A Hybrid WENO Scheme for Conservation Laws

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Abstract

We describe a hybrid method for the solution of hyperbolic conservation laws. A fourth-order total variation diminishing (TVD) finite difference scheme is conjugated with the seven-order WENO scheme. An efficient multi-resolution technique is used to detect the high gradient regions of the numerical solution in order to capture the shock with the seven-order WENO scheme while the smooth regions are computed with the more efficient TVD scheme. The hybrid scheme captures correctly the discontinuities of the solution and saves CPU time. Numerical experiments with one- and two-dimensional problems are presented.

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1. Introduction

The present work is concerned with the numerical solution of the hyperbolic conservation laws. It is well known that the exact solutions to such equations may develop discontinuities in finite time, even when the initial condition is smooth, so that one needs to consider weak solutions. A successful method should compute such discontinuities with the correct position and without spurious oscillations and yet achieve a high order of accuracy in the regions of smoothness. Harten [11] introduced the total variation diminishing (TVD) schemes modified by many others. A fourth order total variation diminishing (TVD) scheme is presented in [11]. The main property of the TVD scheme is that it can be second order (or higher) and oscillations-free across discontinuities. Moreover, TVD schemes are very accurate in smooth parts. The disadvantage of the TVD schemes is that they avoid oscillations near discontinuities by locally reverting to first order of accuracy near discontinuities and extrema and are therefore unsuitable for applications involving long-time evolution of complex structures, such as in acoustic and compressible turbulence. In these applications, extrema are clipped as time evolves and numerical diffusion may become dominant. Shock-captured methods are needed for such
applications, these methods which must be capable of capturing shock waves and resolving a broad range of length scales. The high order weighted essentially non-oscillatory (WENO) schemes are designed for this purpose. The WENO schemes were originally proposed by Liu et al [18], and have been improved by Jiang and Shu [9]. In the past decade, there are various improvements and applications of WENO schemes. Shu [5] summarized the main features of different WENO schemes for their applicabilities and strengths. There are two active areas for WENO schemes development. One is to improve the ENO properties, for example, Wang and Chen [22] proposed optimized WENO schemes for linear waves with discontinuity; Panziani et al [7] developed the optimized WENO schemes to improve the resolution of a class of compressible flows characterized by a wide disparity of scales for compressible turbulence and/or aero-acoustic phenomena, and shock waves; Balsara and Shu[8] developed the monotonicity preserving WENO schemes; Xu and Shu[9] generalized a technique of anti-diffusive flux corrections to WENO schemes; Henric[3] used a mapped-WENO to achieve optimal order near critical points. The other area is to combine WENO with other high order schemes, for example, Pirozzoli[15] proposed a hybrid compact-WENO scheme for shock-turbulence interaction; Kim and Kwon[6] proposed a high-order accurate hybrid scheme using a central flux scheme and a WENO scheme for compressible flow-field analysis, Shen and Yang[20] developed hybrid finite compact-WENO schemes.

Latini et al [13] studied the effects of WENO flux reconstruction order and spatial resolution on re-shocked two-dimensional Richtmyer-Meshkov instability. It is shown that higher-order higher-resolution simulations have lower numerical dissipation. Lower-order simulations preserve large-scale structures and flow symmetry, while higher-order higher-resolution simulations exhibit fragmentation of the structures, symmetry breaking and increased mixing. Their computational scaling shows that increasing the order is more advantageous than increasing the resolution for the flow. The motivation of this article is to combine the cheap fourth order TVD scheme [11] with the more expensive seventh order WENO scheme (k = 4) [19] to produce a fourth order accuracy in both smooth parts and near discontinuities which is less expensive and therefore saves more CPU time. The expensive WENO scheme is used only near discontinuities and the fourth order TVD scheme is used in smooth parts.

The main advantages of the resulting scheme are as follows:

i. It is very cheap, as WENO scheme is used only near discontinuities.
ii. It is very efficient, as it is very high order accurate in both smooth parts and near discontinuities (fourth order accurate in both regions).
iii. Very simple to implement.

