System of Set-Valued Variational Inclusions

Rais Ahmad

Department of Mathematics
Aligarh Muslim University
Aligarh-202002, India
raisain_123@rediffmail.com
farhat.math@gmail.com

Abstract

In this paper, we consider and study a new system of nonlinear multivalued variational inclusions in uniformly smooth Banach spaces. An iterative algorithm for computing the approximate solutions of system of nonlinear multivalued variational inclusions is suggested and analyzed. By using the definition of nonexpansive retraction of the resolvent operator we prove the convergence of iterative algorithm for this system of nonlinear multivalued variational inclusions. Some special cases are also discussed.

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1. Introduction

It is well known that variational inequality theory has become a very effective and powerful tool for studying a wide range of problems arising in mechanics, contact problems in elasticity, optimization and control problems, management science, operations research, general equilibrium problems in economics and transportation, unilateral, obstacle, moving etc, see for example and refernce therein [2-5,8,9,13,16].

A useful and important generalization of variational inequalities is variational inclusions. In 1994, Hassouni and moudafi [9] used the resolvent technique for maximal monotone mapping to study a class of mixed type variational inequalities with single-valued mappings which was called variational
inclusions. Since then several authors have obtained some important extensions and generalizations of the results in [9] from various different directions, for example Adly [2], Ding [4-5], Ding and Lou [6], Huang [7], Noor [12], Noor et al. [13].

Using the concept of resolvent operator technique Park and Jeong [15], studied the behaviour and sensitivity analysis of the solution set for a class of parametric generalized variational inequalities with set valued mappings. Noor and Noor [11], introduced and studied resolvent equations and has established the equivalence between the mixed variational inequalities and the resolvent equations. As generalization of system of variational inequalities, Agarwal et al. [1] introduced a system of generalized nonlinear mixed quasi-variational inclusions and investigate the sensitivity analysis of solutions for their system.

This field is dynamics and is experiencing an explosive growth in both theory and applications. So inspired and motivated by the research work going on in this field, in this paper, we consider and study a new system of nonlinear multivalued variational inclusions in uniformly smooth Banach spaces and constructed an iterative algorithm for solving this kind of system of set-valued variational inclusions. We prove the existence of solutions for the system of nonlinear multivalued variational inclusions and the convergence of iterative sequences generated by the proposed algorithm. Some special cases are also given.

2. Preliminaries

Throughout the paper, we assume that $E$ is a real Banach space with its norm $\|\cdot\|$, $E^*$ is the topological dual of $E$, $d$ is the metric induced by the norm $\|\cdot\|$, $CB(E)$ is the family of all nonempty closed and bounded subsets of $E$, $2^E$ is the family of all nonempty subsets of $E$, $D(\cdot,\cdot)$ is the Hausdorff metric on $CB(E)$ defined by

$$D(A, B) = \max\left\{ \sup_{x \in A} d(x, B), \sup_{y \in B} d(A, y) \right\}. $$

where $d(x, B) = \inf_{y \in B} d(x, y)$ and $d(A, y) = \inf_{x \in A} d(x, y)$.

We also assume that $\langle \cdot, \cdot \rangle$ is the duality pairing between $E$ and $E^*$, $D(T)$ is the domain of $T$ and $\mathcal{F} : E \to 2^{E^*}$ is the normalized duality mapping defined by

$$\mathcal{F}(x) = \{ f \in E^* : \langle x, f \rangle = \| x \| \| f \|, \| x \| = \| f \| \text{ for all } x \in E \}.$$

Now we recall some definitions, notations and results which will be used throughout the paper.
The modulus of smoothness of a Banach space $E$ is the function $	au_E : [0, \infty) \rightarrow [0, \infty)$ defined by

$$
\tau_E(t) = \sup \left\{ \frac{\|x + y\| - \|x - y\|}{2} - 1 : \|x\| \leq 1, \|y\| \leq t \right\}.
$$

A Banach space $E$ is called uniformly smooth if

$$
\lim_{t \to 0} \frac{\tau_E(t)}{t} = 0.
$$
Definition 2.1. A mapping \( g : E \to E \) is said to be;

