[R_+, R] \) and \( m(t) \) is left continuous at \( t_k, k = 1,2,\ldots \)

(A2) \( \forall k = 1,2,\ldots \) and \( t \geq t_0 \)

\[
\begin{cases}
D^+ m(t) \leq g(t, m(t)), & t \neq t_k, \\
m(t_k^+) \leq \psi_k (m(t_k)), \\
m(t_0) \leq w_0
\end{cases}
\]

where \( g : R^2_+ \to R \) is continuous in \( (t_{k-1}, t_k] \times R_+ \) and for each \( w \in R_+ \),

\[
\lim_{(t,z) \to (t_k^+, w)} g(t, z) = g(t_k^+, w)
\]

exists and \( \psi_k : R_+ \to R \) is non-decreasing;

(A3) \( r(t) = r(t,t_0,w_0) \) is the maximal solution of

\[
\begin{cases}
w'(t) = g(t, w), & t \neq t_k \\
w(t_k^+) = \psi_k (w(t_k)), \\
w(t_0) = w_0 \geq 0
\end{cases}
\]

existing on \([t_0, \infty)\). Then

\[
m(t) \leq r(t), \ t \geq t_0
\]

We recall that the maximal solution \( r(t) \) of (3.2) means the following
Impulsive differential equations

\[ \begin{align*}
\begin{bmatrix}
    r_0(t, t_{0}, w_0), & t \in [t_0, t_1] \\
    r_i(t, t_i, r_{i-1}(t^+_i)), & t \in (t_i, t_2] \\
    & \vdots \\
    r_k(t, t_k, r_{k-1}(t^+_k)) & t \in (t_k, t_{k+1}] \\
    & \vdots
\end{bmatrix}
\end{align*} \]

(3.4)

where each \( r_i(t, t_i, r_{i-1}(t^+_i)) \) is the maximal solution of (2.3) on the interval \((t_i, t_{i+1}]\) for each \( i = 1, 2, \ldots \), and \( r_{i-1}(t^+_i) = \psi_i(r_{i-1}(t_i, t_{i-1}, r_{i-2}(t^+_{i-2}))) \).

4. Algorithm

Suppose the IDE (2.5) with start condition \( x_0 = x(t_0) \) and the impulsive operators \( I_k, (k \in \mathbb{Z}) \) is given. The impulsive operators act at the moments of jump happen, \( t_k \) for all \( k \in \mathbb{Z} \) which are described by the quadruple matrices of dimensions \( n \times n \).

The numerical algorithm is different only at the jump point, where we have to apply the operators concern with the particular point. Other than that, we employ the usual manners to solve the IDE using the numerical method chosen.

1. At the moment, \( t = t_0 \), we apply the numerical method to the function with the initial values \( x = x_0 \). The algorithm applies until the first jump point, by now we will get the values for the left limit.
2. At the jump point, \( t = t_k \), we apply the operators to find the values of the right limit.
3. The first step is repeated until the next jump point.
4. Then, we apply the operators concern with the particular jump point.
5. The above steps are repeated and the iteration stop until we encounter with the desired values that has to be found, let assume, \( t_s \) where \( t_s > t_0 \). Notice that we only have the approximate values of the function at \( t = t_s \).

5. Numerical Examples

Example 1 Consider the IDE given in [2]:
\[
\frac{dx}{dt} = f(t, x), \quad t \neq t_k
\]
\[
\Delta x(t_k) = I_k(x(t_k)), \quad t = t_k, \quad k = 1, 2, \ldots
\]
\[
x_0 = x(t_0)
\]
and \( t_0 = 0.0 \),

\[
x = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \quad x_0 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}
\]

\[
f(t, x) = \begin{bmatrix} 0.1666666x_1 + 0.1666666x_2 + 0.1666666 \\ -0.1666666x_1 - 0.1666666x_2 + 0.5833333 \end{bmatrix}
\]

The impulsive operators act at \( t_1 = 1.0 \) and \( t_2 = 2.0 \) are given as follows:

\[
I_1 = \begin{bmatrix} 0.25 & 0.25 \\ 0.0 & -1.0 \end{bmatrix}, \quad I_2 = \begin{bmatrix} 3.0 & 4.0 \\ 0.0 & -1.0 \end{bmatrix}
\]

Here, we wish to approximate the value of \( t_s = 2.3 \).