The question is how to distinguish between those regions? For this purpose, Harten [1] developed a multi-resolutions technique which is performed at every step of the temporal integration process. Here we use a more efficient multi-resolution technique presented in [12]. The resulting scheme ensures that fluxes at grid points around discontinuities will always be computed by a WENO scheme; where as smooth tendencies will not suffer any unnecessary extra damping because they will be treated by a TVD scheme. Another advantage is that the application of the TVD scheme avoids the heavy machinery employed by the characteristics-wise WENO finite difference algorithm such that the evaluation of the Jacobian of the fluxes and characteristic and characteristic decomposition and recomposition.
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The rest of the article is organized as follows. Section 2 briefly reviews the WENO scheme [19]. In Section 3, we describe the fourth order TVD scheme [11] and in Section 4, a TVD Runge-Kutta method is presented. The multi-resolution algorithm is discussed in Section 5. In Section 6, we present the hybrid scheme. Numerical results for one dimensional conservation laws are presented in Section 7. In Section 8 the extension to two dimensional problems is presented.

2. The seven-order WENO scheme

In this section, we describe the WENO finite difference scheme for one dimensional scalar hyperbolic conservation law

\[ u_t + f(u)_x = 0, \quad -\infty < x < \infty, \quad t \geq 0 \]  

Along with the initial condition

\[ u(x, 0) = u_0(x) \]  

The semi-discrete finite difference formulation of (2.1) in a uniformly spaced grid is

\[ \frac{a}{\Delta t} \left[ u_j^t (t) \right] = -\frac{1}{\Delta x} \left[ f_{j+\frac{1}{2}} - f_{j-\frac{1}{2}} \right] - L_j(u) \]  

Where \( \Delta x \) is the grid size, \( u_j(t) \) is the solution within the stencil \( I_j = \left[ x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}} \right] \), and the numerical flux

\[ f_{j+\frac{1}{2}} = f \left( u_{j-r}, \ldots, u_{j+s} \right) \]  

with \( r \) and \( s \), integer parameters defining the set of values used for the computation of the flux \( f(u) \).

In the current WENO schemes, the numerical solutions of (2.2) are advanced in time by means a TVD Runge-Kutta method [14] which will be discussed later.

The key idea of a k-th order ENO scheme is to choose one “smoothest” stencil

\[ s_k = \{ x_{j-s}, \ldots, x_{j+s} \}, \quad \text{with} \quad s = k - r - 1, \quad r = 0 \ldots K - 1, \quad (2.4) \]

among the k candidate to avoid spurious oscillations near shocks. If the stencil \( s_k \) happens to be chosen as the ENO interpolation stencil, then the k-th order ENO reconstruction of \( f_{j+\frac{1}{2}} \) is:

\[ p_r^j \left( x_{j+\frac{1}{2}} \right) = \frac{\hat{f}_{j+\frac{1}{2}}}{\delta x} = \sum_{l=0}^{k-1} C_{rl} f_{j-l+\frac{1}{2}} \]  

where \( C_{rl} \) is the constant coefficients obtained through Lagrangian interpolation process. This process is called the reconstruction step. WENO is an improved over ENO, for it uses a convex combination of all available polynomials for a fixed \( k \). This yields a \( (2k-1) \) order method at smooth parts of the solution. The flux \( f_{j+\frac{1}{2}} \) of WENO method is defined as

\[ f_{j+\frac{1}{2}} = \sum_{l=0}^{k-1} W_r^l \left[ x_{j+\frac{1}{2}} \right] \]  

(2.6)
where \( p_{i}^{\frac{1}{2}} \left( x_{i+\frac{1}{2}} \right) \) is defined in (2.5).

The essentially non-oscillatory property is obtained by requiring that the weights \( \mathcal{W}_{r} \) reflect the relative smoothness of \( f \):

\[
\mathcal{W}_{r} = \frac{\alpha_{r}}{\alpha_{0} + \alpha_{1} + \cdots + \alpha_{k-1}}, \quad r = 0, 1, \ldots, k-1, \quad (2.7a)
\]

where

\[
\alpha_{r} = \frac{d_{r}}{(\varepsilon + \| \mathcal{S}_{r} \|^2)}, \quad (2.7b)
\]

The parameter \( \varepsilon \) is introduced to avoid the denominator to become zero. We take \( \varepsilon = 10^{-6} \) in our numerical tests and \( \mathbf{d}_{r} \) are the optimal weight coefficients given by Balsara and Shu in [8]. \( \| \mathcal{S}_{r} \| \) is the smoothness measurement of the flux function on the \( r \)-th candidate stencil \( \mathcal{S}_{r} \).

