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Norm Bounds for a Transformed Activity Level Vector in Sraffian Systems: A ‘Dual’ Exercise

Nikolaos Rodousakis

Department of Public Administration, Panteion University
136 Syngrou Ave, 17671 Athens, Greece
nrodousakis@gmail.com

George Soklis

Department of Public Administration, Panteion University
136 Syngrou Ave, 17671 Athens, Greece
gsok@hotmail.gr

Abstract

This paper gives lower and upper bounds for the largest and the smallest element of a transformed activity level vector in Sraffian systems. The bounds appear as ‘dual’ to those of the price system, indicated by Mariolis in 2010: (i) they are expressed in terms of the ‘maximum row sum matrix norm’; and (ii) depend on the vertically integrated coefficients and the ratio of the uniform rate of growth to the maximum rate of growth.

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1 Introduction

In a linear production system, the relationship between the wage rate and the rate of profit can be obtained from the price system, whilst the relationship between consumption per head and the growth rate can be obtained from the system of physical quantities. As is it has been pointed out, these two relationships (systems) have exactly the same mathematical form (see, *e.g.*, [6] and [2], p. 114). This fact is referred to the literature as the ‘duality’ of the two relationships (systems).

It is well known that, within the Sraffian framework, the long-run relative prices (the structure of outputs) can change in a complicated way as the rate of profit (growth rate) changes (see, *e.g.*, [7, chs 3 and 6] and [6], respectively).[1] In a recent paper in this *Journal*, Mariolis [3] has shown that, when the price system can be transformed (via a diagonal similarity matrix formed from the elements of the left-hand side Perron-Frobenius eigenvector of the technical coefficients matrix) into a vertically integrated system in which the technical coefficients matrix is a column stochastic matrix, the largest and the smallest element of the transformed (and normalized with Sraffa's 'Standard commodity') price vector admit norm bounds that depend on the socio-technical conditions of production.

The purpose of this paper is to carry out (for reasons of symmetry and completeness) a similar elaboration on the system of physical quantities. Since the quantity side of a Sraffian system is *formally* 'dual' to the price system, it is reasonable to expect that, by following the route suggested by Mariolis [3], one could obtain a *suitably* transformed vector of activity levels that admits norm bounds for its largest and smallest element.[2]

The remainder of the paper is structured as follows. Section 2 deals with the usual circulating capital model and gives norm bounds for a transformed activity level vector. Section 3 concludes the paper.

2 Norm Bounds

Consider a closed linear system, involving only single products, basic commodities (in the sense of Sraffa [7, §6]) and circulating capital. Furthermore, assume that (i) the input-output coefficients are fixed; (ii) the system is viable, *i.e.*, the Perron-Frobenius (P-F hereafter) eigenvalue of the irreducible $n \times n$ matrix of input-output coefficients, $\mathbf{A}$, is less than 1;[3] (iii) the rate of growth, g, is uniform; and (iv) commodities are consumed in proportion to the entries of the $n \times 1$ vector of consumption bundle, $\mathbf{b}\ (\geq \mathbf{0})$, which serves as the unit of consumption.

[1] It should be noted, however, that Steedman [9] has detected a rule that the price (net output) vector follows as the rate of profit (growth rate) varies.

[2] It is should be noted, however, that the formal symmetry between the price and physical quantity systems does not necessarily implies a symmetry in a substantial sense (see [6]).

[3] Matrices (and vectors) are denoted by boldface letters. The transpose of an $n \times 1$ vector $\mathbf{a} \equiv [a_i]$ is denoted by $\mathbf{a}^{\mathrm{T}}$. $\lambda_{\mathbf{A}}$ denotes the P-F eigenvalue of a (semi-) positive matrix $\mathbf{A} \equiv [a_{ij}]$, $(\mathbf{q}_{\mathbf{A}}, \mathbf{y}_{\mathbf{A}}^{\mathrm{T}})$ the corresponding eigenvectors, and $\hat{\mathbf{q}}_{\mathbf{A}}\,(\hat{\mathbf{y}}_{\mathbf{A}})$ the diagonal matrix formed from the elements of $\mathbf{q}_{\mathbf{A}}\,(\mathbf{y}_{\mathbf{A}})$. Finally, $\mathbf{e}$ denotes the summation vector, *i.e.*, $\mathbf{e} \equiv [1,1,...,1]^{\mathrm{T}}$.

