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Scattering of SH Waves in Fluid Saturated Medium

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Abstract

The deviation in SH-waves' velocities is expected once the saturation degree in the medium is asymmetrical. Hence, SH-waves' propagation in the porous medium saturated with asymmetry fluid density is studied for the diffusive profiles. SH-waves are propagated in similar directions and also opposite directions with the mediums fall into two distinctive groups: insoluble as well as soluble mediums. In similar direction of propagation, low density fluid revokes the diffusive characteristics while high density fluid promotes diffusive attribute. However, the diffusive SH-waves are as well found in the medium saturated with low density fluid when the fluid is asymmetrical in density. In the case of opposite direction of propagation, the recurring SH-waves are found in the medium saturated with low and asymmetry density fluid.

Mathematics Subject Classification: 86A15, 74J30, 86A17

Keywords: Diffusion, Scattering waves, Saturated-medium

1 Introduction

S-wave is referred to shear wave when an earthquake is discussed and S-wave cannot move in liquid because the liquid is not rigid enough to transmit an S wave. S-waves are divided into horizontal directed (SH) waves and vertical directed (SV) waves. Although SH-waves cannot move in liquid yet these SH-waves show the effects on horizontal and vertical motions in a layered soil saturated with water [1].

Biot's theory has been used in showing the elastic wave propagation in a fluid saturated porous solid [2]. The attenuation and dispersion of compressional waves in fluid-filled porous rocks with partial gas saturation was studied in accordance to Biot's theory [3]. The studies of fluid saturated method were extended through numerical modelling. Finite element modelling of the

effective elastic properties of partially saturated rocks was studied [4]. Subsequently, this research will exploit the elastic wave equation to study the diffusion of SV-waves.

The SH-wave has been studied for the diffusion in saturated medium [6]. Immediately, the SV-waves' diffusion is explored for the propagation characteristic in porous medium [5].

In investigating the deposited sediment at the reservoir behind the dam, the SV-wave was used for studying the dynamic pressure [7]. The sensitivity of SV-wave scattering to surface breaking cracks was studied and this inspiration has brought to this research [8]. Two setups are done for SH-waves propagation to relate to the scattering of SH-waves (see figure 1(a) and 1(b)). In this paper, the aim is to obtain the diffusion profiles depicted by SH-wave's propagation in the saturated mediums. The derivation of SH-waves will be shown before the analytical discussion.

2 Methodology

The governing equations:

$$\mu\nabla^2 u + (\lambda+\mu)\,\nabla\,(\nabla u) + \rho F = \rho\frac{\partial^2 u}{\partial t^2} \tag{1}$$

Boundary conditions for $z = 0$:

$$\left.\begin{array}{l} \sigma_z \equiv (\lambda+\mu)\,\frac{\partial u_z}{\partial z} + \lambda\left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y}\right) = 0 \\ \tau_{xz} \equiv \mu\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right) = 0 \\ \tau_{yz} \equiv \mu\left(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}\right) = 0 \end{array}\right\} \tag{2}$$

Initial conditions for $t = 0$:

$$u\,(x,y,z) = 0 \tag{3}$$

$$\frac{\partial u_x}{\partial t} = \frac{\partial u_y}{\partial t} = \frac{\partial u_z}{\partial t} = 0 \tag{4}$$

In linear elasticity for isotropic medium, λ and μ denote the Lame parameters for the stress σ_z, τ_{xz}, τ_{yz} and the displacements u_x, u_y and u_z are continuous everywhere. F is the body force in the direction of x, y, z respectively and ρ is the density.

Provided that ω is the angular velocity or frequency, k is the wavenumber, c is the wave velocity along with the dispersion relation $\omega = ck$ the group velocity of a wave is the velocity with the overall shape of the wave's amplitudes which is also known as envelope of the waves. In other words, the phase velocity is the average velocity of the components, given by $V_P = \omega/k$. The group velocity is velocity of the envelope, given by $V_g = d\omega/dk$ [9].

