

# Asymptotics of Integrals of Hermite Polynomials

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## Abstract

Integrals involving products of Hermite polynomials with the weight factor  $\exp(-x^2)$  over the interval  $(-\infty, \infty)$  are considered. A result of Azor, Gillis and Victor (SIAM J. Math. Anal. **13** (1982) 879–890] is derived by analytic arguments and extended to higher order products. An asymptotic expansion in the case of a product of four Hermite polynomials  $H_n(x)$  as  $n \rightarrow \infty$  is obtained by a discrete analogue of Laplace's method applied to sums.

**Keywords:** Hermite polynomials, Moment integrals, Asymptotic expansion

## 1. Introduction

Consider an even set of distinguishable elements divided into  $k$  subsets containing  $n_r$  elements ( $r = 1, 2, \dots, k$ ) with  $\sum_{r=1}^k n_r$  an even integer,  $2s$ . In [2], Azor, Gillis and Victor considered the combinatorial problem of the arrangement of these elements into  $s$  disjoint pairs. They showed that the number of possible pairings of different types is expressed in terms of integrals of the type

$$I(n_1, n_2, \dots, n_k) = \int_{-\infty}^{\infty} e^{-x^2} \prod_{r=1}^k H_{n_r}(x) dx, \quad (1.1)$$

where  $H_n(x)$  denotes a Hermite polynomial of degree  $n$ . The evaluation of the above integral in the case  $k = 4$  when  $n_1 = n_2$  and  $n_3 = n_4$  was obtained by combinatorial arguments and expressed in terms of a terminating

${}_3F_2$  generalised hypergeometric function. In the case  $n_r = n$  ( $1 \leq r \leq 4$ ) the asymptotics of this integral as  $n \rightarrow \infty$  were obtained by means of an elaborate argument involving the use of a generating function derived from a contour integral containing a Legendre function. In this paper, we shall obtain the evaluation of (1.1) in the cases  $k = 4, 5$  and  $6$  by analytic arguments. In addition, we show how an asymptotic expansion in the case  $k = 4$  and  $n_r = n$  can be obtained as  $n \rightarrow \infty$  by a discrete analogue of Laplace's method applied to sums.

We assemble here the main properties of the Hermite polynomials that we shall require. The  $H_n(x)$  are defined by

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} (e^{-x^2}), \quad n = 0, 1, 2, \dots$$

and have the representation

$$H_n(x) = \begin{cases} (-1)^{n/2} \frac{n!}{(\frac{1}{2}n)!} {}_1F_1(-\frac{1}{2}n; \frac{1}{2}; x^2) & (n \text{ even}), \\ (-1)^{(n-1)/2} \frac{n!}{(\frac{1}{2}n - \frac{1}{2})!} 2x {}_1F_1(-\frac{1}{2}n + \frac{1}{2}; \frac{3}{2}; x^2) & (n \text{ odd}), \end{cases} \quad (1.2)$$

where  ${}_1F_1$  denotes the confluent hypergeometric function. These polynomials satisfy the well-known orthogonality property

$$\int_{-\infty}^{\infty} e^{-x^2} H_m(x) H_n(x) dx = 2^n \sqrt{\pi} n! \delta_{m,n}, \quad (1.3)$$

where  $\delta_{m,n}$  is the Kronecker delta.

From [4, p. 838, 7.376], we have the integrals<sup>1</sup>

$$\int_0^{\infty} e^{-x^2} x^\nu H_{2n}(x) dx = \frac{(-1)^n 2^{2n-1}}{\sqrt{\pi}} \Gamma(\frac{1}{2}\nu + \frac{1}{2}) \Gamma(n + \frac{1}{2}) {}_1F_1(-n, \frac{1}{2}\nu + \frac{1}{2}; \frac{1}{2}; 1),$$

$$\int_0^{\infty} e^{-x^2} x^\nu H_{2n+1}(x) dx = \frac{(-1)^n 2^{2n-1}}{\sqrt{\pi}} \Gamma(\frac{1}{2}\nu + 1) \Gamma(n + \frac{3}{2}) {}_1F_1(-n, \frac{1}{2}\nu + 1; \frac{3}{2}; 1),$$

the first integral holding for  $\text{Re}(\nu) > -1$  and the second integral for  $\text{Re}(\nu) > -2$ . Making use of Vandermonde's theorem [5, p. 243]

$${}_1F_1(-n, r + \frac{1}{2}; \frac{1}{2}; 1) = \frac{(-r)_n}{(\frac{1}{2})_n},$$

<sup>1</sup>There are misprints in the second of these evaluations in [4].

where  $(a)_n = \Gamma(a+n)/\Gamma(a)$  is the Pochhammer symbol, we obtain the moment integrals for nonnegative integer  $r$

$$\int_{-\infty}^{\infty} e^{-x^2} x^{2r} H_{2n}(x) dx = (-)^n 2^{2n} \sqrt{\pi} \left(\frac{1}{2}\right)_r (-r)_n, \tag{1.4}$$

$$\int_{-\infty}^{\infty} e^{-x^2} x^{2r+1} H_{2n+1}(x) dx = (-)^n 2^{2n} \sqrt{\pi} \left(\frac{3}{2}\right)_r (-r)_n. \tag{1.5}$$

Since  $(-r)_n = (-)^n r(r-1)\dots(r-n+1)$ , we see that both these moment integrals vanish when  $r < n$ .

