A New Mathematical Model for Time Cost Trade-off Problem with Budget Limitation Based on Time Value of Money

H. Nikoomaram
Dept. of management, Science and Research Branch
Islamic Azad University, Tehran, Iran
nikoomaram@srbiau.ac.ir

F. Hosseinzadeh Lotfi
Dept. of mathematics, Science and Research Branch
Islamic Azad University, Tehran, Iran
farhad@hosseinzadeh.ir

J. Jassbi
Dept. of industrial management, Science and Research Branch
Islamic Azad University, Tehran, Iran
jassbi@srbiau.ac.ir

Mohammad Reza Shahriari*
Dept. of industrial management, Science and Research Branch
Islamic Azad University, Tehran, Iran
shahriari.mr@gmail.com

Abstract

The main objective in the time cost trade-off problems has always been the reduction in the total cost. In the previous researches the total cost function has been defined

* corresponding author e-Mail: shahriari.mr@gmail.com
using some elements such as direct costs, indirect costs and opportunity costs whereas the missing element which has not been so far discussed is Time Value Of Money (TVOM) which is regarded as synonymous with the capital cost. Capital cost can have a great impact on the optimum results in capital-intensive projects, meaning that when we crash the starting activities in a project, the extra investment will be tied in until the end of the project; however, when we crash the final activities the extra investment will be tied in for a much shorter period. The present paper as well as considering the concept of TVOM, it also includes budget limitation.

**Keywords:** crashing, time-cost trade-off, slop cost, time value of money, capital cost

### 1. Introduction

Project managers face special conditions in implementing most of the projects in which they have to shorten the project duration. The said process is called crashing. As in the crashing problems money has to be sacrificed to further shorten the projects duration, it can be said that money is being compromised with time, which is known as time cost trade-off problem.

Reducing the original project duration which is called ‘crashing PERT/CPM networks’ in many studies which is aimed at meeting a desired deadline with the lowest amount of cost is one of the most important and useful concepts for project managers.

Since there is a need to allocate extra resources in PERT/CPM crashing networks, and the project managers are intended to spend the lowest possible amount of money and achieve the maximum crashing time, as a result both direct and indirect costs will be influenced in the project; therefore, in some researches the terms ‘time-cost trade-off’ is also used for this purpose.

The PERT/CPM methods of time-cost trade-offs is concerned with determining how much (if any) to crash each of the activities in order to reduce the anticipated duration of the project to a desired value.

The major reasons which make the project managers interested in using the crashing models are:

- Avoiding unfavorable weather conditions
- Early commissioning and operation
- Improving the project cash flow
- Compensating the delays
- Early utilization
Network crashing was originally developed along with the Critical Path Method (CPM) to plan and control large-scale projects. Network crashing in PERT/CPM is aimed at finding which activities should be crashed with the use of extra resources if we intend to shorten the project duration. Crashing in PERT/CPM means the selection of the lowest cost slope activity or activities, which will shorten the critical path(s).

2. LITERATURE REVIEW

Crashing PERT/CPM networks has been studied by a number of researchers. Salinge presented equations for the lines in a linear planning chart assuming non-interference of crews and continuity of work. Johnston presented applications of the LSM for highway construction projects. Using optimal theory, Handa and Barcia formulated the problem as an optimization one minimizing the project duration. These early LSM models had limitations such as constant rate of production for each task, binding continuity constraints, and no provisions for the use of multiple crews. [1]

George and Schou [2] have explored effective rules to expedite the PERT networks in order to find the activity that must be crashed first. Samman tried to solve this problem using the heuristic methods [3]. Keefer and Verdini also tried to increase the time estimation of PERT activity parameters. This increase gave more precision than the PERT formulas currently being used [4]. Having been criticized by several researchers, the time distribution of PERT networks [4,5] is still considered an effective tool to schedule probabilistic projects. Cho and Yum [6] developed a new method for evaluating what they called the ‘Uncertainty Importance Measure’. Researchers intended to explore some of the problems in developing a rationale to crash stochastic networks. In other words in the TCTP, the objective is to determine the duration of each activity in order to achieve the minimum total direct and indirect costs of the project.