When this is applied to the problem of SH-wave's propagation in fluid saturated medium, the group velocity reflects the apparent velocity of the surface displacement or the overall shape of the SH- wave's amplitude at the boundary of the medium. The envelope is formed by the phase velocity of SH-waves. In this research, the apparent velocity V_{app} or surface displacement velocity is measured along the boundary of the similar density medium in accordance to Snell law:

$$\frac{V_{app}}{\sin f} = \frac{c}{\sin e} \tag{5}$$

e is the incident angle and f is the refraction angle made by the P- and S-waves. However, our objective is to study S-waves in horizontal direction only. For the case of fluid-saturated medium, there exists variation in density within the medium [7, 12]. There exists slowness induced by refraction for fluid-saturated medium [10, 11] i.e. the apparent velocity of the displacement at the boundary is slower than phase velocity of the wave in the medium that reads:

$$V_{app} < c \tag{6}$$

In this study, the elastic wave equation will be solved so as to obtain the SH-wave's displacements in the different types of medium that satisfied (5) and (6). By using the divergence operator,

$$\nabla^2 u = \nabla (\nabla u) - \nabla \times (\nabla \times u) \tag{7}$$

equation (1) is reduced to

$$\alpha^2 \nabla (\nabla \cdot u) - \beta^2 \nabla \times \nabla \times u + F = \frac{\partial^2 u}{\partial t^2} \tag{8}$$

$$\alpha = \sqrt{\frac{\lambda + 2\mu}{\rho}}, \ \beta = \sqrt{\frac{\mu}{\rho}} \tag{9}$$

α and β are the velocities for the P-wave and S-wave and equation (8) is the elastic wave equation [13]. Hence, this modeling only valid for elastic medium and it is necessary to reduce the right hand side (RHS) of (8) by letting

$$u = \exp [i (kx - \omega t)] \tag{10}$$

By inserting second order derivative of (10) into RHS of (8), the equation reads:

$$\alpha^2 \nabla (\nabla \cdot u) - \beta^2 \nabla \times \nabla \times u + F = -\omega^2 u \tag{11}$$

Since elastic wave consists of irrotational by P-wave and solenoid by S-wave [13], the displacement shall be written as:

$$u = u_P + u_S \tag{12}$$

For irrotational P-waves, the vorticity $\nabla \times u_P = 0$. Thus, the relation (12) reads:

$$\nabla u_P \neq 0,\ \ \nabla \times u_P = 0,\ \ \nabla u_S = 0,\ \ \nabla \times u_S \neq 0 \tag{13}$$

By inserting (12) and (13) into (11), the equation yields:

$$\alpha^2 \left(\nabla^2 u_P + {k_P}^2 u_P\right) + \beta^2 \left(\nabla^2 u_S + {k_S}^2 u_S\right) = 0 \tag{14}$$

$$\omega_P = \frac{\alpha}{k_P},\ \ \omega_S = \frac{\beta}{k_S},\ \ F = 0 \tag{15}$$

In the following, the equations (16) are the Helmholtz equations [14] for P- and S-waves:

$$\nabla^2 u_P + {k_P}^2 u_P = 0,\ \ \nabla^2 u_S + {k_S}^2 u_S = 0 \tag{16}$$

Here, the Helmholtz equations are solved by utilizing Hansen vector [10] that gives:

$$u_P = A\left(la_x + na_z\right) \exp\left[i\omega\left(t - \frac{lx + nz}{\alpha}\right)\right] \tag{17}$$

$$u_{SV} = B\left(-na_x + la_z\right) \exp\left[i\omega\left(t - \frac{lx + nz}{\beta}\right)\right] \tag{18}$$

$$u_{SH} = Ca_y \exp\left[i\omega\left(t - \frac{lx + my}{\beta}\right)\right] \tag{19}$$

a_x, a_y and a_z are the unit vectors while l, m and n are the vector components. Only equation (19) will be considered in this paper since the aim is to study the SH-wave only. SH-wave vanishes when the depth go infinity as illustrated in figure 1(a) and 1(b). Thus, the modified equation (19) or SH-wave on which the amplitude reduces with depth is:

$$u_{SH} = C\left(a_y\right) \exp\left[ik\left(ct - x - \eta_\beta y\right)\right],\ \ \eta_\beta = \sqrt{\frac{c^2}{\beta^2} - 1} \tag{20}$$

η_β is always positive. The velocity of the SH-wave, β is measured at the boundary of the medium or $z = 0$ and it is similar to the apparent velocity V_{app} of the surface displacement. c is the SH-wave velocity in the medium.