Finally, with  $2s = n_1 + n_2 + n_3$ , we note the integral (1.1) corresponding to  $k = 3$  is given by [4, p. 838, 7.375]

$$\int_{-\infty}^{\infty} e^{-x^2} H_{n_1}(x) H_{n_2}(x) H_{n_3}(x) dx = \begin{cases} \frac{2^s \sqrt{\pi} n_1! n_2! n_3!}{(s-n_1)!(s-n_2)!(s-n_3)!} & (s \text{ even}), \\ 0 & (s \text{ odd}); \end{cases} \tag{1.6}$$

this integral is also clearly zero when  $s$  is even if any one of the three indices is greater than the sum of the other two.

### 2. The case $k = 4$

We consider the evaluation of the integral (1.1) when  $k = 4$ : this requires the sum of the indices  $n_r$  to be an even integer for a nonzero value. Thus, the indices can be all even or odd, or two of them can be of different parity. With  $n_3 + n_4 = 2p$  even, we can expand the product  $H_{n_3}(x)H_{n_4}(x)$  as a linear combination of  $H_{2k}(x)$  ( $0 \leq k \leq p$ ) in the form

$$H_{n_3}(x)H_{n_4}(x) = \sum_{k=0}^p c_{2k} H_{2k}(x), \tag{2.1}$$

where

$$2^{2k} \sqrt{\pi} (2k)! c_{2k} = \int_{-\infty}^{\infty} e^{-x^2} H_{n_3}(x) H_{n_4}(x) H_{2k}(x) dx$$

by the orthogonality property (1.3). Evaluation of this integral by (1.6) then yields the coefficients  $c_{2k}$  in the form

$$c_{2k} = \frac{2^{(n_3+n_4)/2-k} n_3! n_4!}{\left(\frac{n_3-n_4}{2} + k\right)! \left(\frac{n_4-n_3}{2} + k\right)! \left(\frac{n_3+n_4}{2} - k\right)!}. \tag{2.2}$$

Then the integral

$$\begin{aligned}
 I(n_1, \dots, n_4) &= \sum_{k=0}^p c_{2k} \int_{-\infty}^{\infty} e^{-x^2} H_{n_1}(x) H_{n_2}(x) H_{2k}(x) dx \\
 &= \sum_{k=0}^p c_{2k} \frac{2^{(n_1+n_2)/2+k} \sqrt{\pi} n_1! n_2! (2k)!}{\left(\frac{n_1-n_2}{2} + k\right)! \left(\frac{n_2-n_1}{2} + k\right)! \left(\frac{n_1+n_2}{2} - k\right)!} \\
 &= A \sum_{k=0}^p \frac{\left(-\frac{n_1+n_2}{2}\right)_k \left(-\frac{n_3+n_4}{2}\right)_k (2k)!}{\left(\frac{n_1-n_2}{2} + k\right)! \left(\frac{n_2-n_1}{2} + k\right)! \left(\frac{n_3-n_4}{2} + k\right)! \left(\frac{n_4-n_3}{2} + k\right)!},
 \end{aligned}$$

where

$$A = 2^{\sigma/2} \sqrt{\pi} \frac{n_1! n_2! n_3! n_4!}{\left(\frac{n_1+n_2}{2}\right)! \left(\frac{n_3+n_4}{2}\right)!}, \quad \sigma = \sum_{r=1}^r n_r$$

and we have used the fact that  $(-a)_k = (-)^k a! / (a-k)!$ . Since  $(2k)! = 2^{2k} k! (\frac{1}{2})_k$  by the duplication formula, the above sum can be expressed as a  ${}_5F_4$  generalised hypergeometric function to yield the final result

$$\begin{aligned}
 I(n_1, \dots, n_4) &= \frac{2^{\sigma/2} \sqrt{\pi} n_1! n_2! n_3! n_4!}{\left(\frac{n_1+n_2}{2}\right)! \left(\frac{n_3+n_4}{2}\right)! \left(\frac{n_1-n_2}{2}\right)! \left(\frac{n_2-n_1}{2}\right)! \left(\frac{n_3-n_4}{2}\right)! \left(\frac{n_4-n_3}{2}\right)!} \\
 &\times {}_5F_4 \left( \begin{matrix} -\frac{n_1+n_2}{2}, -\frac{n_3+n_4}{2}, \frac{1}{2}, 1, 1; \\ \frac{n_1-n_2}{2} + 1, \frac{n_2-n_1}{2} + 1, \frac{n_3-n_4}{2} + 1, \frac{n_4-n_3}{2} + 1; \end{matrix} 4 \right). \quad (2.3)
 \end{aligned}$$