For the seventh-order WENO reconstruction (\( k = 4 \)), the corresponding smoothness indicators are given by

\[
\| \mathcal{S}_{r} \|^2 = \sum_{i=1}^{k-r} \beta_{i} \left[ u_{i}^{(l)} \left( i + r - k + 1, \ldots, i + r \right) \Delta x^{l} \right] \quad (2.8)
\]

where \( u_{i}^{(l)} \left( i + r - k + 1, \ldots, i + r \right) \) denotes the differencing approximation of \( l \)-th order derivative \( u_{i}^{(l)} \) by using points \( i + r - k + 1, \ldots, i + r \). Because \( k \) points are used, the highest order approximation of \( u_{i}^{(l)} \) is \((k-l)\)-th order interpolation. The coefficients \( \beta_{i} \) can affect the accuracy of the final scheme. For \( u_{i}^{(l)} \left( i + r - k + 1, \ldots, i + r \right) \), it can always be expressed as a linear combination of \( u_{i+r+1-n} - u_{i+r-n} \).

For the smoothness estimators Eq.(2.8) of seventh-order WENO(\( k = 4 \)) scheme[19], we have

\[
\beta_{1} = 240, \quad \beta_{2} = 1040, \quad \beta_{3} = 9732
\]

Where

For \( r = 0 \)

\[
\begin{align*}
\frac{u_{i}^{(1)}}{6} (i-3, \ldots, i) &= \frac{1}{6\Delta x}\left[ 11(u_{i} - u_{i-1}) - 7(u_{i-1} - u_{i-2}) + 2(u_{i-2} - u_{i-3}) \right] + o(\Delta x^2) \\
\frac{u_{i}^{(2)}}{\Delta x^2} (i-3, \ldots, i) &= \frac{1}{\Delta x^2}\left[ 2(u_{i} - u_{i-1}) - 3(u_{i-1} - u_{i-2}) + (u_{i-2} - u_{i-3}) \right] + o(\Delta x^2) \\
\frac{u_{i}^{(3)}}{\Delta x^3} (i-3, \ldots, i) &= \frac{1}{\Delta x^3}\left[ (u_{i} - u_{i-1}) - 2(u_{i-1} - u_{i-2}) + (u_{i-2} - u_{i-3}) \right] + o(\Delta x)
\end{align*}
\]

For \( r = 1 \)

\[
\begin{align*}
\frac{u_{i}^{(1)}}{6} (i-3, \ldots, i) &= \frac{1}{6\Delta x}\left[ 11(u_{i} - u_{i-1}) - 7(u_{i-1} - u_{i-2}) + 2(u_{i-2} - u_{i-3}) \right] + o(\Delta x^2) \\
\frac{u_{i}^{(2)}}{\Delta x^2} (i-3, \ldots, i) &= \frac{1}{\Delta x^2}\left[ 2(u_{i} - u_{i-1}) - 3(u_{i-1} - u_{i-2}) + (u_{i-2} - u_{i-3}) \right] + o(\Delta x^2) \\
\frac{u_{i}^{(3)}}{\Delta x^3} (i-3, \ldots, i) &= \frac{1}{\Delta x^3}\left[ (u_{i} - u_{i-1}) - 2(u_{i-1} - u_{i-2}) + (u_{i-2} - u_{i-3}) \right] + o(\Delta x)
\end{align*}
\]
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For $r=2$

\[
\begin{align*}
\begin{align*}
\omega_{i}^{(0)}(i-2,\ldots,i+1) &= \frac{1}{6\Delta x} \left[ 2(u_{i+1} - u_{i}) + 5(u_{i} - u_{i-1}) - (u_{i-1} - u_{i-2}) \right] + o(\Delta x^3) \\
\omega_{i}^{(2)}(i,\ldots,i+1) &= \frac{1}{\Delta x^2} \left[ (u_{i+1} - u_{i}) (u_{i} - u_{i-1}) \right] + o(\Delta x^2) \\
\omega_{i}^{(3)}(i-2,\ldots,i+1) &= \frac{1}{\Delta x^3} \left[ (u_{i+1} - u_{i}) - 2(u_{i} - u_{i-1}) + (u_{i-1} - u_{i-2}) \right] + o(\Delta x)
\end{align*}
\end{align*}
\]