For the displacement normal to the propagation direction by the means of quantity η_β and amplitude C, the equation (20) for horizontal direction of displacements with amplitude reduces with depth reads:

$$u_{SH} = C\left(a_y\right) \exp\left(-ik\eta_\beta y\right) \exp\left[ik\left(ct - x\right)\right],\ \ \omega_1 = \eta_\beta k \tag{21}$$

Here, quantity η_β refers to the SH-wave velocities in the medium and the fluid that gives the frequency $\omega_1 = \eta_\beta k$ and the vector a_y shows the displacement

is polarized between $x - y$ directions. The polarization of equation (21) gives the frequency equation that reads [13]:

$$u_{SH} = C_y \exp\left(-ik\eta_\beta y\right) \exp\left[ik\left(ct - x\right)\right] \tag{22}$$

Consequently, the SH-waves propagate in the x-direction but the amplitude or displacement is in y-direction, C_y. From wave terminology, the term η_β in equation (22) will show that the harmonic wave is diffusive if the term $\eta_\beta k$ is in complex [9]. The diffusive waves are associated with attenuation of the amplitudes with the time due to certain dissipation mechanisms present in the system.

Next, the roles of the refracted velocity η_β will be shown. For the fluid-saturated medium, there comes the slowness induced by refraction [10, 11]. The envelope velocity at medium surface is different with wave velocity in the medium after the slowness or $V_{app} \neq c$. For a particular case, the SH-wave velocity in the medium is greater than the envelope's velocity that yields:

$$c > \beta, \;\; \beta = V_{app} \tag{23}$$

Here, we propose the relation for another two types of medium conditions such that:

$$c = \beta, \;\; \beta = V_{app} \tag{24}$$

$$c < \beta, \;\; \beta = V_{app} \tag{25}$$

Here, we presume that the relations (23) and (25) are meant for the insoluble medium such that the variation of velocities c and β is remarkable. When the velocity c is similar to β, we deduce this will explain the soluble medium such that the fluid mixes well with the medium to give similar velocity.

Yang and Tadanobu [1] and Kahraman [15] show that the high density medium promotes high wave's velocity. Hence, the relation (23) will only be presented when the low density fluid is saturated in the insoluble medium; the low density fluid will reshuffle the ray velocity or reduce the SH-wave velocity. Eventually, the relation (25) is meant for the high density fluid saturated in the insoluble medium.

The detail explanations about the tie between medium's solubility and the fluid's density will be discussed next for showing the vital roles play by the relations (23) to (25) especially the displacements characteristic. When condition (25) is applied to the quantity η_β in equation (22), a complex solution will be obtained for η_β that reads:

$$\eta_\beta \rightarrow i\eta_\beta \tag{26}$$

This indicates an amendment is required for equation (22) to give (27). Hence, the SH-wave's displacements for three types of mediums read:

$$u_{SH} = C_y \exp\left(k\eta_\beta y\right) \exp\left[ik\left(ct - x\right)\right] \;\; \text{for} \;\; c < \beta \tag{27}$$

$$u_{SH} = C_y \exp\left(-ik\eta_\beta y\right) \exp\left[ik\left(ct - x\right)\right] \text{ for } c = \beta \tag{28}$$

$$u_{SH} = C_y \exp\left(-ik\eta_\beta y\right) \exp\left[ik\left(ct - x\right)\right] \text{ for } c > \beta \tag{29}$$

Equations (27) to (29) show that the SH-waves propagation in -direction while the diffusion is normal to x-direction. The equations (27) to (29) are arranged such that the density of the saturated fluid reduces from $c < \beta$ to $c > \beta$.