On the basis of these assumptions, the quantity side of the system is described by the following relation

$$\mathbf{x} = (1+g)\mathbf{A}\mathbf{x} + c\mathbf{b} \tag{1}$$

where $\mathbf{x}$ denotes the vector of activity levels and c the index of consumption. Relation (1) after rearrangement gives:

$$\mathbf{x} = g\mathbf{G}\mathbf{x} + c\mathbf{f} \tag{2}$$

where $\mathbf{G} \equiv [\mathbf{I}-\mathbf{A}]^{-1}\mathbf{A}$ denotes the 'vertically integrated matrix' ([4]) the jth column of which represents the vector of activity levels whose operation would produce, as a net product, just the capital stock required (directly) to support the operation of the jth process at unit level, and $\mathbf{f} \equiv [\mathbf{I}-\mathbf{A}]^{-1}\mathbf{b}$ $(> \mathbf{0})$ denotes the activity vector required to support one unit of consumption, when $g = 0$.[4]

If we evaluate the physical quantities in terms of 'Standard prices' ([6]), *i.e.*, $\mathbf{y}_{\mathbf{A}}^{\mathrm{T}}[\mathbf{I}-\mathbf{A}]\mathbf{x} = 1$, with $\mathbf{y}_{\mathbf{A}}^{\mathrm{T}}\mathbf{b} = 1$, then (1) implies that[5]

$$c = 1 - (g / g_{\max})$$

or

$$c = 1 - \gamma \tag{3}$$

where $g_{\max} \equiv (1/\lambda_{\mathbf{A}}) - 1$ $(= 1/\lambda_{\mathbf{G}})$ represents the maximum rate of growth and $\gamma \equiv g / g_{\max}$, $0 \le \gamma \le 1$, the 'relative rate of growth'. Substituting (3) in (2) yields

$$\mathbf{x} = \gamma\mathbf{Z}\mathbf{x} + (1-\gamma)\mathbf{f} \tag{4}$$

or, if $\gamma < 1$,

$$\mathbf{x} = (1-\gamma)[\mathbf{I}-\gamma\mathbf{Z}]^{-1}\mathbf{f} = (1-\gamma)\Big(\sum_{k=0}^{\infty}\gamma^k\mathbf{Z}^k\Big)\mathbf{f} \tag{5}$$

where $\mathbf{Z} \equiv g_{\max}\mathbf{G}$ and $\lambda_{\mathbf{Z}} = 1$.

Given that

$$[\hat{\mathbf{q}}_{\mathbf{A}}^{-1}\mathbf{Z}\hat{\mathbf{q}}_{\mathbf{A}}]\mathbf{e} = \hat{\mathbf{q}}_{\mathbf{A}}^{-1}\mathbf{Z}\mathbf{q}_{\mathbf{A}} = \hat{\mathbf{q}}_{\mathbf{A}}^{-1}\mathbf{q}_{\mathbf{A}} = \mathbf{e}$$

it follows that $\mathbf{Z}$ is similar to the *row stochastic* matrix $\mathbf{M} \equiv [m_{ij}] \equiv \hat{\mathbf{q}}_{\mathbf{A}}^{-1}\mathbf{Z}\hat{\mathbf{q}}_{\mathbf{A}}$ $(> \mathbf{0})$, the elements of which are independent of the choice of physical measurement units (and the normalization of $\mathbf{q}_{\mathbf{A}}$). Substituting $\mathbf{Z} = \hat{\mathbf{q}}_{\mathbf{A}}\mathbf{M}\hat{\mathbf{q}}_{\mathbf{A}}^{-1}$, with $\mathbf{y}_{\mathbf{A}}^{\mathrm{T}}[\mathbf{I}-\mathbf{A}]\mathbf{q}_{\mathbf{A}} = 1$, in (4) yields

$$\boldsymbol{\varphi} = \gamma\mathbf{M}\boldsymbol{\varphi} + (1-\gamma)\boldsymbol{\zeta} \tag{6}$$