(i) \( k \)-strongly accretive \( k \in (0, 1) \), if for any \( x, y \in E \), there exists \( f(x - y) \in \mathcal{F}(x - y) \) such that
\[
\langle g(x) - g(y), f(x - y) \rangle \geq k\|x - y\|^2;
\]

(ii) Lipschitz continuous, if for any \( x, y \in E \), there exist a constant \( \delta > 0 \) such that
\[
\|g(x) - g(y)\| \leq \delta\|x - y\|.
\]

Definition 2.2. A set-valued mapping \( A : D(A) \subseteq E \to 2^E \) is said to be;

(i) accretive, if for any \( x, y \in D(A) \), there exists \( f(x - y) \in \mathcal{F}(x - y) \) such that
\[
\langle u - v, f(x - y) \rangle \geq 0, \quad \text{for all } u \in Ax, v \in Ay;
\]

(ii) \( k \)-strongly accretive \( k \in (0, 1) \), if for any \( x, y \in D(A) \), there exists \( f(x - y) \in \mathcal{F}(x - y) \) such that for any \( u \in Ax, v \in Ay \),
\[
\langle u - v, f(x - y) \rangle \geq k\|x - y\|^2;
\]

(iii) \( m \)-accretive, if \( A \) is accretive and \((I + \rho A)(D(A)) = E\), for all (equivalently, for some) \( \rho > 0 \), where \( I \) is the identity mapping, (equivalently, if \( A \) is accretive) and \((I + A)(D(A)) = E\).

Remark 2.2. If \( E = H \) is a Hilbert space, then \( A : D(A) \subseteq E \to 2^E \) is an \( m \)-accretive mapping if and only if it is a maximal monotone mapping.

Definition 2.3. Let \( A : D(A) \subseteq E \to 2^E \) be an \( m \)-accretive mapping. For any \( \rho > 0 \), the mapping \( J^A_\rho : E \to D(A) \) associated with \( A \) is defined by,
\[
J^A_\rho(u) = (I + \rho A)^{-1}(u), \quad \text{for all } u \in E,
\]
is called the resolvent operator.

Remark 2.3. It is well known that \( J^A_\rho \) is a single-valued and nonexpansive mapping.

Definition 2.4. The resolvent operator \( J^A_\rho : E \to D(A) \) is said to be;

(i) retraction if \((I + \rho A)^{-1} \circ (I + \rho A)^{-1}(u) = (I + \rho A)^{-1}(u)\) for all \( u \in E \), where \( I \) is the identity operator;
(ii) nonexpansive retraction if
\[ \| J_\rho^A(z_1) - J_\rho^A(z_2) \| \leq \| z_1 - z_2 \| \quad \text{for all} \quad z_1, z_2 \in E. \]

Lemma 2.1. Let \( g : E \to E \) be a continuous and \( k \)-strongly accretive mapping. Then \( g \) maps \( E \) onto \( E \).

Definition 2.5. A set-valued mapping \( T : E \to \text{CB}(E) \) is said to be \( D \)-Lipschitzian continuous if for any \( x, y \in E \) there exist a constant \( \mu > 0 \) such that
\[ D(Tx, Ty) \leq \mu \| x - y \|. \]

Definition 2.6. Let \( T, F : E \to 2^E \) be set-valued mappings. The mapping \( \theta : E \times E \to E \) is said to be;

(i) Lipschitz continuous in first argument with respect to \( T \) if there exists a constant \( \lambda_{\theta_1} > 0 \) such that
\[ \| \theta(w_1, .) - \theta(w_2, .) \| \leq \lambda_{\theta_1} \| w_1 - w_2 \| \]
for all \( w_1 \in T(x_1) \), \( w_2 \in T(x_2) \) and \( x_1, x_2 \in E \);

(ii) Lipschitz continuous in second argument with respect to \( F \) if there exists a constant \( \lambda_{\theta_2} \) such that
\[ \| \theta(., q_1) - \theta(., q_2) \| \leq \lambda_{\theta_2} \| q_1 - q_2 \| \]
for all \( q_1 \in F(x_1) \), \( q_2 \in F(x_2) \), and \( x_1, x_2 \in E \).