Next, the equations (27), and (29) will be amended to depict the circumstances in figures 1(a) and 1(b). Two different SH-waves with different diffusive characteristics which are propagated in similar direction reads:

$$u_{SH} = \left[C_y \exp\left(-ik\eta_{\beta 1} y\right) + C_y \exp\left(-ik\eta_{\beta 2} y\right)\right] \exp\left[ik\left(ct - x\right)\right], \ c > \beta \tag{30}$$

$$u_{SH} = \left[C_y \exp\left(k\eta_{\beta 1} y\right) + C_y \exp\left(k\eta_{\beta 2} y\right)\right] \exp\left[ik\left(ct - x\right)\right], \ c < \beta \tag{31}$$

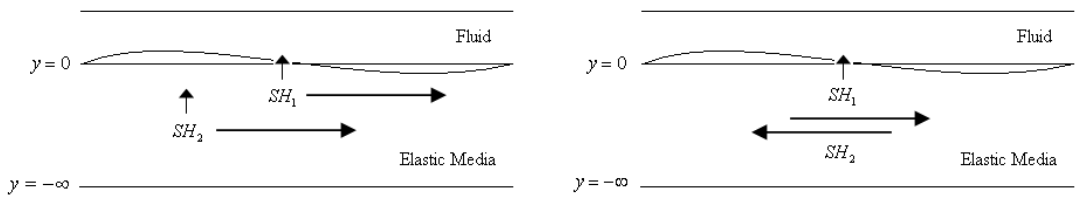
$$\eta_{\beta 1} = \sqrt{\frac{c_1^2}{\beta_1^2} - 1}, \ \eta_{\beta 2} = \sqrt{\frac{c_2^2}{\beta_2^2} - 1}$$

For the cases of opposite direction of propagation, the equations read:

$$u_{SH} = \left[C_y \exp\left(ik\eta_{\beta 1} y\right) + C_y \exp\left(ik\eta_{\beta 2} y\right)\right] \exp\left[\pm ik\left(ct - x\right)\right], \ c > \beta \tag{32}$$

$$u_{SH} = \left[C_y \exp\left(k\eta_{\beta 1} y\right) + C_y \exp\left(k\eta_{\beta 2} y\right)\right] \exp\left[\pm ik\left(ct - x\right)\right], \ c < \beta \tag{33}$$

$$\eta_{\beta 1} = \sqrt{\frac{c_1^2}{\beta_1^2} - 1}, \ \eta_{\beta 2} = \sqrt{\frac{c_2^2}{\beta_2^2} - 1}$$



(a) (a)

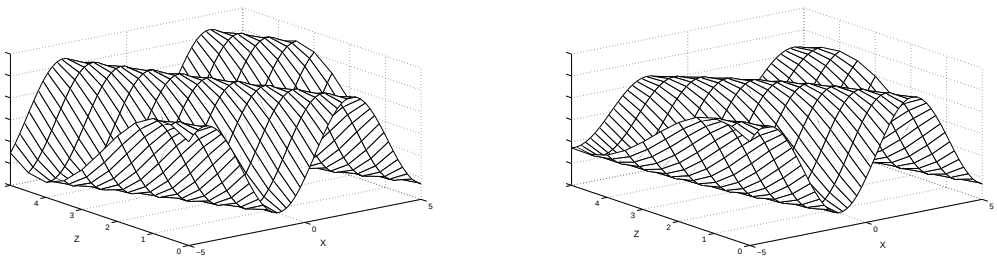
Figure 1: The propagation of SH waves. (a) The SH-waves propagation in similar direction. (b)The SH-waves propagate in opposite direction

3 Results

Equations (30), (31), (32) and (33) are plotted in accordance to Table 1 for showing the influence of different velocities towards the diffusive characteristics or SH-waves scattering.

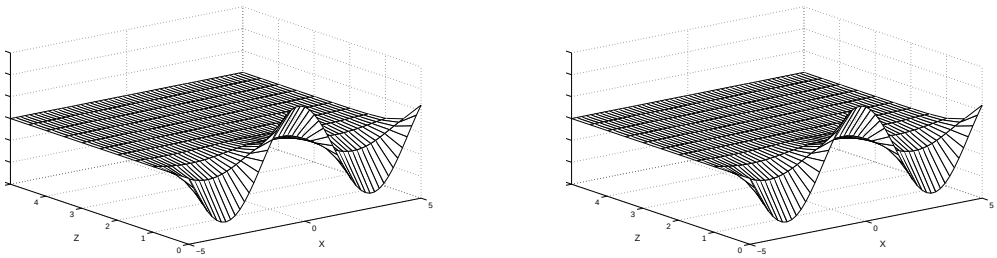
Table 1: SH-waves' velocities in high density fluid saturated medium and low density fluid saturated medium for SH-waves propagation in similar direction and opposite direction.