Research on TCTP has been conducted considering the total cost function. Various kinds of cost functions such as linear [8,9], discrete [10], convex [11,12], and concave [13] have been used in the studies on TCTP so far. Dynamic programming (DP) approach was suggested by Robinson [14] and Elmaghraby [15] incase the cost functions are arbitrary.

Recently computational optimization techniques, such as genetic algorithms and simulated annealing have been adopted by some researchers to solve TCTP. JIN Chao-guang et al. [16] and Feng et al. [17] and Chua et al. [18] proposed models using genetic algorithms and the Pareto front approach to solve construction time-cost trade-off problems; moreover, Uncertain TCTP are categorized into two types: probabilistic models and fuzzy models. One of the probabilistic models is Ang’s model [19]. In an attempt to incorporate the variability of funding, which is quantified by the coefficient of variation, Yang [20] proposed a chance-constrained
programming model. The said model formulates financial feasibility as a stochastic constraint.


As mentioned above many researchers have been carried out in this regard so far in which they have focused on reducing direct and indirect costs to the lowest possible amounts.

In other words in most studies the slope cost has been used as the main criterion in selecting activities for crashing, but the Time Value Of Money (TVOM) in crashing has not been considered in any of the said researches.

TVOM can have a great impact on the optimum results in capital-intensive projects, meaning that when we crash the starting activities in a project, the extra investment will be tied in until the end of the project; however, when we crash the final activities the extra investment will be tied in for a much shorter period.

Since crashing involves extra investment; therefore, it is one of the major preoccupations of project managers to reduce the aforementioned cost.

This paper intends to present a new mathematical approach based on integer programming to add TVOM concept in crashing PERT/CPM network problems and budget limitation as a main managerial constraint will considered as well.

### 3. Model modification

In most studies 'slop cost' has been used as the main criterion in selecting activities for crashing.

Slop cost is shown in figure 1:
Mathematical model for time cost trade-off problem

The parameters of the above-mentioned equation are as follows:
- Slope cost
- Crashing cost
- Normal cost
- Crashed time
- Normal time

Now to determine the best combination of activities to decrease the length of the project and finish it by the due date, the following parameters and variables are defined to prepare the model.

- Activity normal time
- Activity crashed time
- Activity slope cost
- Activity planned time

The total direct costs of the project in case all the activities are done during a normal time
- The indirect cost that is depended to duration of project
- The planned time for the event to happen
- Number of events in the network
- Interest rate

Fig.1: Slope Cost
The supposed budget to crash the project

\( G = (V, E) \), consists of a set of objects \( V = \{1,2,3,\ldots,n\} \) called \textit{events} and another set \( E = \{\text{all defined edges}\} \) that called \textit{activities}.

Therefore the crash cost function can be defined as:

\[
C_0 = \sum_i \sum_j C_{ij} (D_{n(ij)} - d_{ij}) + \sum_i \sum_j C_{ij} (t_n - t_i) \cdot l_0 \tag{2}
\]

In which \( \sum_i \sum_j C_{ij} (D_{n(ij)} - d_{ij}) \) is the crashing cost and \( \sum_i \sum_j C_{ij} (t_n - t_i) \cdot l_0 \) is the TVOM cost. Now the \( y_{ij} \) binary variable must be added to \( T\mathcal{C}_0 \) to consider the TVOM cost of the activities which have been crashed. As where \( B \) is the maximum available budget; therefore, \( C_0 \) will incline the model to crash the final activities, because in this case the cost of the capital tied in the project will be less, but to consider the said cost only for the crashed activities so to assist the model to assign the appropriate value to \( C_0 \), it is convenient to introduce the following inequalities based on the binary \( y_{ij} \) variables into the model:

\[
D_{ij} - d_{ij} \leq y_{ij} \cdot M \tag{3}
\]

\[
y_{ij} \cdot \varepsilon \leq D_{ij} - d_{ij} \tag{4}
\]