The following proposition plays an important role in proving our main results.

Proposition 2.1.\textsuperscript{[16]} Let \( E \) be uniformly smooth Banach space and let \( F : E \to 2^{E^*} \) be a normalized duality mapping. Then for any \( x, y \in E \), the following holds,

(i) \[ \| x + y \|^2 \leq \| x \|^2 + 2 \langle y, f(x + y) \rangle \quad \text{for all} \quad f(x + y) \in \mathcal{F}(x + y); \]

(ii) \[ \langle x - y, f(x) - f(y) \rangle \leq 2d^2 \tau_E(4\| x - y \|/d), \quad \text{where} \quad d = \sqrt{(\| x \|^2 + \| y \|^2)/2}. \]

3. Variational Inclusion System And Iterative Algorithm

In this section, we shall introduce a new system of nonlinear multivalued variational inclusions and construct a new iterative algorithm for solving this kind of system of set-valued variational inclusions in Banach spaces.
Let $E_1$ and $E_2$ be any two real Banach spaces. Let $S : E_1 \times E_2 \to E_1$, $T : E_1 \times E_2 \to E_2$, $p : E_1 \to E_1$ and $q : E_2 \to E_2$ be single-valued mappings, $G : E_1 \to CB(E_1)$, $F : E_2 \to CB(E_2)$, $M : E_1 \times E_1 \to 2^{E_1}$ and $N : E_2 \times E_2 \to 2^{E_2}$ be multivalued mappings, $f : E_1 \to E_1$ and $g : E_2 \to E_2$ be nonlinear mappings with $f(E_1) \cap D(M) \neq \emptyset$ and $g(E_2) \cap D(N) \neq \emptyset$. Then the problem is to find $(x, y) \in E_1 \times E_2$, $u \in G(x)$ and $v \in F(y)$ such that

$$0 \in S(x - p(x), v) + M(f(x), x)$$
$$0 \in T(u, y - q(y)) + N(g(y), y)$$

is called the system of set-valued variational inclusion problem.

Below are some special cases of problem (3.1).

(i) If $x = 2p(x)$, $y = 2q(y)$, $M(f(x), x) = M(f(x))$ and $N(g(y), y) = N(g(y))$, then problem (3.1) reduces to the problem of finding $(x, y) \in E_1 \times E_2$, $u \in G(x)$, $v \in F(y)$ such that

$$0 \in S(p(x), v) + M(f(x)),$$

$$0 \in T(u, q(y)) + N(g(y)).$$

Problem (3.2) is considered by Lan et al. [10] in Hilbert spaces with A-monotone operators.

(ii) If $p(x) = 0 = q(y)$, $M(f(x), x) = M(x)$ and $N(g(y), y) = N(y)$, then problem (3.1) reduces to the problem of finding $(x, y) \in E_1 \times E_2$, $u \in G(x)$, $v \in F(y)$ such that

$$0 \in S(x, v) + M(x),$$
$$0 \in T(u, y) + N(y).$$

Problem (3.3) is considered by Huang and Fang [8] in Hilbert spaces.

First we established the following equivalence between the problem (3.1) and a fixed point problem.

**Lemma 3.1.** $(x, y, u, v)$, where $(x, y) \in E_1 \times E_2$, $u \in G(x)$ and $v \in F(y)$ is a solution of the problem (3.1) if and only if $(x, y, u, v)$ satisfies

$$f(x) = J_{\rho}^{M(x)}(f(x) - \rho S(x - p(x), v))$$
$$g(y) = J_{\gamma}^{N(y)}(g(y) - \gamma T(u, y - q(y)))$$

where $\rho > 0$ and $\gamma > 0$ are constants.

