Effect of Pollution Tax on Population in a Polluted Environment

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Abstract

This paper establishes a mathematical model to study the effect of pollution tax on the long behavior of the population living in a polluted environment. We give the sufficient condition for persistence of the population under some conditions, and give some numerical simulation.

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1. Introduction

Due to the rapid development of industrialization, presence of toxicant in the environment decreases the growth rate of species and its carrying capacity. It is well known that the pollution of environment is a very serious problem in the world today. The study of the effect of toxicant on population is quite important. One efficient way to control the environment pollution is imposing pollution tax on toxicant emitters. In this paper, we introduce a mathematical model to analyze this problem.

The study of deterministic dynamic population models with toxicant effect was proposed by Hallam and his colleagues in 1980s[1,2,3]. This model has been improved by many scholars[4,5]. He Zerong et al in [6] studied the survival of a single logistic species in a polluted environment, considering the organism's uptake of toxicant from the environment and egestion of toxicant into the environment. In [7], Ma et al studied a Leslie resource-consumer model, and obtained the threshold between the persistence and extinction of the consumer. In [8], the authors improved the results of [7]. In [9], Buonomo et al studied the effect of variation of the population on the toxicant concentration in the

organism and environment. In [10] and [11], we gave some concrete analysis of the ODE system in [9], and improved the results into Gallopin resource-consumer system.

In this paper, we suppose that pollution tax is imposed on toxicant emitters if their emission crosses the permissible limit, limit up to which there is no harm to the population. In this paper, we assume that all the emission cross the permissible limit. What's more, we also consider the negative effect of tax evasion. We obtain the sufficient condition for persistence of the population, and give some numerical simulation.

2 The Model

Our study relies on the hypothesis of a complete spatial homogeneous environment, and no migration. Let $x(t)$ represent the population of the consumer at time t , $C_e(t)$ be the toxicant concentration of the environment at time t , $C_0(t)$ be the toxicant concentration in a body, and $I(t)$ be the environmental tax imposed on the emitters. We assume that the living organisms absorb part of this toxicant into their bodies so that the dynamics of the population is affected by this (internal) toxicant. We will study their relationship, and give some sufficient conditions gurantee the persistence of the population in a polluted environment. The single species model in a polluted environment is given as follows:

$$\dot{x}(t) = x(t)(r_0 - \alpha C_0(t) - f x(t)), \quad (1)$$

$$\dot{C}_0(t) = k C_e(t) - (g + m) C_0(t), \quad (2)$$

$$\dot{C}_e(t) = -k C_e(t) x(t) + g x(t) C_0(t) - h C_e(t) + Q - \rho I(t), \quad (3)$$

$$\dot{I}(t) = \theta(C_e(t) - A) - \theta_0 I(t). \quad (4)$$

The initial values $x(0) = x_0 > 0$, $C_0(0) \geq 0$, $C_e(0) \geq 0$, $I(0) \geq 0$. All the coefficients in the model are positive. r_0 is the intrinsic growth rate in absence of toxicant, α is the response of population to the toxicant, f is a positive constant representing intra specific competition of the population. k represents the uptake rate of the toxicant from the environment by the population. g is the egestion rate of toxicant input to the environment from the population. m represents metabolism rate of the toxicant. h denotes the losses rate of the toxicant in the environment due to natural degradation. Q represents the exogenous rate of toxicant input into the environment beyond the initial toxicant. ρ is the tax repulsion coefficient. Tax is imposed only if C_e crosses the permissible limit A (limit up to which there is no harm to the population). θ and θ_0 are constants, θ_0 denotes some practical difficulties on implementing the fool proof tax system.

3 Main Results

Lemma 1 If $h > k, g+m-gr_0/f > 0$, then the region $\Omega = \{(x(t), C_0(t), C_e(t), I(t)) : 0 \leq x(t) \leq r_0/f, 0 \leq C_0(t) + C_e(t) \leq Q/\delta, 0 \leq I \leq \frac{\theta Q}{\theta_0 \delta}\}$ is the invariant set of model (M_1) here $\delta = \min\{h - k, g + m - gr_0/f\}$.

Proof: $x = 0$ is the solution of equation (1) obviously. If $x_0 > 0$, then $x(t) > 0, t \in R^+$. From

$$\dot{C}_0(t)|_{C_0=0, C_e>0} > 0, \quad \dot{C}_e(t)|_{C_e=0, C_0>0} = gx(t)C_0(t) + Q - \rho I(t) > 0, \quad \dot{I}(t)|_{I=0, C_e(t)>A} > 0,$$

we have

$$x(t) \geq 0, \quad C_e(t) \geq 0, \quad I(t) \geq 0. \quad (5)$$

From equation (1) we get

$$\dot{x}(t) \leq x(t)(r_0 - fx(t)),$$

the comparison theorem gives

$$\limsup_{t \rightarrow \infty} x(t) \leq r_0/f. \quad (6)$$

Equations (2) and (3) give that

$$\begin{aligned} \dot{C}_0(t) + \dot{C}_e(t) &\leq Q + kC_e(t) - gC_0(t) - mC_0(t) - hC_e(t) + gx(t)C_0(t) \\ &\leq Q - (g + m - \frac{gr_0}{f})C_0(t) - (h - k)C_e(t) \\ &< Q - \delta(C_0(t) + C_e(t)). \end{aligned}$$

Then we have

$$\limsup_{t \rightarrow \infty} (C_0(t) + C_e(t)) \leq \frac{Q}{\delta}. \quad (7)$$

From equation (4) we have $\dot{I}(t) \leq \theta C_e(t) - \theta_0 I \leq \theta \frac{Q}{\delta} - \theta_0 I(t)$. Using the comparison theorem again, we can get

$$\limsup_{t \rightarrow \infty} I(t) \leq \frac{\theta Q}{\theta_0 \delta}. \quad (8)$$

From (5), (6), (7), (8) we get Ω is the invariant set of model (M_1) .