For $r=3$

\[
\begin{align*}
\begin{align*}
\omega_{i}^{(0)}(i,\ldots,i+3) &= \frac{1}{6\Delta x} \left[ 11(u_{i+2} - u_{i+1}) - 7(u_{i+2} - u_{i+1}) + 2(u_{i+1} - u_{i}) \right] + o(\Delta x^3) \\
\omega_{i}^{(2)}(i,\ldots,i+3) &= \frac{1}{\Delta x^2} \left[ -7(u_{i+2} - u_{i+1}) + 3(u_{i+2} - u_{i+1}) - 2(u_{i+1} - u_{i}) \right] + o(\Delta x^2) \\
\omega_{i}^{(3)}(i,\ldots,i+3) &= \frac{1}{\Delta x^3} \left[ (u_{i+3} - u_{i+2}) - 2(u_{i+2} - u_{i+1}) + (u_{i+1} - u_{i}) \right] + o(\Delta x)
\end{align*}
\end{align*}
\]

The weights $\omega_r$ are given by [8]:

\[
d_0 = \frac{1}{35}, \quad d_1 = \frac{12}{35}, \quad d_2 = \frac{18}{35}, \quad d_3 = \frac{4}{35},
\]

For systems the reconstruction is carried out in local characteristic variables rather than the conservative variables and (2.6) is applied to each characteristic field [2]. One first transforms to characteristic variable and then applies (2.6) to each component of these variables. The final values are obtained by transforming back to conservative variables. It can be shown that the use of conservative variables (in component-wise manner) in the reconstruction results in oscillations even for simple test problem [2].
3. TVD-Finite difference scheme

In this section, the fourth order explicit TVD schemes presented in [11] is reviewed. First let us consider the linear case \( f'(u) = au \) in (2.1) so that \( f'(u) = \alpha \) is a constant wave speed. The fourth order conservative TVD numerical fluxes introduced in [11] have the form:

\[
f_{i+\frac{1}{2}} = \frac{1}{2}(au_i + au_{i+1}) - \frac{1}{2}|\alpha|\Delta_{i+\frac{1}{2}}u + |\alpha|\left(A_{0}\Delta_{i+\frac{1}{2}}u + A_{1}\Delta_{i+L_{i+\frac{1}{2}}}u\right)\phi_i + |\alpha|A_{2}\Delta_{i+M_{i+\frac{1}{2}}}u \phi_{i+M}
\]

(3.1)

where \( L = -1, M = 1 \) for \( c > 0 \) and \( L = 1, M = -1 \) for \( c < 0 \). Here \( \frac{\Delta t}{\Delta x} \alpha \) is the Courant number, \( \Delta t \) is the time step, and \( \Delta_{i+\frac{1}{2}}u_j = u_{j+1} - u_j \).

Where

\[
A_0 = \frac{1}{2} - |\alpha| + \frac{|\alpha|^2}{12}, \quad A_1 = \frac{3}{24} - \frac{|\alpha|^2}{12} - \frac{|\alpha|^3}{24}, \quad A_2 = -\frac{3}{24} + \frac{|\alpha|}{12} + \frac{|\alpha|^2}{12} - \frac{|\alpha|^3}{24}
\]

(3.2)

Here \( \varphi_{i} \) and \( \varphi_{i+M} \) are flux limiter functions defined by

\[
\varphi_i = \begin{cases} 
\varphi(A_2 A_0 - A_2), & \text{for } 0 \leq \theta_i \leq \theta^L \\
1, & \text{for } \theta^L \leq \theta_i \leq \theta^R \\
\varphi(A_1 A_1 - A_0), & \text{for } \theta_i > \theta^R
\end{cases}
\]

(3.3a)

\[
\varphi_{i+M} = \begin{cases} 
\gamma \frac{\theta_{i+M}}{1 - |\alpha| - \gamma A_1} & \text{for } 0 \leq \theta_{i+M} < 0.5 \\
1 & \text{for } \theta_{i+M} > 0.5 \\
0 & \text{for } \theta_i = 0
\end{cases}
\]

(3.3b)

Where

\[
\theta^L = \frac{\gamma(A_0 - A_2)}{1 - |\alpha| - \gamma A_1}, \quad \theta^R = \frac{1 - |\alpha| - \gamma(A_0 - A_2 \varphi_{i+M} / \theta^*)}{\gamma A_1},
\]

here \( \theta^*_i \) is called the local flow parameter and is defined by

\[
\theta^*_i = \frac{\Delta t \phi_{i+\frac{1}{2}}}{A_{i+\frac{1}{2}}}
\]

(3.4a)

and \( \theta^*_j \) is called the upwind-downward flow parameter and is given by
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\[ \theta_i^* = \frac{\Delta_{i+\frac{1}{2}+\frac{1}{2}N}}{\Delta_{i+\frac{1}{2}-\frac{1}{2}N}} \]  