$c_{1,2} > \beta_{1,2}$				$c_{1,2} < \beta_{1,2}$			
β_1	β_2	c_1	c_2	β_1	β_2	c_1	c_2
2	2	7	7	7	7	2	2
2	2	5	7	7	7	2	5



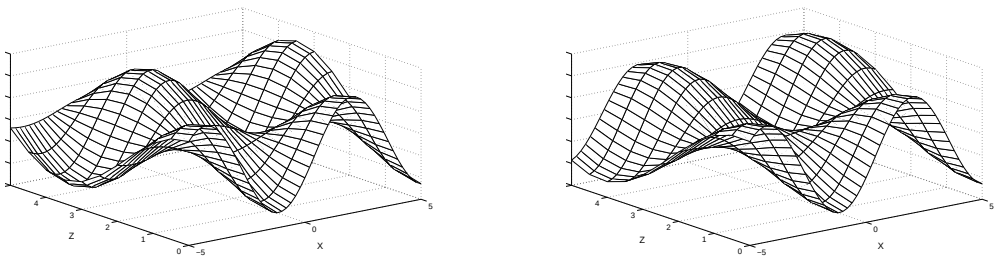
(a) (a)

Figure 2: The diffusive profiles generated by SH-waves propagation in the medium saturated with low density fluid. (a) The non-diffusive characteristic by SH-waves' propagate in similar direction. $c > \beta$, $-5 \leq x \leq 5$, $k = 1$, $\beta_1 = 2$, $\beta_2 = 2$, $c_1 = 7$, $c_2 = 7$. (b) The diffusive characteristic by SH-waves' propagate in similar direction. $c > \beta$, $-5 \leq x \leq 5$, $k = 1$, $\beta_1 = 2$, $\beta_2 = 5$, $c_1 = 7$, $c_2 = 7$.



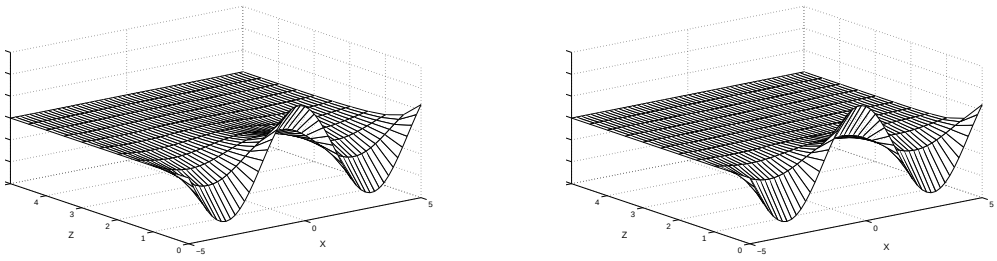
(a) (a)

Figure 3: The diffusive profiles generated by SH-waves propagation in the medium saturated with high density fluid. (a) The non-diffusive characteristic by SH-waves' propagate in similar direction. $c < \beta$, $-5 \leq x \leq 5$, $k = 1$, $\beta_1 = 7$, $\beta_2 = 7$, $c_1 = 2$, $c_2 = 2$. (b) The diffusive characteristic by SH-waves' propagate in similar direction. $c < \beta$, $-5 \leq x \leq 5$, $k = 1$, $\beta_1 = 7$, $\beta_2 = 5$, $c_1 = 2$, $c_2 = 2$.



(a) (a)

Figure 4: The recursive profiles generated by SH-waves propagation in the medium saturated with low density fluid. (a) The recursive characteristic by SH-waves' propagate in opposite direction. $c > \beta$, $-5 \leq x \leq 5$, $k = 1$, $\beta_1 = 2$, $\beta_2 = 2$, $c_1 = 7$, $c_2 = 7$. (b) The recursive characteristic by SH-waves' propagate in opposite direction. $c > \beta$, $-5 \leq x \leq 5$, $k = 1$, $\beta_1 = 2$, $\beta_2 = 5$, $c_1 = 7$, $c_2 = 7$.