Where:

\[
y_{ij} \in \{0,1\}, M \text{ is a very large number}, \varepsilon \text{ is a very small number} \tag{5}
\]

And in case crashing is not applied to \( i - j \) activities:

\[
D_{ij} = d_{ij} \rightarrow D_{ij} - d_{ij} = 0 \rightarrow 0 \leq y_{ij} \cdot M, y_{ij} \cdot \varepsilon \leq 0 \rightarrow y_{ij} = 0 \tag{6}
\]

And in case crashing is applied to \( i - j \) activities:

\[
D_{ij} - d_{ij} > 0 \rightarrow 0 < y_{ij} \cdot M \rightarrow y_{ij} > 0 \rightarrow y_{ij} = 1 \tag{8}
\]

Including these relations the capital cost will only apply to the crashed activities. It is obvious that the planned time to do \( i - j \) activities is at least equal to \( t_j - t_i \), thus it is necessary to add \( t_j - t_i \geq d_{ij} \) to the model and also all decision variables are restricted to nonnegative values.

On the other hand the planned time to do \( i - j \) activities will be at least equal to the allowed crashing time and at most equal to the normal time.
As described above the model to find the least expensive way of crashing activities based on TVOM element can be rephrased in a form more familiar to Integer Programming as follows:

\[
Min Z = H(t_n - t_1) + K_n
\]

subject to:
\[
\begin{align*}
t_j - t_i & \geq d_{ij} \\
D_{f(ij)} & \leq d_{ij} \leq D_{n(ij)} \\
\sum_i \sum_j C_{ij}(D_{n(ij)} - d_{ij}) + \sum_i \sum_j C_{ij}(t_n - t_i) * y_{ij} * I_0 & \leq B \\
D_{ij} - d_{ij} & \leq y_{ij} * M \\
y_{ij} * \varepsilon & \leq D_{ij} - d_{ij} \\
t_i & \geq 0, y_{ij} \in \{0,1\}, \quad i,j \in G
\end{align*}
\]

Using the suggested model to crash the PERT/CPM networks instead of the below classic model, will bring us more realistic results.

\[
Min Z = H(t_n - t_1) + K_n
\]

subject to:
\[
\begin{align*}
t_j - t_i & \geq d_{ij} \\
D_{f(ij)} & \leq d_{ij} \leq D_{n(ij)} \\
\sum_i \sum_j C_{ij}(D_{n(ij)} - d_{ij}) & \leq B \\
t_i & \geq 0, \quad i,j \in G
\end{align*}
\]

4. Numerical example

To show the difference between the classical models (10) and proposed model (9), firstly we solve a numerical problem regardless of the TVOM concept and then the proposed model will be applied to solve the same problem.

Example:
It is supposed that the available budget for crashing the project which its details are listed in Table1 is $10000 and $0 = 0.05. According to the current scheduling it will be finished in 36 month and the indirect cost is equal to $8 per month. Also figure 2 shows the project network.
Table 1- The example details

<table>
<thead>
<tr>
<th>Activity</th>
<th>predecessor</th>
<th>Crashed time (month)</th>
<th>Normal time (month)</th>
<th>Slope cost</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>-</td>
<td>2</td>
<td>4</td>
<td>800</td>
</tr>
<tr>
<td>B</td>
<td>A</td>
<td>1</td>
<td>3</td>
<td>750</td>
</tr>
<tr>
<td>C</td>
<td>B</td>
<td>5</td>
<td>7</td>
<td>920</td>
</tr>
<tr>
<td>D</td>
<td>A</td>
<td>7</td>
<td>8</td>
<td>800</td>
</tr>
<tr>
<td>E</td>
<td>D</td>
<td>4</td>
<td>6</td>
<td>880</td>
</tr>
<tr>
<td>F</td>
<td>D</td>
<td>3</td>
<td>5</td>
<td>750</td>
</tr>
<tr>
<td>G</td>
<td>C, E</td>
<td>6</td>
<td>9</td>
<td>900</td>
</tr>
<tr>
<td>H</td>
<td>G, F</td>
<td>8</td>
<td>10</td>
<td>940</td>
</tr>
</tbody>
</table>

*The amount of money that should be spent to crash each activity for one week.