**Proof.** The fact is directly follow from the Definition of resolvent operator.
Algorithm 3.1. Let $S : E_1 \times E_2 \to E_1$, $T : E_1 \times E_2 \to E_2$, $p : E_1 \to E_1$ and $q : E_2 \to E_2$ be single-valued mappings, $G : E_1 \to CB(E_1)$, $F : E_2 \to CB(E_2)$, $M : E_1 \times E_1 \to 2^{E_1}$ and $N : E_2 \times E_2 \to 2^{E_2}$ be multivalued mappings, $f : E_1 \to E_1$ and $g : E_2 \to E_2$ be nonlinear mappings with $f(E_1) \cap D(M) \neq \phi$ and $g(E_2) \cap D(N) \neq \phi$. For any given $(x_o, y_o) \in E_1 \times E_2$, we choose $u_o \in G(x_o)$, $v_o \in F(x_o)$ and let

$$x_1 = x_o - f(x_o) + J^M_{\rho}(x_o)(f(x_o) - \rho S(x_o - p(x_o), v_o)),$$

$$y_1 = y_o - g(y_o) + J^N_{\gamma}(y_o)(g(y_o) - \gamma T(u_o, y_o - q(y_o))).$$

Since $u_o \in G(x_o)$, $v_o \in F(x_o)$ for any $(x_1, y_1) \in E_1 \times E_2$, by Nadler's Theorem [14], there exists $u_1 \in G(x_1)$, $v_1 \in F(x_1)$ such that

$$\|u_0 - u_1\| \leq (1 + 1)D(G(x_0), G(x_1)),$$

$$\|v_0 - v_1\| \leq (1 + 1)D(F(y_0), F(y_1)).$$

let

$$x_2 = x_1 - f(x_1) + J^M_{\rho}(x_1)(f(x_1) - \rho S(x_1 - p(x_1), v_1)),$$

$$y_2 = y_1 - g(y_1) + J^N_{\gamma}(y_1)(g(y_1) - \gamma T(u_1, y_1 - q(y_1))).$$

Continuing the above process inductively, we can obtain the following iterative sequences $\{x_n\}, \{y_n\}, \{u_n\}$ and $\{v_n\}$ for solving problem (3.1) as follows:

$$x_{n+1} = x_n - f(x_n) + J^M_{\rho}(x_n)(f(x_n) - \rho S(x_n - p(x_n), v_n)), \quad (3.3)$$

$$y_{n+1} = y_n - g(y_n) + J^N_{\gamma}(y_n)(g(y_n) - \gamma T(u_n, y_n - q(y_n))), \quad (3.4)$$

$$u_n \in G(x_n), \quad \|u_n - u_{n+1}\| \leq (1 + (n + 1)^{-1}D(G(x_n), G(x_{n+1})), \quad (3.5)$$

$$v_n \in F(y_n), \quad \|v_n - v_{n+1}\| \leq (1 + (n + 1)^{-1}D(F(y_n), F(y_{n+1})). \quad (3.6)$$

$n = 1, 2, 3, \ldots$ and $\rho, \gamma > 0$ are constants.

4. Existence and Convergence Result

Theorem 4.1. Let $E_1$ and $E_2$ be any two real uniformly smooth Banach spaces with module of smoothness $\tau_{E_1}(t) \leq C_1t^2$ and $\tau_{E_2}(t) \leq C_2t^2$ for some $C_1, C_2 > 0$. Let $M : E_1 \times E_1 \to 2^{E_1}$ and $N : E_2 \times E_2 \to 2^{E_2}$ be $m$-accretive mappings, $S : E_1 \times E_2 \to E_1$ and $T : E_1 \times E_2 \to E_2$ be single-valued mappings such that $S$ and $T$ are Lipschitz continuous in first argument with constants $\lambda_{S_1}$ and $\lambda_{T_1}$, respectively; and Lipschitz continuous in second argument with constants $\lambda_{S_2}$ and $\lambda_{T_2}$, respectively. Let $f : E_1 \to E_1$, $g : E_2 \to E_2$, $p : E_1 \to E_1$ and $q : E_2 \to E_2$ be strongly accretive mappings with constants $\delta_f, \delta_g, \delta_p$ and $\delta_q$, respectively; and Lipschitz continuous with constants $\lambda_f$, $\lambda_g$, $\lambda_p$ and $\lambda_q$, respectively such that $f(E_1) \cap D(M) \neq \phi$ and $g(E_2) \cap D(N) \neq \phi$. Let
$G : E_1 \to CB(E_1)$ and $F : E_2 \to CB(E_2)$ be $D$-Lipschitz continuous mappings with constants $\lambda_{DG}$ and $\lambda_{DF}$, respectively. Suppose that there exists constants $\psi, \varphi > 0$ and $\rho, \gamma > 0$ such that for each $x \in E_1$, $y \in E_2$, $x^* \in E^*$