(3.4b)

and \( \gamma \) is defined by

\[ \gamma = \begin{cases} 1 - |c| & \text{for } 0 \leq |c| < \frac{1}{2} \\ |c| & \text{for } \frac{1}{2} \leq |c| \leq 1 \end{cases} \]  

(3.5)

For nonlinear scalar problems \( a = a(u) \), we define the wave speed

\[ a_{t+\frac{1}{2}} = \begin{cases} \frac{\Delta_{i+\frac{1}{2}+\frac{1}{2}N}}{\Delta_{i+\frac{1}{2}-\frac{1}{2}N}} & \Delta_{i+\frac{1}{2}+\frac{1}{2}N} \\ \frac{\partial f}{\partial u} |_{u_i} & \Delta_{i+\frac{1}{2}+\frac{1}{2}N} = 0 \end{cases} \]  

(3.6)

Now we redefine \( \theta_i \) in (3.4a) as

\[ \theta_i = \frac{\Delta_{i+\frac{1}{2}+\frac{1}{2}N}^N}{\Delta_{i+\frac{1}{2}-\frac{1}{2}N}^N} \]  

(3.7)

here \( c \) is replaced by \( c_{t+\frac{1}{2}} = \frac{\Delta}{\Delta x} a_{t+\frac{1}{2}} \).

The numerical flux (3.1) takes the form

\[ f_{t+\frac{1}{2}} = \frac{1}{2}(f_t + f_{t+1}) - \frac{1}{2} \left| a_{t+\frac{1}{2}} \right| \Delta_{i+\frac{1}{2}+\frac{1}{2}N}^N \left( -A_0 \Delta_{i+\frac{1}{2}+\frac{1}{2}N} + A_4 \Delta_{i+\frac{1}{2}-\frac{1}{2}N}^N \right) \theta_i \]  

(3.8)

The flux limiter becomes the same (3.3) with replacing \( c \) by \( c_{t+\frac{1}{2}} \).

With stability condition CFL \( \leq 1 \), here CFL denotes the maximum Courant number over all cells at a given time step.

4. Time discretization

In this section, we consider the issue of time discretization. The time discretization will be implemented by a class of high order TVD Runge-Kutta methods developed in [14]. These Runge-Kutta methods are used to solve a system of initial value problems of ordinary differential equations ODE written as
\[
\frac{du}{dt} = L(u) 
\]  \hspace{2cm} (4.1)

where \( L(u) \) is an approximation to the derivative \( (-f(u))_x \) in the differential Eq. (2.1a).

In [14], schemes up to third order were found to satisfy the TVD conditions. The optimal third order TVD Runge-Kutta method is given by

\[
\begin{align*}
    u^{1} &= u^{n} + \frac{3}{4} \Delta t L(u^{n}) \\
    u^{2} &= \frac{1}{4} u^{n} + \frac{3}{4} u^{1} + \frac{1}{4} \Delta t L(u^{1}) \\
    u^{n+1} &= \frac{1}{3} u^{n} + \frac{2}{3} u^{2} + \frac{2}{3} \Delta t L(u^{2})
\end{align*} 
\]  \hspace{2cm} (4.2)

In [3], it has been shown that, even with a very nice second order TVD spatial discretization, if the time discretization is by a non-TVD but linearly stable Runge-Kutta method, the result may be oscillatory. Thus it would always be safer to use TVD Runge-Kutta methods for hyperbolic problems.

5. Multi-Resolution Analysis

The successful implementation of the Hybrid method depends on the ability to obtain accurate information on the smoothness of a function. In this work, we employ the Multi-Resolution (MR) algorithms by Harten [1, 2] to detect the smooth and rough parts of the numerical solution. The general idea is to generate a coarser grid of averages of the point values of a function and measure the differences (MR coefficients) \( d^l_k \) between the interpolated values from this sub-grid and the point values themselves. A tolerance parameter MR is chosen in order to classify as smooth those parts of the function that can be well interpolated by the averaged function and as rough those where the differences \( d^l_k \) are larger than the parameter MR. We shall see that the order of interpolation is relevant and the ratio between \( d^l_k \) of distinct orders may also be taken as an indication of smoothness.