In the case  $n_1 = n_2 = m, n_3 = n_4 = n$ , the  ${}_5F_4$  function reduces to a  ${}_3F_2$  function to yield the evaluation

$$\int_{-\infty}^{\infty} e^{-x^2} H_m^2(x) H_n^2(x) dx = 2^{m+n} \sqrt{\pi} m! n! {}_3F_2 \left( \begin{matrix} -m, -n, \frac{1}{2}; \\ 1, 1; \end{matrix} 4 \right) \quad (2.4)$$

as found in [2] by means of combinatorial arguments. A similar reduction arises when  $n_1 = n_2 = n_3 = n, n_4 = m$ , where  $m$  and  $n$  have the same parity, to produce

$$\int_{-\infty}^{\infty} e^{-x^2} H_m(x) H_n^3(x) dx = \frac{2^{(m+3n)/2} \sqrt{\pi} (n!)^2 m!}{\left(\frac{m+n}{2}\right)!} F(m, n), \quad (2.5)$$

where

$$F(m, n) := \frac{1}{\left(\frac{m-n}{2}\right)! \left(\frac{n-m}{2}\right)!} {}_3F_2 \left( \begin{matrix} -n, -\frac{m+n}{2}, \frac{1}{2}; \\ \frac{m-n}{2} + 1, \frac{n-m}{2} + 1; \end{matrix} 4 \right). \quad (2.6)$$

We remark that the above hypergeometric functions have natural cut-offs. For example, the function  $F(m, n)$  can be written as

$$F(m, n) = \sum_{k=k_0}^{k_1} \frac{(-n)_k \left(-\frac{m+n}{2}\right)_k \left(\frac{1}{2}\right)_k 2^{2k}}{k! \left(\frac{m-n}{2} + k\right)! \left(\frac{n-m}{2} + k\right)!}, \quad (2.7)$$

where

$$k_0 = \lfloor \frac{1}{2}(m - n) \rfloor, \quad k_1 = \min\{n, \frac{1}{2}(m + n)\}.$$

From this, it is seen that when  $m > 3n$  we have  $k_0 > k_1$ , so that the sum (2.7) is empty and the integral in (2.5) is therefore zero; when  $n > m$  we have  $k_0 < k_1$ , so that the integral (2.5) is nonzero.

### 3. Examples of (1.1) when $k = 5$ and $k = 6$

The same procedure used in Section 2, combined with the result in (2.5), can be employed to evaluate cases of (1.1) corresponding to  $k = 5$  and  $k = 6$ . Consider the integral (1.1) with  $k = 5$  and  $n_r = 2n$  ( $1 \leq r \leq 5$ ) given by

$$I_n = \int_{-\infty}^{\infty} e^{-x^2} H_{2n}^5(x) dx. \tag{3.1}$$

Then, since by (2.1) and (2.2)

$$H_{2n}^2(x) = \sum_{k=0}^{2n} c_{2k} H_{2k}(x), \quad c_{2k} = \frac{2^{2n-k} ((2n)!)^2}{(k!)^2 (2n - k)!},$$

we have upon employing (2.5)

$$\begin{aligned} I_n &= \sum_{k=0}^{2n} c_{2k} \int_{-\infty}^{\infty} e^{-x^2} H_{2n}^3(x) H_{2k}(x) dx \\ &= 2^{3n} \sqrt{\pi} ((2n)!)^2 \sum_{k=0}^{2n} c_{2k} \frac{2^k (2k)!}{(n + k)!} F(2k, 2n), \end{aligned}$$

where  $F(2k, 2n)$  is defined in (2.6). We therefore have the evaluation

$$I_n = 2^{5n} \sqrt{\pi} ((2n)!)^4 \sum_{k=0}^{2n} \frac{(\frac{1}{2})_k 2^{2k} F(2k, 2n)}{k! (2n - k)! (n + k)!}. \tag{3.2}$$

In a similar manner we can evaluate (1.1) when  $k = 6$  and  $n_r = n$  ( $1 \leq r \leq 6$ ), namely the integral

$$J_n = \int_{-\infty}^{\infty} e^{-x^2} H_n^6(x) dx. \tag{3.3}$$

If we first suppose that  $n$  is even, we have (upon replacing  $n$  by  $2n$ )

$$H_{2n}^3(x) = \sum_{k=0}^{3n} c_{2k} H_{2k}(x),$$

where by (1.3)

$$2^{2k} \sqrt{\pi} (2k)! c_{2k} = \int_{-\infty}^{\infty} e^{-x^2} H_{2n}^3(x) H_{2k}(x) dx. \tag{3.4}$$