$$\| J^M_{\rho}(x_n)(x^*) - J^M_{\rho}(x_{n-1})(x^*) \| \leq \psi \| x_n - x_{n-1} \|$$

$$\| J^N_{\gamma}(y_n)(x^*) - J^N_{\gamma}(y_{n-1})(x^*) \| \leq \varphi \| y_n - y_{n-1} \|$$

and the following conditions are satisfied:
\[
\sqrt{1 - 2\delta_f + 64C_1\lambda_f^2} + \lambda_f + \psi + \rho \lambda_S \sqrt{1 - 2\delta_p + 64C_1\lambda_p^2} + \gamma \lambda_{T_1} \lambda_{D_G} < 1
\]

\[
\sqrt{1 - 2\delta_q + 64C_1\lambda_q^2} + \lambda_g + \varphi + \gamma \lambda_{T_2} \sqrt{1 - 2\delta_q + 64C_1\lambda_q^2} + \rho \lambda_{S_2} \lambda_{D_F} < 1
\] (4.1)

Then problem (3.1) admits a solutions \((x, y, u, v)\) and the iterative sequences \(\{x_n\}, \{y_n\}, \{u_n\}\) and \(\{v_n\}\) generated by Algorithm (3.1) converge strongly to \(x, y, u\) and \(v\) respectively.

**Proof.** From Algorithm (3.1), nonexpansiveness of the operator \(J_p^M\) and by assumption, we have

\[
\|x_{n+1} - x_n\| = \|x_n - f(x_n) + J_p^M(\cdot, x_n)(f(x_n) - \rho S(x_n - p(x_n), v_n)) - (x_{n-1} - f(x_{n-1}) + J_p^M(\cdot, x_{n-1})(f(x_{n-1}) - \rho S(x_{n-1} - p(x_{n-1}), v_{n-1}))\|
\]

\[
\leq \|x_n - x_{n-1} - (f(x_n) - f(x_{n-1}))\|
\]

\[
+ \|J_p^M(\cdot, x_n)(f(x_n) - \rho S(x_n - p(x_n), v_n)) - J_p^M(\cdot, x_{n-1})(f(x_{n-1}) - \rho S(x_{n-1} - p(x_{n-1}), v_{n-1}))\|
\]

\[
\leq \|x_n - x_{n-1} - (f(x_n) - f(x_{n-1}))\| + \|f(x_n) - \rho S(x_n - p(x_n), v_n) - (f(x_{n-1}) - \rho S(x_{n-1} - p(x_{n-1}), v_{n-1}))\|
\]

\[
+ \|S(x_n - p(x_n), v_n) - S(x_{n-1} - p(x_{n-1}), v_n)\| + \psi \|x_n - x_{n-1}\|
\]

\[
\leq \|x_n - x_{n-1} - (f(x_n) - f(x_{n-1}))\| + \|f(x_n) - f(x_{n-1})\|
\]

\[
+ \|S(x_n - p(x_n), v_n) - S(x_{n-1} - p(x_{n-1}), v_n)\|
\]

\[
+ \psi \|x_n - x_{n-1}\|
\] (4.2)

By proposition 2.1, we have

\[
\|x_n - x_{n-1} - (f(x_n) - f(x_{n-1}))\|^2 \leq (1 - 2\delta_f + 64C_1\lambda_f^2)\|x_n - x_{n-1}\|^2
\] (4.3)