Theorem 1 If $\delta > \max\{\frac{\rho\theta}{\theta_0}, \frac{\alpha Q}{r_0}\}$, $\frac{Q(1-\frac{\rho\theta}{\delta\theta_0})}{h+kr_0/f} > A$, then model (M_1) is persistent uniformly.

Proof. From equation (1) we have $\dot{x}(t) \geq x(t)(r_0 - \alpha Q/\delta - fx)$, then

$$\liminf_{t \rightarrow \infty} x(t) \geq (r_0 - \alpha Q/\delta)/f. \quad (9)$$

Equation (3) gives that

$$\dot{C}_e(t) \geq Q - \rho \frac{Q\theta}{\delta\theta_0} - \frac{kr_0}{f} C_e(t) - h C_e(t) = Q(1 - \frac{\rho\theta}{\delta\theta_0}) - (h + \frac{kr_0}{f}) C_e(t),$$

Using the comparison theorem, we can get

$$\liminf_{t \rightarrow \infty} C_e(t) \geq Q(1 - \frac{\rho\theta}{\delta\theta_0}) / (h + \frac{kr_0}{f}). \quad (10)$$

From equation (2) we get

$$\dot{C}_0(t) \geq kQ(1 - \frac{\rho\theta}{\delta\theta_0}) / (h + kr_0/f) - (g + m)C_0(t),$$

then

$$\liminf_{t \rightarrow \infty} C_0(t) \geq \frac{kQ(1 - \frac{\rho\theta}{\delta\theta_0})}{(g + m)(h + kr_0/f)}. \quad (11)$$

Equation (4) implies that

$$\dot{I}(t) \geq \theta(\frac{Q(1 - \frac{\rho\theta}{\delta\theta_0})}{h + kr_0/f} - A) - \theta_0 I(t),$$

then we have

$$\liminf_{t \rightarrow \infty} I(t) \geq \frac{\theta}{\theta_0}(\frac{Q(1 - \frac{\rho\theta}{\delta\theta_0})}{h + kr_0/f} - A). \quad (12)$$

And then from (9), (10), (11), (12) and Lemma 1, we get

$$(r_0 - \alpha Q/\delta)/f \leq \liminf_{t \rightarrow \infty} x(t) \leq \limsup_{t \rightarrow \infty} x(t) \leq r_0/f, \quad (13)$$

$$\frac{kQ(1 - \frac{\rho\theta}{\delta\theta_0})}{(g + m)(h + kr_0/f)} \leq \liminf_{t \rightarrow \infty} C_0(t) \leq \limsup_{t \rightarrow \infty} C_0(t) \leq \frac{Q}{\delta} \quad (14)$$

$$Q(1 - \frac{\rho\theta}{\delta\theta_0}) / (h + \frac{kr_0}{f}) \leq \liminf_{t \rightarrow \infty} C_e(t) \leq \limsup_{t \rightarrow \infty} C_e(t) \leq \frac{Q}{\delta} \quad (15)$$

$$\frac{\theta}{\theta_0}(\frac{Q(1 - \frac{\rho\theta}{\delta\theta_0})}{h + kr_0/f} - A) \leq \liminf_{t \rightarrow \infty} I(t) \leq \limsup_{t \rightarrow \infty} I(t) \leq \frac{\theta Q}{\theta_0 \delta} \quad (16)$$

Then from (13) – (16) we know that (M_1) is persistent uniformly.

4 Numerical Simulation

In this section, we give some examples. Set $r_0 = 1, \alpha = 1, f = 0.5, k = 1, g = 1, m = 2, h = 1, \rho = 0.5, \theta = 2, \theta_0 = 0.5, A = 0.2$, and the initial values are $x(0) = 0.5, C_0(0) = 0.1, C_e(0) = 0.2, I(0) = 0$, then the numerical solution of model (M_1) is as following figure 1. If there is no environmental tax, the solution of model (M_1) is illustrated in figure 2.

Fig. 1. $Q = 1$ Fig. 2. $Q = 1, \rho = 0$

From these plots, we can see that the toxicant concentration of environment increases to 0.5 from 0.3 and the amount of the population decreases to 1.7 from 1.8.

Fig. 3. $Q = 3$ Fig. 4. $Q = 3, \theta = 10$

In fig. 1 and fig. 3, we can find that when Q increases from 1 to 3, which will lead the increase the tax and sharp decrease of the amount of the population. But when the heavy tax ($\theta = 10$) is imposed on the emitters, the balanced amount of the population will recover to 1.8 quickly. θ is the key parameter which need to choose very carefully in order to maintain the equilibrium level of population.

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