It then follows that

$$J_{2n} = \sum_{k=0}^{3n} c_{2k} \int_{-\infty}^{\infty} e^{-x^2} H_{2n}^3(x) H_{2k}(x) dx = \sqrt{\pi} \sum_{k=0}^{3n} 2^{2k} (2k)! c_{2k}^2.$$

Evaluating the integral in (3.4) by means of (2.5), combined with a similar procedure for odd  $n$ , we then obtain the result

$$J_n = 2^{3n} \sqrt{\pi} ((n)!)^4 \begin{cases} \sum_{k=0}^{3n/2} \frac{(\frac{1}{2})_k k! 2^{2k}}{((\frac{1}{2}n + k)!)^2} F^2(2k, n) & (n \text{ even}), \\ \sum_{k=0}^{(3n-1)/2} \frac{(\frac{3}{2})_k k! 2^{2k}}{((\frac{1}{2}n + \frac{1}{2} + k)!)^2} F^2(2k + 1, n) & (n \text{ odd}). \end{cases} \tag{3.5}$$

### 4. Moment integrals

In this section we evaluate the moment integrals involving a product of two Hermite polynomials defined by

$$K(m, n, r) = \int_{-\infty}^{\infty} e^{-x^2} x^r H_m(x) H_n(x) dx, \tag{4.1}$$

where  $r$  is a nonnegative integer and  $m + n + r$  is even.

From (1.2) and (1.4) we find, for even  $m$  and  $n$ ,

$$\begin{aligned} K(2m, 2n, 2r) &= (-)^n \frac{(2n)!}{n!} \sum_{k=0}^n \frac{(-n)_k}{k! (\frac{1}{2})_k} \int_{-\infty}^{\infty} e^{-x^2} x^{2k+2r} H_{2m}(x) dx \\ &= (-)^{m+n} 2^{2m} \sqrt{\pi} \frac{(2n)!}{n!} \sum_{k=0}^n \frac{(-n)_k}{k! (\frac{1}{2})_k} (\frac{1}{2})_{k+r} (-k-r)_m. \end{aligned}$$

Upon noting that

$$(\frac{1}{2})_{k+r} = (\frac{1}{2})_r (r + \frac{1}{2})_k, \quad (-k-r)_m = (-1)^m \frac{(r+k)!}{(r-m+k)!},$$

we then obtain

$$K(2m, 2n, 2r) = (-)^n 2^{2m} \sqrt{\pi} \frac{(2n)!}{n!} \left(\frac{1}{2}\right)_r \sum_{k=0}^n \frac{(-n)_k (r + \frac{1}{2})_k (r + k)!}{k! (\frac{1}{2})_k (r - m + k)!}. \tag{4.2}$$

It is readily seen that when  $r < m - n$  ( $m > n$ ) all terms in the above sum vanish. Since  $m$  and  $n$  are interchangeable, it then follows that  $K(2m, 2n, 2r) = 0$  when  $r < |m - n|$ .

An alternative representation is given by expressing the sum in (4.2) as a  ${}_3F_2$  hypergeometric function of unit argument to obtain

$$K(2m, 2n, 2r) = (-)^n 2^{2m} \sqrt{\pi} \frac{(2n)!r!}{n!(r - m)!} \left(\frac{1}{2}\right)_r {}_3F_2 \left( \begin{matrix} -n, r + \frac{1}{2}, r + 1; \\ \frac{1}{2}, r - m + 1; \end{matrix} 1 \right). \tag{4.3}$$

A similar procedure can be applied to the remaining two cases of odd  $m, n$  and  $m, n$  of different parity with  $r$  odd to yield

$$K(2m + 1, 2n + 1, 2r) = (-)^n 2^{2m+1} \sqrt{\pi} \frac{(2n + 1)!r!}{n!(r - m)!} \left(\frac{3}{2}\right)_r {}_3F_2 \left( \begin{matrix} -n, r + 1, r + \frac{3}{2}; \\ \frac{3}{2}, r - m + 1; \end{matrix} 1 \right) \tag{4.4}$$

and

$$K(2m + 1, 2n, 2r + 1) = (-)^n 2^{2m} \sqrt{\pi} \frac{(2n)!r!}{n!(r - m)!} \left(\frac{3}{2}\right)_r {}_3F_2 \left( \begin{matrix} -n, r + 1, r + \frac{3}{2}; \\ \frac{1}{2}, r - m + 1; \end{matrix} 1 \right). \tag{4.5}$$

It is found that  $K(2m + 1, 2n + 1, 2r)$  in (4.4) similarly vanishes when  $r < |m - n|$ , whereas  $K(2m + 1, 2n, 2r + 1)$  in (4.5) vanishes when  $r < m - n$  ( $m > n$ ) and  $r < n - m - 1$  ( $n > m + 1$ ).