As \(f\) is Lipschitz continuous with constant \(\lambda_f\), we have

\[
\|(f(x_n) - f(x_{n-1})\| \leq \lambda_f \|x_n - x_{n-1}\|
\] (4.4)
By using the Lipschitz continuity of $S$ in second argument and $F$ is $D$-Lipschitz continuous, we have

$$
\|S(x_{n-1} - p(x_{n-1}), v_n) - S(x_{n-1} - p(x_{n-1}), v_{n-1})\|
\leq \lambda_{S_2} \|v_n - v_{n-1}\|
\leq \lambda_{S_2} (1 + n^{-1}) D(F(y_n, F(y_{n-1})))
\leq \lambda_{S_2} \lambda_{D_F} (1 + n^{-1}) \|y_n - y_{n-1}\|
$$

(4.5)

By using the Lipschitz continuity of $S$ in first argument, we have

$$
\|S(x_n - p(x_n), v_n) - S(x_{n-1} - p(x_{n-1}), v_n)\|
\leq \lambda_{S_1} \|x_n - x_{n-1} - (p(x_n) - p(x_{n-1}))\|
$$

(4.6)

Using the same arguments as for (4.3), we have

$$
\|x_n - x_{n-1} - (p(x_n) - p(x_{n-1}))\|^2 \leq (1 - 2\delta_p + 64C_1\lambda_p^2) \|x_n - x_{n-1}\|^2
$$

(4.7)

By (4.6) and (4.7), we have

$$
\|S(x_n - p(x_n), v_n) - S(x_{n-1} - p(x_{n-1}), v_n)\|
\leq \lambda_{S_1} \sqrt{1 - 2\delta_p + 64C_1\lambda_p^2} \|x_n - x_{n-1}\|
$$

(4.8)

By using (4.3)-(4.5), (4.8), (4.2) becomes

$$
\|x_{n+1} - x_n\| \leq \sqrt{1 - 2\delta_f + 64C_1\lambda_f^2} \|x_n - x_{n-1}\| + \lambda_f \|x_n - x_{n-1}\|
+ \rho \lambda_{S_1} \sqrt{1 - 2\delta_p + 64C_1\lambda_p^2} \|x_n - x_{n-1}\|
\leq [\sqrt{1 - 2\delta_f + 64C_1\lambda_f^2} + \lambda_f + \sqrt{1 - 2\delta_p + 64C_1\lambda_p^2} \|x_n - x_{n-1}\|
+ \lambda_{S_1} \lambda_{D_F} (1 + n^{-1}) \|y_n - y_{n-1}\|]
$$

(4.9)

Similarly,

$$
\|y_{n+1} - y_n\| = \|y_n - g(y_n) + J_{\gamma}^{N(y_n)}(g(y_n) - \gamma T(u_n, y_n - q(x_n)))
- [y_{n-1} - g(x_{n-1}) + J_{\gamma}^{N(y_{n-1})}(g(y_{n-1}) - \gamma T(u_{n-1}, y_{n-1} - q(y_{n-1})))\|]
\leq \|y_n - y_{n-1} - (g(y_n) - g(y_{n-1}))\|
+ \|J_{\gamma}^{N(y_n)}(g(y_n) - \gamma T(u_n, y_n - q(y_n)))\|
$$
By using the Lipschitz continuity of $T$ in first argument and $G$ is $D$-Lipschitz continuous, we have
\[
\|T(u_n, y_n - q(x_n)) - T(u_{n-1}, y_n - q(x_n))\| \\
\leq \lambda_{T_1} \|u_n - u_{n-1}\| \\
\leq \lambda_{T_1} (1 + n^{-1})D(G(x_n, G(x_{n-1})) \\
\leq \lambda_{T_1} \lambda_{D_G} (1 + n^{-1})\|x_n - x_{n-1}\| \tag{4.13}
\]

By using the Lipschitz continuity of $T$ in second argument, we have
\[
\|T(u_{n-1}, y_n - q(y_n)) - T(u_{n-1}, y_{n-1} - q(y_{n-1}))\| \\
\leq \lambda_{T_2} \|y_n - y_{n-1} - (q(y_n) - q(y_{n-1}))\| \tag{4.14}
\]

Using the same argument as for (4.3), we have
\[
\|y_n - y_{n-1} - (g(y_n) - g(y_{n-1}))\|^2 \leq (1 - 2\delta_g + 64C_2\lambda_g^2)\|y_n - y_{n-1}\|^2 \tag{4.11}
\]