(a) (a)

Figure 5: The recursive profiles generated by SH-waves propagation in the medium saturated with low density fluid. (a) The recursive characteristic by SH-waves' propagate in opposite direction. $c < \beta$, $-5 \leq x \leq 5$, $k = 1$, $\beta_1 = 7$, $\beta_2 = 3$, $c_1 = 2$, $c_2 = 2$. (b) The recursive characteristic by SH-waves' propagate in opposite direction. $c < \beta$, $-5 \leq x \leq 5$, $k = 1$, $\beta_1 = 7$, $\beta_2 = 5$, $c_1 = 2$, $c_2 = 2$.

4 Discussion

In this study, the endeavors are to study the behavior of SH-waves when the medium is saturated. In particular, diffusive profile is explored for the possessions by low and high densities fluids in elastic mediums in view of the fact that the SH-wave equation is derived from momentum equation with elasticity (1).

Equations (27) to (29) are presumed for the medium with symmetrical density such that there is no deviation in the SH-wave's velocities in both the fluid and the medium. Once there is deviation in saturation degree in the

medium, it is expected to see the variation of SH-waves' velocities when the medium porosity and the density are asymmetrical as depicted by equations (30) to (33).

When the low density fluid is saturated to the insoluble medium, equation (29) is referred to a non diffusive wave since the term $k\eta_\beta$ is in real. However, the diffusive wave exists when there is variation in fluid density. For similar direction of SH-waves' propagation, figures 2(a) and 2(b) are generated from equation (30) for showing the diffusive profiles generated by SH-waves propagate in the medium saturated with low density fluid. By letting the SH waves' celerity in fluid, $\beta_1 = 2$ and $\beta_2 = 2$ in equation (30), figure 2(a) shows that the SH-waves are non diffusive when there is no deviation in fluid density which will induce dissimilar SH waves' celerity. However, the SH-waves are diffusive when the SH-waves' celerity in fluid is different with $\beta_1 = 2$ and $\beta_2 = 5$ (see figure 2(b)). The consequence implies that the variation of fluid density in insoluble medium is capable of generating different waves' velocities and thus the diffusive characteristic for SH-waves propagate in low density fluid in the insoluble medium is formed.

In contrast with equation (29), equation (27) is referred to a diffusive wave since the term $k\eta_\beta$ is in complex by referring to (26). For similar direction of SH-waves' propagation, figures 3(a) and 3(b) are generated from equation (31) to show the diffusive profiles generated by SH-waves' propagation in the medium saturated with high density fluid. There is no significant different between the diffusive profiles showed in figures 3(a) and 3(b). The density of fluid higher than the medium is presumed of capable in preserving the diffusive characteristic of SH-waves.

For the opposite direction of propagating SH-waves, figures 4(a) and 4(b) are generated from equation (32) for showing the SH-waves' propagation in the medium saturated with low density fluid. Both the figures 4(a) and 4(b) depict the recursive SH-waves. From equation (32), the SH-waves suppose to be non-diffusive when both the term $k\eta_\beta$ is in real. By means of SH-waves propagating in opposite direction, these non diffusive waves stack and initiate the recursive SH-waves as shown in figures 4(a) and 4(b). In other words, the recursive SH-waves in the medium saturated with low density fluid are non-diffusive.

Figures 5(a) and 5(b) are generated from equation (33) for showing SH-waves' propagation in the medium saturated with high density fluid. As expected, the high density fluid will promote the diffusive characteristic (see figures 5(a) and 5(b)) since equation (33) shows the term $k\eta_\beta$ is in complex by referring to (26). Again, the density of fluid higher than the medium is found capable in preserving the diffusive characteristic of SH-waves.

For soluble medium with $c = \beta$, equation (28) gives

$$u_{SH} = C_y \exp\left[ik\left(ct - x\right)\right] \tag{34}$$

Consequently, the equation (34) implies that the SH-wave's in the soluble medium is non diffusive sinusoidal plan waves. An example of sinusoidal plane wave is acoustic waves (Bhatnagar, 1979).

5 Conclusion

The SH-waves in the medium saturated with low density fluid is non diffusive. However, the variation of fluid density is capable of generating different waves' velocities and thus the diffusive SH-waves propagating in low density fluid is formed. Nonetheless, the recursive waves are found when the SH-waves are propagated in opposite direction.

For further research, the SH-waves will be implemented to tsunamis model as one of the initial conditions for studying the effects of SH-waves in displacement.

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