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[4] It may be noted that the matrix $\mathbf{G}$ is 'dual' to the well-known 'vertically integrated coefficients matrix' ([4]), $\mathbf{H} \equiv \mathbf{A}[\mathbf{I}-\mathbf{A}]^{-1}$, whilst $\mathbf{f}$ is 'dual' to the vector of the 'vertically integrated labour coefficients' (*ibid.*), $\mathbf{v}^{\mathrm{T}} (\equiv \mathbf{l}^{\mathrm{T}}[\mathbf{I}-\mathbf{A}]^{-1})$ (where $\mathbf{l}$ $(> \mathbf{0})$ is the vector of direct labour coefficients). Furthermore, for the significance of the matrix $\mathbf{G}$ in economic theory, see [8].

[5] It is known that the vector of 'Standard prices' is dual to vector of physical quantities that represent Sraffa's 'Standard commodity' (see [6]). Furthermore, the vector of 'Standard prices', $\mathbf{y}_{\mathbf{A}}^{\mathrm{T}}$, corresponds to a particular price vector which expresses a 'pure capital theory of value' ([5], pp. 76-78), as opposed to the well known 'pure labour theory of value'.

or,[6] if $\gamma < 1$,

$$\boldsymbol{\varphi} = [\mathbf{B}(\gamma)]^{-1}\boldsymbol{\zeta} \tag{7}$$

where $\boldsymbol{\varphi} \equiv \hat{\mathbf{q}}_{\mathbf{A}}^{-1}\mathbf{x}$, $\boldsymbol{\zeta} \equiv \hat{\mathbf{q}}_{\mathbf{A}}^{-1}\mathbf{f}$, $\mathbf{B}(\gamma) \equiv (1-\gamma)^{-1}[\mathbf{I} - \gamma\mathbf{M}]$, and $[\mathbf{B}(\gamma)]^{-1}$ is a row stochastic matrix, since $[\mathbf{B}(\gamma)]^{-1} \geq \mathbf{0}$ and

$$[\mathbf{B}(\gamma)]^{-1}\mathbf{e} = (1-\gamma)(1-\gamma)^{-1}\mathbf{e} = \mathbf{e}$$

From relations (6)-(7), and the normalization conditions, we derive the following:

(i). $\boldsymbol{\varphi} = \boldsymbol{\zeta}$ at $\gamma = 0$, and $\boldsymbol{\varphi} = \mathbf{e}$ at $\gamma = 1$. In the trivial case in which $\boldsymbol{\zeta} = \mathbf{e}$, then $\boldsymbol{\varphi} = \mathbf{e}$.[7] Furthermore, since $(\mathbf{y}_{\mathbf{A}}^{\mathrm{T}}\hat{\mathbf{q}}_{\mathbf{A}})(\boldsymbol{\varphi} - \boldsymbol{\zeta}) = 0$ for each γ, it follows that $(\mathbf{y}_{\mathbf{A}}^{\mathrm{T}}\hat{\mathbf{q}}_{\mathbf{A}})(\mathbf{e} - \boldsymbol{\zeta}) = 0$, which in its turn implies

$$\min\{\zeta_i\} \leq 1 \leq \max\{\zeta_i\} \tag{8}$$

(if $\boldsymbol{\zeta} \neq \mathbf{e}$, then both inequalities in (8) are strict).

(ii). Relation (7) implies that $\boldsymbol{\varphi}_i$, $i = 1, 2, ..., n$, is a convex combination of the elements of $\boldsymbol{\zeta}$. Thus, we may write

$$\min\{\zeta_i\} \leq \phi_i \leq \max\{\zeta_i\}$$

or

$$\|\boldsymbol{\varphi}\| = \max\{\phi_i\} \leq \|\boldsymbol{\zeta}\| = \max\{\zeta_i\} \tag{9}$$

and

$$\min\{\zeta_i\} = 1/\left\|\hat{\boldsymbol{\zeta}}^{-1}\right\| \leq \min\{\phi_i\} = 1/\left\|\hat{\boldsymbol{\varphi}}^{-1}\right\| \tag{10}$$

where $\|\bullet\|$ denotes the 'maximum row sum matrix norm'.