As $g$ is Lipschitz continuous with constants $\lambda_g$, we have
\[
\|(g(y_n) - g(y_{n-1}))\| \leq \lambda_g \|y_n - y_{n-1}\| \tag{4.12}
\]

By using the Lipschitz continuity of $T$ in first argument and $G$ is $D$-Lipschitz continuous, we have
\[
\|T(u_n, y_n - q(x_n)) - T(u_{n-1}, y_n - q(x_n))\| \\
\leq \lambda_{T_1} \|u_n - u_{n-1}\| \\
\leq \lambda_{T_1} (1 + n^{-1})D(G(x_n, G(x_{n-1})) \\
\leq \lambda_{T_1} \lambda_{D_G} (1 + n^{-1})\|x_n - x_{n-1}\| \tag{4.13}
\]

By using the Lipschitz continuity of $T$ in second argument, we have
\[
\|T(u_{n-1}, y_n - q(y_n)) - T(u_{n-1}, y_{n-1} - q(y_{n-1}))\| \\
\leq \lambda_{T_2} \|y_n - y_{n-1} - (q(y_n) - q(y_{n-1}))\| \tag{4.14}
\]

Using the same argument as for (4.3), we have
\[
\|y_n - y_{n-1} - (g(y_n) - g(y_{n-1}))\|^2 \leq (1 - 2\delta_g + 64C_2\lambda_g^2)\|y_n - y_{n-1}\|^2 \tag{4.15}
\]

By (4.14) and (4.15), we have
\[
\|T(u_{n-1}, y_n - q(y_n)) - T(u_{n-1}, y_{n-1} - q(y_{n-1}))\| \\
\leq \lambda_{T_2} \sqrt{1 - 2\delta_g + 64C_2\lambda_g^2}\|y_n - y_{n-1}\| \tag{4.16}
\]
Using (4.11)-(4.13), (4.16), (4.10) becomes

\[
\|y_{n+1} - y_n\| \leq \sqrt{1 - 2\delta_g + 64C_2\lambda_f^2}\|y_n - y_{n-1}\| + \lambda_g\|y_n - y_{n-1}\|
\]

\[
\gamma\lambda_{T_1}\lambda_{D_C}(1+n^{-1})\|x_n - x_{n-1}\| + \gamma\lambda_{T_2}\sqrt{1 - 2\delta_g + 64C_2\lambda_f^2}\|y_n - y_{n-1}\|
\]

\[
+ \varphi\|y_n - y_{n-1}\|
\]

\[
\leq \left[ \sqrt{1 - 2\delta_g + 64C_2\lambda_f^2}\lambda_f + \lambda_g + \varphi + \gamma\lambda_{T_2}\sqrt{1 - 2\delta_g + 64C_2\lambda_f^2}\|y_n - y_{n-1}\| \right]
\]

\[
\times\|x_n - x_{n-1}\|
\]

\[
+ \left[ \sqrt{1 - 2\delta_g + 64C_2\lambda_f^2}\lambda_g + \varphi + \gamma\lambda_{T_2}\sqrt{1 - 2\delta_g + 64C_2\lambda_f^2}\|y_n - y_{n-1}\| \right]
\]

\[
\times\|y_n - y_{n-1}\|
\]

\[
\leq \theta_n(\|x_n - x_{n-1}\| + \|y_n - y_{n-1}\|)
\]

(4.17)

Equation (4.9) and (4.17) implies that

\[
\|x_{n+1} - x_n\| + \|y_{n+1} - y_n\|
\]

\[
\leq \left[ \sqrt{1 - 2\delta_g + 64C_1\lambda_f^2}\lambda_f + \lambda_g + \psi + \rho\lambda_{S_1}\sqrt{1 - 2\delta_g + 64C_1\lambda_f^2} + \gamma\lambda_{T_1}\lambda_{D_C}(1+n^{-1}) \right]
\]

\[
\times\|x_n - x_{n-1}\|
\]

\[
+ \left[ \sqrt{1 - 2\delta_g + 64C_2\lambda_f^2}\lambda_g + \varphi + \gamma\lambda_{T_2}\sqrt{1 - 2\delta_g + 64C_2\lambda_f^2} + \rho\lambda_{S_2}\lambda_{D_F}(1+n^{-1}) \right]
\]

\[
\times\|y_n - y_{n-1}\|
\]

\[
\leq \theta_n(\|x_n - x_{n-1}\| + \|y_n - y_{n-1}\|)
\]