Consider a set of dyadic grids on the form

\[
V^j = \{x^l_k \in R : x^l_k = 2^{-j} k, k \in \mathbb{Z} \}, j \in \mathbb{Z}, \hspace{2cm} (5.1)
\]

where \( j \) identifies the resolution level and \( k \) the spatial location, as illustrated in Fig. 1. Assume that the solution is known on grid \( V^j \) for \( j_{\text{min}} \leq j \leq j_{\text{max}} \) and we want to extend it to the finer grid \( V^{j+1} \). Values on the even-numbered grid points are known from the corresponding values on the lower resolution grid:

\[
u_{2k}^{j+1} = u_k^j. \hspace{2cm} (5.2)\]
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Fig. 1. Example of points in a dyadic grid.

Whereas the function values in the odd numbered grid points in \( V^{j+1} \) are computed using a suitable interpolation scheme, based on the known even-numbered grid points.

The interpolative error coefficient (or multi-resolution coefficient), \( d_k^j \), is defined as the difference between the interpolated value, \( I^j(\tilde{u}_{2k+1}^{j+1}) \), and the real one, \( u_{2k+1}^{j+1} \).

\[
d_k^j = \left| u_{2k+1}^{j+1} - I^j(\tilde{u}_{2k+1}^{j+1}) \right| / u_{ref}^j,
\]

(5.3)

Where \( u_{ref}^j \) is a reference value of the dependent variable such that,

\[
u_{ref}^j = \max \left( \left| u_{2k+1}^{j+1} \right| \right), \quad k = 0, 1, ..., 2^j,
\]

(5.4)

The calculation of the interpolated values is illustrated for the case of odd numbered grid point:

a- Calculate the face velocity

\[
a_{k+\frac{1}{2}}^j = \frac{a_k^j + a_{k+1}^j}{2}
\]

(5.5)

b- Calculate the normalized face value \( \tilde{u}_{k+\frac{1}{2}}^j \), using the SMART high resolution scheme [21], the normalized face value is given by
c- Calculate the interpolated value:

\[ I^j(\mathbf{u}^{k+1}_{2k+1}) = \begin{cases} 
\hat{u}^j_{k-1} + \hat{u}^j_{k+2} \frac{(\mathbf{u}^j_{k+1} - \mathbf{u}^j_{k-1})}{\hat{u}^j_{k+1} - \hat{u}^j_{k-1}}, & \text{if } a_{k+\frac{1}{2}} \geq 0 \\
\hat{u}^j_{k+2} + \hat{u}^j_{k+1} \frac{(\mathbf{u}^j_{k+1} - \mathbf{u}^j_{k+2})}{\hat{u}^j_{k+2} - \hat{u}^j_{k+1}}, & \text{if } a_{k+\frac{1}{2}} < 0
\end{cases} \]  

(5.7)

The maximum level of resolution is specified by the user so that grid coalescence is avoided in problematic regions (typically in this work we used \( J_{\text{max}} = 12 \)). The user also supplies the minimum level of resolution, and all the grid points pertaining to this level of resolution are always conserved throughout the computations (typically in this work we set \( J_{\text{min}} = 4 \)).

6. Hybrid method

In this section, we describe the hybrid TVD-WENO scheme. It is defined as a grid-based adaptive method in which the choice of the numerical scheme is determined by the smoothness of the solution at each grid point which is measured by the multi-resolution procedure mentioned in Section 5. The fourth-order TVD scheme is used at those grid points where the solution is flagged as smooth in lieu of the standard high order WENO scheme. The hybrid scheme is summarized in the following steps:

1. Assume that the function values \( \mathbf{u}^j_k \) in the grid \( V^j \) at time \( t = t^j \), compute the multi-resolution coefficients \( d^j_k \) for \( J_{\text{min}} \leq j \leq J_{\text{max}} \) from Equation (5.3).
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(2) A grid point is flagged as non-smooth when \( |d_k^j| > \varepsilon \) where \( \varepsilon \) is a tolerance parameter defined by the user:

\[
\text{flag}_{i} = \begin{cases} 
1, & \text{if } |d_k^j| > \varepsilon \\
0, & \text{otherwise}
\end{cases}
\]

(3) Once the flags are set, a number of neighbouring points around each flagged points \( x_i \), depending on the number of the ghost points needed for a given order of the difference scheme and WENO scheme, are also flagged to 1. In particular, if \( n_f \) and \( n_w \) are the orders of the TVD-difference and WENO schemes respectively, the number of ghost points required by the TVD-difference and WENO schemes are \( \frac{2}{3} n_f \) and \( \frac{5}{4} (n_w + 1) \), respectively. At any given point, say \( x_i \) flagged as non-smooth, its \( r = \max \left( \frac{1}{3} n_f, \frac{1}{2} (n_w + 1) \right) \) neighboring points \( \{x_{i-r}, \ldots, x_i, \ldots, x_{i+r}\} \) will also be designated as non-smooth, that is, \( \text{flag}_{i-j} = 1, j = i - r, \ldots, i + r \). This procedure avoids computing the derivative of the solution by the difference scheme using non-smooth functional values. Furthermore, the same WENO flag will be used at the Runge-Kutta stages and will be updated at the next time step.