(iii). Relation (6) can be restated as

$$(1-\gamma)\boldsymbol{\zeta} = [\mathbf{I} - \gamma\mathbf{M}]\boldsymbol{\varphi} \tag{11}$$

Taking norms of (11), and using the Hölder's inequality, we obtain

$$(1-\gamma)\|\boldsymbol{\zeta}\| \leq \|\boldsymbol{\varphi}\|(\max\{1 - \gamma m_{ii} + \sum_{\substack{j=1 \\ j \neq i}}^{n} \gamma m_{ij}\}),\ i = 1, 2, ..., n \tag{12}$$

or, given that $\sum_{\substack{j=1 \\ j \neq i}}^{n} m_{ij} = 1 - m_{ii}$,

[6] Relation (6) represents a transformed vertically integrated physical quantity system in which the technical coefficients matrix, $\mathbf{M}$, is a row stochastic matrix. Note that $\mathbf{M}$ is dual to the column stochastic technical coefficients matrix, $\mathbf{K}$ $(\equiv \hat{\mathbf{y}}_{\mathbf{A}}\mathbf{J}\hat{\mathbf{y}}_{\mathbf{A}}^{-1})$, (where $\mathbf{J} \equiv R\mathbf{H}$ and R represents the maximum rate of profit) of the transformed vertically integrated price system obtained by Mariolis ([3], relation 6), whilst $\boldsymbol{\zeta}$ is dual to the transformed vector of 'vertically integrated labour coefficients', $\boldsymbol{\omega}^{\mathrm{T}}$ $(\equiv \mathbf{v}^{\mathrm{T}}\hat{\mathbf{y}}_{\mathbf{A}}^{-1})$, that also appears in the transformed price system (*ibid.*).

[7] This is the case in which the proportions of the economic system are those of Sraffa's 'Standard system' (see, *e.g.*, [6], p. 271).

$$(1-\gamma)\|\boldsymbol{\zeta}\| \le \|\boldsymbol{\varphi}\|(\max\{1+\gamma(1-2m_{ii})\})$$

or

$$(1-\gamma)\|\boldsymbol{\zeta}\| \le \|\boldsymbol{\varphi}\|[1+\gamma(1-2\mu)]$$

or

$$f(\gamma) \le \|\boldsymbol{\varphi}\| / \|\boldsymbol{\zeta}\| \tag{13}$$

where $\mu \equiv \min\{m_{ii}\}$, $0<\mu<1$, and $f(\gamma) \equiv (1-\gamma)/[1+\gamma(1-2\mu)]$, $0< f(\gamma) \le 1$, a strictly decreasing function of γ, which is strictly convex to the origin for $\mu < 0.5$ and tends to 1 as μ tends to 1.[8]

(iv). Pre-multiplying (6) by $\hat{\boldsymbol{\varphi}}^{-1}\hat{\boldsymbol{\zeta}}^{-1} (= \hat{\boldsymbol{\zeta}}^{-1}\hat{\boldsymbol{\varphi}}^{-1})$ gives

$$\hat{\boldsymbol{\zeta}}^{-1}\mathbf{e} = \gamma\hat{\boldsymbol{\varphi}}^{-1}\hat{\boldsymbol{\zeta}}^{-1}\mathbf{M}\boldsymbol{\varphi} + (1-\gamma)\hat{\boldsymbol{\varphi}}^{-1}\mathbf{e}$$