In the case  $m = n$ , we define

$$k_r(n) := K(n, n, 2r) = \int_{-\infty}^{\infty} e^{-x^2} x^{2r} H_n^2(x) dx \tag{4.6}$$

for nonnegative integer  $n$ . If we apply Thomae's transformation [1, p. 143]

$${}_3F_2 \left( \begin{matrix} a, b, c; \\ d, e; \end{matrix} 1 \right) = \frac{\Gamma(d)\Gamma(e)\Gamma(s)}{\Gamma(a)\Gamma(s + b)\Gamma(s + c)} {}_3F_2 \left( \begin{matrix} d - a, e - a, s; \\ s + b, s + c; \end{matrix} 1 \right),$$

where in this section  $s = d + e - a - b - c$  denotes the parametric excess, to the hypergeometric functions appearing in (4.3) and (4.4) we find that

$$k_r(n) = (-)^r 2^n \sqrt{\pi n!} \begin{cases} \left(\frac{1}{2}\right)_r {}_3F_2 \left( \begin{matrix} -r, r + 1, \frac{1}{2}n + \frac{1}{2}; \\ \frac{1}{2}, 1; \end{matrix} 1 \right) & (n \text{ even}), \\ \left(\frac{3}{2}\right)_r {}_3F_2 \left( \begin{matrix} -r, r + 1, \frac{1}{2}n + 1; \\ 1, \frac{3}{2}; \end{matrix} 1 \right) & (n \text{ odd}). \end{cases} \tag{4.7}$$

Application of Sheppard's transformation for nonnegative integer  $r$  [1, p. 141]

$${}_3F_2 \left( \begin{matrix} -r, a, b; \\ d, e; \end{matrix} ; 1 \right) = \frac{(d-a)_r(e-a)_r}{(d)_r(e)_r} {}_3F_2 \left( \begin{matrix} -r, a, 1-s; \\ a-r+d+1, a-r-e+1; \end{matrix} ; 1 \right)$$

shows that the ratio of the two  ${}_3F_2$  functions appearing in (4.7) equals  $(\frac{3}{2})_r/(\frac{1}{2})_r$ . Hence we finally obtain the evaluation

$$k_r(n) = (-)^r 2^{n-r} \sqrt{\pi n!} \wp_r(n) \quad (4.8)$$

for nonnegative integers  $n$  and  $r$ , where the polynomial  $\wp_r(n)$  is given by

$$\wp_r(n) = 2^r \left(\frac{1}{2}\right)_r {}_3F_2 \left( \begin{matrix} -r, r+1, \frac{1}{2}n + \frac{1}{2}; \\ \frac{1}{2}, 1; \end{matrix} ; 1 \right).$$

The first few polynomials  $\wp_r(n)$  are then found to be

$$\begin{aligned} \wp_0(n) &= 1, & \wp_1(n) &= 2n + 1, \\ \wp_2(n) &= 3(2n^2 + 2n + 1), \\ \wp_3(n) &= 5(4n^3 + 6n^2 + 8n + 3), \\ \wp_4(n) &= 35(2n^4 + 4n^3 + 10n^2 + 8n + 3), \\ \wp_5(n) &= 63(4n^5 + 10n^4 + 40n^3 + 50n^2 + 46n + 15), \dots \end{aligned}$$

## 5. An asymptotic expansion for $n \rightarrow \infty$

The asymptotic behaviour of the integral (1.1) when one or more of the indices  $n_r$  is large and  $k = 2, 3$  follows immediately from (1.3) and (1.6), respectively. In [2], Azor *et al.* considered the particular case of the next integral in the sequence corresponding  $k = 4$  and  $n_r = n$  ( $1 \leq r \leq n$ ), namely

$$I_n = \int_{-\infty}^{\infty} e^{-x^2} H_n^4(x) dx,$$

and derived by an elaborate argument an asymptotic estimate of this integral as  $n \rightarrow \infty$ . They expressed  $I_n$  as integral involving a Legendre function taken round a contour surrounding the origin in the complex plane and from this constructed a generating function to which they applied Darboux's method.



The asymptotics of  $I_n$  as  $n \rightarrow \infty$  were then deduced from the behaviour of the generating function at its singularities on its circle of convergence.