Fig. 2 – The project network

Firstly the classical model, without considering TVOM has been used as follows:

\[ Min \, Z = 8(t_7 - t_1) \]  

(11)

Subject to:
\[
\begin{align*}
t_2 - t_1 &\geq d_A, \quad t_3 - t_2 \geq d_B, \quad t_5 - t_3 \geq d_C, \quad t_4 - t_2 \geq d_D, \\
t_5 - t_4 &\geq d_E, \quad t_6 - t_4 \geq d_F, \quad t_6 - t_5 \geq d_G, \quad t_7 - t_6 \geq d_H, \\
2 &\leq d_A \leq 4, \quad 1 \leq d_B \leq 3, \quad 5 \leq d_C \leq 7, \quad 7 \leq d_D \leq 8, \\
4 &\leq d_E \leq 6, \quad 3 \leq d_F \leq 5, \quad 6 \leq d_G \leq 9, \quad 8 \leq d_H \leq 10, \\
800(4 - d_A) + 750(3 - d_B) + 920(7 - d_C) + 800(8 - d_D) + 880(6 - d_E) + 750(5 - d_F) + 900(9 - d_G) + 940(10 - d_H) &\leq 10000 \\
t_1, t_2, \ldots, t_7 &\geq 0
\end{align*}
\]
The results of above model have been shown in table 2.

Then the proposed model (9) will be applied as follows considering TVOM:

\[
\begin{align*}
\text{Min } Z &= 8(t_7 - t_1) \\
\text{Subject to:} \\
t_2 - t_1 &\geq d_A , \quad t_3 - t_2 \geq d_B , \quad t_5 - t_3 \geq d_C , \quad t_4 - t_2 \geq d_D \\
t_5 - t_4 &\geq d_E , \quad t_6 - t_4 \geq d_F , \quad t_6 - t_5 \geq d_G , \quad t_7 - t_6 \geq d_H \\
2 \leq d_A &\leq 4 , \quad 1 \leq d_B \leq 3 , \quad 5 \leq d_C \leq 7 , \quad 7 \leq d_D \leq 8 \\
4 \leq d_E &\leq 6 , \quad 3 \leq d_F \leq 5 , \quad 6 \leq d_G \leq 9 , \quad 8 \leq d_H \leq 10 \\
800(4 - d_A) + 750(3 - d_B) + 920(7 - d_C) + 800(8 - d_D) + 880(6 - d_E) + 750(5 - d_F) + 900(9 - d_G) + 940(10 - d_H) + 0.05 \cdot 800y_A(t_7 - t_1) + 0.05 \cdot 750y_B(t_7 - t_2) + 0.05 \cdot 880y_C(t_7 - t_3) + 0.05 \cdot 900y_D(t_7 - t_4) + 0.05 \cdot 940y_E(t_7 - t_5) + 0.05 \cdot 940y_F(t_7 - t_6) &\leq 10000 \\
4 - d_A &\leq y_A \times 10^5 , \quad 3 - d_B \leq y_B \times 10^5 , \quad 7 - d_C \leq y_C \times 10^5 \\
8 - d_D &\leq y_D \times 10^5 , \quad 6 - d_E \leq y_E \times 10^5 , \quad 7 - d_F \leq y_F \times 10^5 \\
9 - d_G &\leq y_G \times 10^5 , \quad 10 - d_H \leq y_H \times 10^5 , \quad 4 - d_A \geq 10^{-5} \times y_A \\
3 - d_B &\geq 10^{-5} \times y_B , \quad 7 - d_C \geq 10^{-5} \times y_C , \quad 8 - d_D \geq 10^{-5} \times y_D \\
6 - d_E &\geq 10^{-5} \times y_E , \quad 7 - d_F \geq 10^{-5} \times y_F , \quad 9 - d_G \geq 10^{-5} \times y_G \\
10 - d_H &\geq 10^{-5} \times y_H , \quad t_1, t_2, ..., t_7 \geq 0 , \quad y_k \in \{0,1\} \text{ for } k = A, B, C, D, E, F, G, H
\end{align*}
\]
As the results show the result of decision variables will be so different in case the TVOM concept has been considered in comparison to the case in which this concept has not been applied.