(4.18)

where

\[
\theta_n = \max\left\{ \sqrt{1 - 2\delta_g + 64C_1\lambda_f^2}\lambda_f + \lambda_g + \psi + \rho\lambda_{S_1}\sqrt{1 - 2\delta_g + 64C_1\lambda_f^2} + \gamma\lambda_{T_1}\lambda_{D_C}(1+n^{-1}), \right. \]

\[
\left. \sqrt{1 - 2\delta_g + 64C_2\lambda_f^2}\lambda_g + \varphi + \gamma\lambda_{T_2}\sqrt{1 - 2\delta_g + 64C_2\lambda_f^2} + \rho\lambda_{S_2}\lambda_{D_F}(1+n^{-1}) \right\}
\]

Let

\[
\theta = \max\left\{ \sqrt{1 - 2\delta_g + 64C_1\lambda_f^2}\lambda_f + \lambda_g + \psi + \rho\lambda_{S_1}\sqrt{1 - 2\delta_g + 64C_1\lambda_f^2} + \gamma\lambda_{T_1}\lambda_{D_C}, \right. \]

\[
\left. \sqrt{1 - 2\delta_g + 64C_2\lambda_f^2}\lambda_g + \varphi + \gamma\lambda_{T_2}\sqrt{1 - 2\delta_g + 64C_2\lambda_f^2} + \rho\lambda_{S_2}\lambda_{D_F} \right\}
\]

Then \(\theta_n \to \theta\) as \(n \to \infty\). By condition (4.1) we know that \(0 < \theta < 1\) and so (4.18) implies that \(\{x_n\}\) and \(\{y_n\}\) are both Cauchy sequences. Thus there exists \(x \in E_1\) and \(y \in E_2\) such that \(x_n \to x\) and \(y_n \to y\) as \(n \to \infty\).
Now we prove that $u_n \to u \in G(x)$ and $v_n \to v \in F(y)$. In fact it follows from (4.5) and (4.13) that $\{u_n\}$ and $\{v_n\}$ are also Cauchy sequences. Let $u_n \to u$ and $v_n \to v$, respectively. We will show that $u \in G(x)$ and $v \in F(y)$.
Since $u_n \in G(x_n)$ and
\[
d(u_n, G(x)) \leq \max \left\{ d(u_n, G(x)), \sup_{v \in G(x)} d(G(x_n), v) \right\}
\]
\[
\leq \max \left\{ \sup_{y \in G(x_n)} d(y, G(x)), \sup_{v \in G(x)} d(G(x_n), v) \right\}
\]
\[= D(G(x_n), G(x))
\]
we have
\[
d(u, G(x)) \leq \|u - u_n\| + d(u_n, G(x))
\]
\[
\leq \|u - u_n\| + D(G(x_n), G(x))
\]
\[
\leq \|u - u_n\| + \lambda_{DC} \|x_n - x\| \to 0, \text{ as } n \to +\infty,
\]
since $G(x)$ is closed, we have $u \in G(x)$. Similarly $v \in F(x)$. By continuity and Algorithm (3.1), we know that $x, y, u,$ and $v$ satisfy the following relation:
\[
f(x) = J^M_{\rho}(f(x) - \rho S(x - p(x), v))
\]
\[
g(y) = J^N_{\gamma}(g(y) - \gamma T(u, y - q(y))
\]
By Lemma (3.1), $(x, y, u, v)$ is a solution of problem (3.1). This completes the proof. □

References


8. N.J. Huang and Y.P. Fang, Fixed point theorem and a new system of multivalued generalized order complementarity problems; *Positivity.*, 7 (2003), 257-265.


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