Here, we shall obtain an asymptotic expansion for  $I_n$  by means of the discrete analogue of Laplace’s method applied to sums. This method was employed by Stokes [6] in his determination of the leading behaviour of the generalised hypergeometric function  ${}_pF_q(x)$  for  $x \rightarrow +\infty$ . An example of the application of this method can also be found in [3, p. 304]. From (2.4) together with  $(-n)_k = (-)^k n! / (n - k)!$ , we find

$$I_n = 2^{2n} (n!)^4 \sum_{k=0}^n \frac{2^{2k} \Gamma(k + \frac{1}{2})}{(k!)^3 ((n - k)!)^2}. \tag{5.1}$$

This sum consists of positive terms which are easily shown to possess a maximum for large  $n$  at  $k \simeq \frac{2}{3}n \equiv m$ . As  $n \rightarrow \infty$ , the terms in the sum (5.1) peak sharply near the maximum term. For arbitrary  $\epsilon > 0$  we then have

$$S_n := \sum_{k=0}^n \frac{2^{2k} \Gamma(k + \frac{1}{2})}{(k!)^3 ((n - k)!)^2} \sim \sum_{k=[m(1-\epsilon)]}^{[m(1+\epsilon)]} \frac{2^{2k} \Gamma(k + \frac{1}{2})}{(k!)^3 ((n - k)!)^2} \tag{5.2}$$

with an error that is subdominant with respect to every power of  $1/n$  as  $n \rightarrow \infty$ .

The terms retained in the sum on the right-hand side of (5.2) can now be approximated by means of the well-known expansion for  $\Gamma(z)$  given by

$$\Gamma(z) \sim \sqrt{2\pi} z^{z-\frac{1}{2}} e^{-z} \sum_{s=0}^{\infty} (-)^s \gamma_s z^{-s}, \quad z \rightarrow +\infty, \tag{5.3}$$

where the first few Stirling coefficients  $\gamma_s$  are given by  $\gamma_0 = 1$ ,  $\gamma_1 = -\frac{1}{12}$ ,  $\gamma_2 = \frac{1}{288}$ ,  $\gamma_3 = \frac{139}{51840}$ . Then, by the duplication formula for the gamma function, we have for large  $k$

$$\begin{aligned} \Gamma(k + \frac{1}{2}) &= \frac{\sqrt{2\pi} \Gamma(2k)}{2^{2k-\frac{1}{2}} \Gamma(k)} \sim \sqrt{2\pi} k^k e^{-k} \frac{\sum_{s=0}^{\infty} (-)^s \gamma_s (2k)^{-s}}{\sum_{s=0}^{\infty} (-)^s \gamma_s k^{-s}} \\ &\sim \sqrt{2\pi} k^k e^{-k} \left( 1 - \frac{1}{24k} + \frac{1}{1152k^2} + \frac{1003}{414720k^3} + \dots \right) \end{aligned} \tag{5.4}$$

and

$$\Gamma(k + 1) = k\Gamma(k) \sim \sqrt{2\pi} k^{k+\frac{1}{2}} e^{-k} \left( 1 + \frac{1}{12k} + \frac{1}{288k^2} - \frac{139}{51840k^3} + \dots \right). \tag{5.5}$$

From (5.4) and (5.5) we therefore find

$$\frac{2^{2k} \Gamma(k + \frac{1}{2})}{(k!)^3} \sim \frac{2^{2k} k^{-2k} e^{2k}}{2\pi k^{3/2}} \left( 1 - \frac{7}{24k} + \frac{49}{1152k^2} + \frac{2749}{414720k^3} + \dots \right)$$

as  $k \rightarrow +\infty$ .

We now set

$$k = m + t, \quad m = \frac{2}{3}n, \quad \tau = t/m,$$

where  $|t|$  is small compared with  $m$ . We find from (5.3) that

$$(n - k)! = (\frac{1}{3}n - t)! \sim \sqrt{2\pi}(\frac{1}{3}n - t)^{\frac{1}{3}n - t + \frac{1}{2}} e^{-\frac{1}{3}n + t} \sum_{s=0}^{\infty} \frac{(-)^s 2^s \gamma_s}{m^s (1 - 2\tau)^s}.$$

Some routine algebra then shows that the terms in the sum on the right-hand side of (5.2) can be written as

$$\begin{aligned} \frac{2^{2k} \Gamma(k + \frac{1}{2})}{(k!)^3 ((n - k)!)^2} &\sim \frac{3}{4\pi^2 n} \left(\frac{3}{2n}\right)^{3/2} \left(\frac{n}{3}\right)^{-2n} e^{2n - 3t^2/m} \\ &\times \frac{(1 + \tau)^{-3/2} \exp[3m\tau^2 - 2m(1 + \tau) \log(1 + \tau)]}{1 - 2\tau \exp[m(1 - 2\tau) \log(1 - 2\tau)]} G(\tau, m), \end{aligned}$$

where

$$G(\tau, m) = \frac{\left(1 - \frac{7(1+\tau)^{-1}}{24m} + \frac{49(1+\tau)^{-2}}{1152m^2} + \frac{2749(1+\tau)^{-3}}{414720m^3} + \dots\right)}{\left(1 + \frac{(1-2\tau)^{-1}}{6m} + \frac{(1-2\tau)^{-2}}{72m^2} - \frac{139(1-2\tau)^{-3}}{6480m^3} + \dots\right)^2}.$$