(4) For the grid points flagged zero (smooth), we compute \( u_j^{n+1} \) by solving the ODE (2.2) using the numerical flux \( f_{j+\frac{1}{2}} \) (3.1) and the Runge–Kutta scheme.

(5) For the grid points designated as non-smooth we compute \( u_j^{n+1} \) by WENO scheme.

7. Numerical results

In this section, four examples are presented to illustrate the efficiency and robustness of the proposed scheme. For all tests we use a uniform mesh; \( N \) denotes the number of cells and the exact solution is shown by the solid line and the numerical solution by symbols.

7.1. Scalar equations

We study the performance of our schemes by applying them to the following problems.

Example 1 (Burgers’ equation)

This example considers the numerical solution of the inviscid Burgers’ equation

\[
\frac{\partial u}{\partial t} + \left( \frac{u^2}{2} \right) \frac{\partial}{\partial x} = 0 \tag{7.1}
\]

with initial condition

\[
\]
The breakdown of the initial discontinuity results in a shock wave with speed 0.5 and a rarefaction with a sonic point at $x = 0.5$. The exact solution consists of rarefaction wave (left) and shock wave (right). At $t = \frac{2}{3}$, the rarefaction hits the shock and then the solution has a rarefaction wave only. The numerical solution is displayed at $t = 0.4$ (before collision of the head of the rarefaction with the shock) and $t = 1.1$ (after collision). Results are shown in Figure 2, with $2^6 + 1$ grid points, Multi-resolution tolerance $\varepsilon = 10^{-2}$ and CFL=0.45. Note that the hybrid method reproduces the exact solution.

Example 2
We solve the equation

$$u_t + u_x = 0, \quad x \in [-1,1]$$

Subjected to periodic initial data [9]

$$u(x, 0) = \begin{cases} 
\frac{1}{2}[c(x, a - 8) + d(x, a + 8) + 4G(x, a)] - 0.85 & 
\text{if } 0 < x < 0.2, \\
1 - 10(x - 0.1) & 
\text{if } 0.2 < x < 0.8, \\
\frac{1}{2}[F(x, a - 8) + F(x, a + 8) + 4F(x, a)] & 
\text{otherwise,}
\end{cases}$$

with periodic boundary condition on $[-1,1]$, where $G(x, z) = \exp(-\beta(x - z)^2)$, $F(x, a) = \max(1 - \infty^2 (x - a)^2, 0)^{\frac{1}{2}}$.

The constants are taken as $a=0.5$, $z=-0.7$, $\delta = 0.005\times10^{-2}$, $\varepsilon = 0.05$, $\beta = (\log 2)/36\delta^2$. 

\[
\begin{align*}
\text{(7.2)} & \\
& u(x, 0) = \begin{cases} 
-1, \quad |x| \geq 0.5 \\
2, \quad |x| < 0.5 
\end{cases} \\
\text{(7.3)} & \\
& u_t + u_x = 0, \quad x \in [-1,1] \\
\text{(7.4)} & \\
& u(x, 0) = \begin{cases} 
\frac{1}{2}[c(x, a - 8) + d(x, a + 8) + 4G(x, a)] - 0.85 & 
\text{if } 0 < x < 0.2, \\
1 - 10(x - 0.1) & 
\text{if } 0.2 < x < 0.8, \\
\frac{1}{2}[F(x, a - 8) + F(x, a + 8) + 4F(x, a)] & 
\text{otherwise,}
\end{cases}
\end{align*}
\]
This initial condition consists of several shapes that are difficult for numerical methods to resolve correctly. Some of these shapes are not smooth and others are smooth but very sharp. Here we take CFL number equal to 0.45 and \( N = 2^3 + 1 \) grid points with multi-resolution tolerance \( \varepsilon = 10^{-2} \). Figures 3–5 show the numerical solutions at \( t =20 \) units obtained by the third-order TVD scheme, fourth-order TVD scheme and the hybrid method, respectively. We observe, from the figures, that the TVD schemes produce satisfied results while the numerical solution obtained by the hybrid scheme is almost indistinguishable from the exact solution.