Taking norms, and recalling $\|\mathbf{M}\| = 1$, we obtain

$$\|\hat{\boldsymbol{\zeta}}^{-1}\| \le \gamma\|\hat{\boldsymbol{\varphi}}^{-1}\|\|\hat{\boldsymbol{\zeta}}^{-1}\|\|\boldsymbol{\varphi}\| + (1-\gamma)\|\hat{\boldsymbol{\varphi}}^{-1}\|$$

or, dividing both sides by $\|\hat{\boldsymbol{\varphi}}^{-1}\|$ and recalling (9) and (10),

$$\|\hat{\boldsymbol{\zeta}}^{-1}\| / \|\hat{\boldsymbol{\varphi}}^{-1}\| \le \rho(\gamma) \le h(\gamma) \tag{14}$$

where

$$\rho(\gamma) \equiv \gamma(\|\hat{\boldsymbol{\zeta}}^{-1}\|\|\boldsymbol{\varphi}\| - 1) + 1 \ (\ge 1) \tag{14a}$$

$$\rho(1) = \|\hat{\boldsymbol{\zeta}}^{-1}\|\|\boldsymbol{\varphi}(1)\| = \|\hat{\boldsymbol{\zeta}}^{-1}\| / \|\hat{\boldsymbol{\varphi}}^{-1}(1)\| = \|\hat{\boldsymbol{\zeta}}^{-1}\| \tag{14b}$$

(since $\boldsymbol{\varphi} = \mathbf{e}$ at $\gamma = 1$), and

$$1 \le h(\gamma) \equiv \gamma(\|\hat{\boldsymbol{\zeta}}^{-1}\|\|\boldsymbol{\zeta}\| - 1) + 1 \le \|\hat{\boldsymbol{\zeta}}^{-1}\|\|\boldsymbol{\zeta}\| \tag{14c}$$

Combining (9) and (13) gives

$$f(\gamma) \le \|\boldsymbol{\varphi}\| / \|\boldsymbol{\zeta}\| \le 1 \tag{15}$$

whilst combining (10) and (14) gives

$$1 \le \|\hat{\boldsymbol{\zeta}}^{-1}\| / \|\hat{\boldsymbol{\varphi}}^{-1}\| \le h(\gamma) \tag{16}$$

We therefore conclude that the upper (lower) bound for $\|\boldsymbol{\varphi}\| / \|\boldsymbol{\zeta}\|$ (for $\|\hat{\boldsymbol{\zeta}}^{-1}\| / \|\hat{\boldsymbol{\varphi}}^{-1}\|$) equals 1 and the lower (upper) bound decreases (increases) with increasing γ.[9] Furthermore, these bounds are perfectly dual to those of the price system (see [3]). More specifically, we observe the following 'dualities' between the bounds of the two systems: (i) the bounds of the price system are expressed in terms of the 'maximum column sum matrix norm', whilst those of the physical quantity

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[8] It may be noted that the 'condition number' (see, e.g., [1], pp. 399-400) of $\mathbf{B}(\gamma)$, defined as $\|\mathbf{B}(\gamma)\|\|[\mathbf{B}(\gamma)]^{-1}\|$, equals $1/f(\gamma)$.

[9] The monotonicity of $\rho(\gamma)$ is *a priori* unknown.

system are expressed in terms of the 'maximum row sum matrix norm'; and (ii) the bounds of the price system depend on the vertically integrated coefficients that correspond to the transformed vertically integrated price system (*i.e.*, the elements of $\mathbf{K}$ and $\boldsymbol{\omega}^{\mathrm{T}}$) and the relative rate of profit, whilst those of the physical quantity system depend on the vertically integrated coefficients that correspond to the transformed vertically integrated physical quantity system (*i.e.*, the elements of $\mathbf{M}$ and $\boldsymbol{\zeta}$) and the relative rate of growth.

3 Concluding Remarks

It has been shown that, when the quantity side of a Sraffian production system can be transformed (via a diagonal similarity matrix formed from the elements of the right-hand side Perron-Frobenius eigenvector of the technical coefficients matrix) into a vertically integrated system in which the technical coefficients matrix is a row stochastic matrix, the largest and smallest element of the transformed (and expressed in terms of 'Standard prices' ([6])) activity level vector admit lower and upper bounds. The bounds appear, not quite unexpectedly, as perfectly dual to those of Mariolis [3] regarding the price system: (i) they are expressed in terms of the 'maximum row sum matrix norm'; and (ii) depend on the vertically integrated coefficients and the relative rate of growth.

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