This produces the expansion for large  $n$  in the form

$$\frac{2^{2k} \Gamma(k + \frac{1}{2})}{(k!)^3 ((n - k)!)^2} \sim \frac{3}{4\pi^2 n} \left(\frac{3}{2n}\right)^{3/2} \left(\frac{n}{3}\right)^{-2n} e^{2n - 3t^2/m} \sum_{s=0}^{\infty} c_s(m) \tau^s,$$

where, omitting the odd coefficients (as these will not be required),

$$\begin{aligned} c_0(m) &= 1 - \frac{5}{8}m^{-1} + \frac{25}{128}m^{-2} + O(m^{-3}), & c_2(m) &= \frac{23}{8} - \frac{231}{64}m^{-1} + O(m^{-2}), \\ c_4(m) &= -2m + \frac{1435}{128} + O(m^{-1}), & c_6(m) &= \frac{1}{2}m^2 + O(1), \\ c_8(m) &= \frac{77}{16}m^2 + O(1), \dots \end{aligned}$$

We now extend the range of summation in (5.2) to  $\pm\infty$  to obtain

$$S_n \sim \frac{3e^{2n}}{4\pi^2 n} \left(\frac{3}{2n}\right)^{3/2} \left(\frac{n}{3}\right)^{-2n} \sum_{t=-\infty}^{\infty} e^{-3t^2/m} (c_0(m) + c_1(m)\tau + c_2(m)\tau^2 + \dots). \tag{5.6}$$

The sums in (5.6) may be evaluated using the Poisson-Jacobi transformation [7, p. 124]

$$\sum_{n=-\infty}^{\infty} e^{-an^2} = \sqrt{\frac{\pi}{a}} \left(1 + 2 \sum_{n=1}^{\infty} e^{-\pi^2 n^2/a}\right), \quad \text{Re}(a) > 0,$$

so that for  $a \rightarrow 0+$  we have (neglecting exponentially small terms of order  $e^{-\pi^2/a}$ )

$$\sum_{n=-\infty}^{\infty} n^{2s} e^{-an^2} \sim (-)^s \left(\frac{d}{da}\right)^s \sqrt{\frac{\pi}{a}} = \Gamma\left(s + \frac{1}{2}\right) a^{-s-\frac{1}{2}}$$

for  $s = 0, 1, 2, \dots$ . Since odd powers of  $\tau$  yield zero contribution to the sum in (5.6), we then find from (5.1) and (5.6)

$$I_n \sim \frac{3}{4n} \sqrt{\frac{3}{\pi}} \frac{n^{-2n} e^{2n}}{2\pi n} 6^{2n} (n!)^4 \sum_{s=0}^{\infty} \frac{c_{2s}(m) \Gamma\left(s + \frac{1}{2}\right)}{(3m)^s \sqrt{\pi}}.$$

Evaluation of this sum with the above values of  $c_{2s}(m)$  ( $0 \leq s \leq 3$ ) produces the value  $1 - \frac{5}{12}n^{-1} + O(n^{-2})$ . Continuation of this process with the help of *Mathematica* then yields the expansion

$$I_n \sim \frac{3}{4n} \sqrt{\frac{3}{\pi}} 6^{2n} (n!)^2 \left(\sum_{s=0}^{\infty} b_s n^{-s}\right) \left(\sum_{s=0}^{\infty} (-)^s \gamma_s n^{-s}\right)^2,$$

where we have removed a factor of  $(n!)^2$  with the help of (5.3) and

$$b_0 = 1, \quad b_1 = -\frac{5}{12}, \quad b_2 = \frac{17}{144}, \quad b_3 = \frac{287}{12960},$$

$$b_4 = \frac{1301}{62208}, \quad b_5 = \frac{1371821}{26127360}, \quad \dots$$

We then finally obtain the expansion<sup>2</sup>

$$I_n \sim \frac{3}{4n} \sqrt{\frac{3}{\pi}} 6^{2n} (n!)^2 \sum_{s=0}^{\infty} a_s n^{-s} \quad (n \rightarrow \infty), \tag{5.7}$$

where

$$a_0 = 1, \quad a_1 = -\frac{1}{4}, \quad a_2 = \frac{1}{16}, \quad a_3 = \frac{1}{32},$$

$$a_4 = \frac{7}{256}, \quad a_5 = \frac{59}{1024}, \quad \dots$$

The first three terms of this expansion were obtained in [2], where the third coefficient was incorrectly given as  $a_2 = \frac{3}{16}$ . To demonstrate the validity of this expansion we present in Table 1 the absolute relative error in the computation of  $I_n$  in (5.1) by means of (5.7) for different values of  $n$  and truncation index  $s$ .

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<sup>2</sup>An obvious misprint in [2, Eq. (58)] has the factor  $(n!)^2$  in the denominator.