We applied the algorithm by using the Euler method,

\[
x_{i+1} = x_i + hf(t, x_i)
\]

where \( i \in Z \) is the index of iteration, and \( h \) is the step size of each iteration. Here, the step size \( h = 0.1 \). Then we compared the results obtained by using the analytical expression that is the solution of IDE (5.1).

\[
x = \begin{cases} 
  x_1(t) = 0.0625t^2 - 1.0, & t \in (-\infty, 1) \\
  x_2(t) = -0.0625t^2 + 0.75t, & t \in (-\infty, 1) \\
  x_1(t) = 0.0625t^2 - 1.0625, & t \in [1, 2) \\
  x_2(t) = -0.0625t^2 + 0.75t - 0.6875, & t \in [1, 2) \\
  x_1(t) = 0.0625t^2 - 1.25, & t \in [2, \infty) \\
  x_2(t) = -0.0625t^2 + 0.75t - 1.25, & t \in [2, \infty) 
\end{cases}
\]

The numerical values of solution are obtained by using the Matlab programming and the results of the Euler method as well as the analytical expression are compared in Table 1. Absolute errors are also given by taking the approximate numerical values of \( t_s \). Let assume \( t_s = 2.3 \).
Table 1: The approximate values at $t = 2.3$

<table>
<thead>
<tr>
<th>$t_k$</th>
<th>Euler $x_1(t_k)$</th>
<th>Analytical $x_1(t_k)$</th>
<th>Euler $x_2(t_k)$</th>
<th>Analytical $x_2(t_k)$</th>
</tr>
</thead>
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<td>-1.0</td>
<td>0</td>
<td>0.0</td>
</tr>
<tr>
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<td>0.0744</td>
</tr>
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</tr>
<tr>
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<td>-0.9944</td>
<td>0.2212</td>
<td>0.2194</td>
</tr>
<tr>
<td>0.4</td>
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<td>-0.9900</td>
<td>0.2925</td>
<td>0.2900</td>
</tr>
<tr>
<td>0.5</td>
<td>-0.9875</td>
<td>-0.9844</td>
<td>0.3625</td>
<td>0.3594</td>
</tr>
<tr>
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<tr>
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<td>0.4987</td>
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</tr>
<tr>
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<td>0.5600</td>
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<tr>
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<tr>
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<td>0.1444</td>
</tr>
</tbody>
</table>

error at $t = 2.3$ | 0.1032 | 0.0781
Figure 1 The approximate values of $x_1$ versus time, $t$ between the euler and analytical method for Example 1

Figure 2 The approximate values of $x_2$ versus time, $t$ between the euler and analytical method for Example 1
Example 2  
Consider the IDE,

\[
\frac{dx}{dt} = f(t, x), \quad t \neq t_k
\]

\[
\Delta x(t_k) = I_k(x(t_k)), \quad t = t_k, \quad k = 1, 2, ...
\]

\[
x_0 = x(t_0)
\]

and \( t_0 = 0.0 \),

\[
x = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \quad x_0 = \begin{bmatrix} 2 \\ -2 \end{bmatrix},
\]

\[
f(t, x) = \begin{bmatrix} x_1 - 2x_2 \\ -x_1 \end{bmatrix}
\]

At \( k = 1 \), \( I_k = \begin{bmatrix} 1.0000 & 3.6565 \\ 0.0000 & -0.8020 \end{bmatrix} \). Here, we wish to determine the approximate value at \( t = 1 \).

For that purpose we apply the Euler method (5.3). Then we compared the results obtained by using the analytical expression that is the solution of IDE (5.1).

\[
x = \begin{cases} 
  x_1(t) = -\frac{2}{3} e^{-t} + \frac{8}{3} e^{3t} & t \in (0, 1] \\
  x_2(t) = -\frac{2}{3} e^{-t} - \frac{4}{3} e^{3t} & \\
  x_1(t) = -\frac{2}{3} e^{-t} + \frac{8}{3} e^{3t} - 17.4619 & t \in (1, 2] \\
  x_2(t) = -\frac{2}{3} e^{-t} - \frac{4}{3} e^{3t} + 8.0980 & 
\end{cases}
\]
Table 2: The approximate values at $t = 1$.

<table>
<thead>
<tr>
<th>$t_k$</th>
<th>Euler $x_1(t_k)$</th>
<th>Analytical $x_1(t_k)$</th>
<th>Euler $x_2(t_k)$</th>
<th>Analytical $x_2(t_k)$</th>
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Figure 3: The approximate values of $x_1$ versus time, $t$ between the euler and analytical method for Example 2.
6. Concluding remarks

The accuracy of the results can be improved by investigating the solutions of the other numerical methods. We proposed a general numerical procedure for treating the impulsive differential equations at fixed moments. We interpreted the numerical algorithm following the theory of impulsive differential equations and started with the Euler method. Although it is not the most accurate methods we will study, it is by far the simplest, and analyzing Euler’s method in detail will hopefully carries over to the other methods with higher accuracy without a lot of difficulty. Solving the impulsive differential equations numerically has not been done by many researchers. Therefore, many studies have to be done in order to enhance and verify the existing results. In this paper, we have shown better results with diagrams to the convergence and the behavior of the solutions.

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References


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