<table>
<thead>
<tr>
<th>Variable</th>
<th>Value considering TVOM model (12)</th>
<th>Value Without considering TVOM-model(11)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$t_7$</td>
<td>29.00</td>
<td>27.00</td>
</tr>
<tr>
<td>$t_6$</td>
<td>21.00</td>
<td>19.00</td>
</tr>
<tr>
<td>$t_5$</td>
<td>15.00</td>
<td>13.00</td>
</tr>
<tr>
<td>$t_4$</td>
<td>11.00</td>
<td>9.00</td>
</tr>
<tr>
<td>$t_3$</td>
<td>7.00</td>
<td>7.37</td>
</tr>
<tr>
<td>$t_2$</td>
<td>4.00</td>
<td>2.00</td>
</tr>
<tr>
<td>$t_1$</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>$d_A$</td>
<td>4.00</td>
<td>2.00</td>
</tr>
<tr>
<td>$d_B$</td>
<td>3.00</td>
<td>3.00</td>
</tr>
<tr>
<td>$d_C$</td>
<td>7.00</td>
<td>5.36</td>
</tr>
<tr>
<td>$d_D$</td>
<td>7.00</td>
<td>7.00</td>
</tr>
<tr>
<td>$d_E$</td>
<td>4.00</td>
<td>4.00</td>
</tr>
<tr>
<td>$d_F$</td>
<td>5.00</td>
<td>5.00</td>
</tr>
<tr>
<td>$d_G$</td>
<td>6.00</td>
<td>6.00</td>
</tr>
<tr>
<td>$d_H$</td>
<td>8.00</td>
<td>8.00</td>
</tr>
<tr>
<td>$y_A$</td>
<td>0.00</td>
<td>-</td>
</tr>
<tr>
<td>$y_B$</td>
<td>0.00</td>
<td>-</td>
</tr>
<tr>
<td>$y_C$</td>
<td>0.00</td>
<td>-</td>
</tr>
<tr>
<td>$y_D$</td>
<td>1.00</td>
<td>-</td>
</tr>
<tr>
<td>$y_E$</td>
<td>1.00</td>
<td>-</td>
</tr>
<tr>
<td>$y_F$</td>
<td>0.00</td>
<td>-</td>
</tr>
<tr>
<td>$y_G$</td>
<td>1.00</td>
<td>-</td>
</tr>
<tr>
<td>$y_H$</td>
<td>1.00</td>
<td>-</td>
</tr>
<tr>
<td>$Z$</td>
<td>232.00</td>
<td>216.00</td>
</tr>
</tbody>
</table>
It is obvious that the more capital-intensive the project, the greater the said difference will be and also if the costs of capital (considering the TVOM) add to the objective function of model (11) then real cost of crashing will go over than the current value.

5. CONCLUSION AND SUGGESTIONS

A new mathematical model for crashing PERT networks considering budget limitation was presented in this paper. The main advantages of the said model was to use the concept of time value of money (TVOM) to develop a cost function with more applied components in comparison to other cost functions used in other studies. The main idea of this paper based of which the mathematical model was formed was to answer why project managers attach less importance to cash flow which is one the serious problems in crashing PERT networks. This basic idea led us to the TVOM concept in crashing PERT networks which can have a close relation to cash flow; moreover, it can be said that applying TVOM will reduce the real total cost of the project for those involved in it. It is recommended to apply other factors such as quality and risk for PERT network crashing to other studies in future and to investigate the impacts of these factors in combination with TVOM. It is also possible to further expand the model by using imprecise data.

References


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