**Fig.3.** Solution of Example 2 using the third-order TVD scheme at \( t =20 \).

**Fig.4.** Solution of Example 2 using the fourth-order TVD scheme at \( t =20 \).
7.2. Systems of equations
Now we test our hybrid scheme on the system of Euler equations of gas dynamics
\[ U_t + \mathbf{F}(U)_x = 0 \]  
(7.5)
where \( \mathbf{U} = (\rho, \rho u, \rho E)^T \) and \( \mathbf{F}(\mathbf{U}) = (\rho u, \rho u^2 + P, u(E + P))^T \), where \( \rho \) is the density, \( u \) is the velocity, \( P \) is the pressure, \( E = \rho e + \frac{1}{2} \rho u^2 \) is the total energy (sum of internal energy and kinetic energy); \( e \) is the specific internal energy \( e = P/\rho(\gamma - 1) \) and \( \gamma \) is the ratio of specific heats.

Example 3 (Shock reflection problem)
We consider the test problem concerning shock reflection in one dimension \( 0 \leq x \leq 1 \), governed by Euler equations of monatomic gas \( \gamma = \frac{5}{3} \) with initial data [10]:
\[ \rho = \rho_0, \quad u = u_0, \quad e = e_0. \]
This represents a gas of constant density and pressure moving towards \( x = 0 \). The boundary \( x = 0 \) is a rigid wall and exact solution describes shock reflection from the wall. The gas is brought to rest at \( x = 0 \) and
\[ \rho_0 = 1, \quad u_0 = 1, \quad P_0 = 3 \]  
(7.6)
\( \phi(x, t) \) is chosen such that the pressure jump across the shock equals 2, i.e. \( e_0 = 4.5 \).
Figure 6 illustrates the results at \( t = 0.15 \) and mesh size of \( 2^N + 1 \) grid points with multi-
resolution tolerance $\varepsilon = 10^{-2}$ and CFL = 0.45. We observe that the hybrid scheme resolves the discontinuity exactly.

![Figure 6](image)

**Figure 6.** Solution of Example 3 using third-order scheme (left) and hybrid scheme (right) at $t = 0.15$.

**Example 4 (Blast wave problem)**

The blast problem introduced by Woodward and Colella [17] is a severe test problem and therefore a good problem to test the robustness of numerical schemes. This problem has the initial condition which consists of three states

$$U(x, 0) = \begin{cases} 
(\rho_L, u_L, P_L) = (1, 0, 1000), & x < 0.1 \\
(\rho_M, u_M, P_M) = (1, 0, 0.01), & 0.1 < x < 0.9 \\
(\rho_R, u_R, P_R) = (1, 0, 100), & x > 0.9 
\end{cases}$$

with $\gamma = 1.4$. Boundary conditions are reflective. The solution of this problem contains the propagation of strong shock waves into low pressure regions, the collision of strong shock waves and interaction of shock waves and rarefactions, and is thus a good test of the schemes. Figures 7 and 8 show the density and velocity obtained by the hybrid scheme at $t = 0.025$ and 0.038, respectively, with $2^8 + 1$ grid points with multi-resolution tolerance $\varepsilon = 10^{-3}$ and CFL = 0.45. It is noted that the hybrid scheme is able to obtain such sharp resolution of the complex double-blast problem, particularly; the density peaks have almost the correct value. Comparing the given results with the results obtained with the reference solution shown in [17], fifth-order scheme in [4] with 400 cells and pure TVD schemes in [16] we note that our scheme is more accurate and more economical than the fifth-order scheme and TVD schemes.
8- Conclusions

We have presented an efficient, accurate and high-resolution hybrid scheme. In this scheme we use the fourth-order TVD scheme in the smooth region and the seven-order WENO scheme near discontinuities. The numerical solution is advanced in time by the third-order Runge–Kutta method. The main advantages of the scheme are reduction of CPU time and improvement in overall accuracy over the classical TVD schemes. This is due to the use of more accurate seven-order WENO scheme near discontinuities and high-order TVD scheme in the smooth region. We use an efficient multi-resolution technique to detect the discontinuities. This scheme is tested and validated by solving one- and two-dimensional problems.
References


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