$s$	$n = 50$	$n = 100$	$n = 200$
0	$5.000 \times 10^{-3}$	$2.500 \times 10^{-3}$	$1.250 \times 10^{-3}$
1	$2.538 \times 10^{-5}$	$6.297 \times 10^{-6}$	$1.568 \times 10^{-6}$
2	$2.558 \times 10^{-7}$	$3.161 \times 10^{-8}$	$3.928 \times 10^{-9}$
3	$4.592 \times 10^{-9}$	$2.801 \times 10^{-10}$	$1.729 \times 10^{-11}$
4	$1.956 \times 10^{-10}$	$5.931 \times 10^{-12}$	$1.827 \times 10^{-13}$
5	$1.026 \times 10^{-11}$	$1.552 \times 10^{-13}$	$2.387 \times 10^{-15}$

Table 1: Values of the absolute relative error in the computation of  $I_n$  by the asymptotic expansion (5.7) for different values of  $n$  and truncation index  $s$ .

The integral (1.1) with  $n_1 = n_2 = m$ ,  $n_3 = n_4 = n$  is much more straightforward to estimate asymptotically as  $n \rightarrow \infty$  when  $m$  is finite. From (2.5) and (2.7), we have when  $n > m$

$$I_{m,n} := \int_{-\infty}^{\infty} e^{-x^2} H_m^2(x) H_n^2(x) dx = 2^{m+n} (m!n!)^2 \sum_{k=0}^m \frac{\Gamma(k + \frac{1}{2}) 2^{2k}}{(k!)^3 (m-k)! (n-k)!}$$

and, for large  $n$ , the maximum term in the sum corresponds to  $k = m$ . With  $k = m - j$ , we can rewrite the above sum as

$$I_{m,n} = 2^{3m+n} \Gamma(m + \frac{1}{2}) n! \binom{n}{m} \sum_{j=0}^m \frac{d_j(m)}{(n-m+1)_j},$$

where

$$d_j(m) = \frac{(-)^j 2^{-2j} (j!)^2}{(\frac{1}{2} - m)_j} \binom{m}{j}^3.$$

This is in the form of an inverse factorial expansion in  $n$  which is suitable for computation as  $n \rightarrow \infty$ .

The behaviour of the integrals with  $n_1 = n_2 = n_3 = m$ ,  $n_4 = n$  and  $n_1 = m$ ,  $n_2 = n_3 = n_4 = n$  (where  $m$  and  $n$  have the same parity) as  $n \rightarrow \infty$  can be obtained from (2.5) and (2.7). We have

$$\int_{-\infty}^{\infty} e^{-x^2} H_m^3(x) H_n(x) dx = 0 \quad (n > 3m),$$

and

$$\int_{-\infty}^{\infty} e^{-x^2} H_m(x) H_n^3(x) dx = 2^{(m+3n)/2} \sqrt{\pi} \frac{(n!)^2 m!}{(\frac{m+n}{2})!} F(m, n),$$

where  $F(m, n)$  is defined in (2.6). If we suppose  $m$  and  $n$  are both even and define  $(a, j) := (a + j)!(a - j)!$ , we can express  $F(m, n)$  in the form

$$F(m, n) = \frac{2^n n!}{\sqrt{\pi}} \left(\frac{1}{2}m + \frac{1}{2}n\right)! \sum_{j=-m/2}^{m/2} \frac{2^{2j} \Gamma(\frac{1}{2}n + \frac{1}{2} + j)}{(\frac{1}{2}n, j)(\frac{1}{2}m, j)(n - \frac{1}{2}m + j)!};$$

a similar result applies for  $m, n$  odd. A difficulty arises in the estimation of  $F(m, n)$  as  $n \rightarrow \infty$  since, for finite  $m$ , the discrete analogue of Laplace's method cannot be employed. Since  $(\frac{1}{2}n, j+1)/(\frac{1}{2}n, j) \simeq 1$  as  $n \rightarrow \infty$ , the ratio of consecutive terms in this sum is controlled by  $e_j := 2^{2j} \Gamma(\frac{1}{2}n + \frac{1}{2} + j)/(n - \frac{1}{2}m + j)!$ . It is then easy to see that  $e_{j+1}/e_j \simeq 2$  as  $n \rightarrow \infty$ , so that there is no clear maximum term in the sum.

Finally, we can consider the integral

$$\int_{-\infty}^{\infty} e^{-x^2} H_n^2(x) H_{n+m}^2(x) dx = 2^{m+2n} (m+n)! n! \sum_{k=0}^n \frac{2^{2k} \Gamma(k + \frac{1}{2})}{(k!)^3 (n-k)! (m+n-k)!}$$

by (2.5) and (2.7). As  $n \rightarrow \infty$ , the discrete analogue of Laplace's method can be used to show that that this integral possesses the leading behaviour

$$\frac{3}{4n} \sqrt{\frac{3}{\pi}} 6^{2n+m} (m+n)! n^{-m} \quad (n \rightarrow \infty).$$

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