A Stochastic Model for the Expected Time to Recruitment in a Two Grade Manpower System Having Correlated Inter-Decision Times and Constant Combined Thresholds

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Abstract

In this paper, an organization having two grades in which depletion of manpower occurs at every decision epoch is considered. Using univariate max policy of recruitment, based on shock model approach mathematical model is constructed. It is assumed that the system has constant combined threshold and the inter-decision times are exchangeable and constantly correlated exponential random variable. The mean and variance of the time to recruitment are derived for the above model. The analytical results are substantiated with numerical illustrations.

Mathematics Subject Classification: Primary: 90B70, Secondary: 91B40, 91D35

Keywords: Manpower planning, Univariate recruitment policy, Shock models, correlated inter-decision times, max policy of recruitment ,Mean and variance of the time to recruitment

1. Introduction

Consider an organization having two grades in which the depletion of manpower occurs at every decision epoch. Instead of recruiting people after each occasion of manpower loss, the organization is sustained with the maximum loss of manpower and as and when the maximum loss crosses a constant combined threshold level, it signals the organization to go for recruitment. In [4], [9] and [10] the authors have obtained the mean time to recruitment in a two graded organization using univariate cum policy of recruitment. In [7] and
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the authors have obtained the mean time to recruitment using univariate max policy of recruitment. In all the above cited research works, the threshold levels are not combined and hence only one of the two thresholds can be crossed at a time but not both since transfer of persons are permitted. In [1], [6] and [11] the authors have obtained the mean and variance of the time to recruitment for a two graded manpower system when the thresholds for the loss of manpower in the two grades are combined together. For the combined threshold model, the authors in [1] have obtained the mean and variance of the time to recruitment using univariate cum policy of recruitment when i) the loss of manpower and the thresholds for the two grades are discrete random variables and ii) the inter-decision times for the two grades are i.i.d random variables forming the same renewal process. In [2] the author have obtained the distribution of the maximum of arithmetic mean for correlated random variables and in [5], the mean time to recruitment for correlated renewal sequence is obtained. The objective of the present paper is to obtain the mean and variance of the time to recruitment using univariate max policy of recruitment when inter-decision times are correlated.

2. Model description

Consider an organization with two grades A and B taking decisions at random epochs in \([0, \infty)\) and at every decision making epoch a random number of persons quit the organization. There is an associated loss of manpower if a person quits. It is assumed that the loss of manpower is linear and cumulative. For \(i=1,2,3,\ldots\), let \(X_i^A\) and \(X_i^B\) be the number of exits due to the \(i^{th}\) decision in grades A and B respectively. Let \(\{X_i^A\}\) and \(\{X_i^B\}\) be a sequence of independent and identically distributed random variables with respective distributions \(\phi_A(\cdot)\) and \(\phi_B(\cdot)\). Write \(Z_k^A = \sum_{i=1}^{k} X_i^A\) and \(Z_k^B = \sum_{i=1}^{k} X_i^B\) for the cumulative number of exits in grades A and B in the first \(k\) decisions respectively. It is assumed that the inter-decision times for the two grades are exchangeable and constantly correlated exponential random variable with density function \(f(x) = \lambda e^{-\lambda x}\). Let \(f^*(\cdot)\) be the Laplace transform of \(f(\cdot)\). Let \(c > 0\) be a constant combined threshold level for the two grades.Let R be the correlation between any \(X_i\) and \(X_j, i \neq j\) and \(\psi(n, x) = \int_0^x e^{-\tau^x} \tau^{n-1} d\tau\). Let \(b = a(1 - R)\) where a is a mean of \(X_i, i=1,2,\ldots\) and let \(m = 1/(1 + bs)\). Let \(T\) be a continuous random variable denoting the time to recruitment in the organization with probability density function \(l(\cdot)\) and distribution function \(L(\cdot)\). The recruitment policy employed in this paper is as follows. Recruitment is done whenever the maximum of the maximum total number of exits due to the policy decisions in grades A and B crosses the combined constant threshold level \(c\). Let \(V_k(t)\) be the probability that there are exactly \(k\)-decision epochs in \((0, t]\). Since the number of decisions made in \((0, t]\) form a renewal process, from [2] we note that \(V_k(t) = F_k(t) - F_{k+1}(t)\) where \(F_0(t) = 1\). Let \(E(T)\) and
$V(T)$ be the mean and variance of the time for recruitment respectively.

3. Main Results

The survivor function of $T$ is given by

$$P(T > t) = \sum_{k=0}^{\infty} V_k(t) P(Z^A_k < c) P(Z^B_k < c)$$

$$= 1 - (1 - \phi_A(c)\phi_B(c)) \sum_{k=1}^{\infty} F_k(t)(\phi_A(c)\phi_B(c))^{k-1}$$

Therefore on simplification it can be shown that

$$L(t) = (1 - \phi_A(c)\phi_B(c)) \sum_{k=1}^{\infty} F_k(t)(\phi_A(c)\phi_B(c))^{k-1}$$

(1)

Taking Laplace-Stieltjes transform on both sides

$$L^*(s) = (1 - \phi_A(c)\phi_B(c)) \sum_{k=1}^{\infty} F_k^*(s)(\phi_A(c)\phi_B(c))^{k-1}$$

(2)

As in Gurland [4],

$$F_k(x) = (1 - R) \sum_{i=0}^{\infty} \left[ \frac{(kR)^i}{(1 - R + kR)^{i+1}} \right] \left[ \psi(k+1, x/b) \right]$$

(3)

From (3) it can be shown that

$$F_k^*(s) = \frac{m^k}{1 + \frac{kR}{1 - R}}$$

and

$$\left[ \frac{d}{ds} F_k^*(s) \right]_{s=0} = -ak, \quad \left[ \frac{d^2}{ds^2} F_k^*(s) \right]_{s=0} = a^2((1 + R^2)k^2 + (1 - R^2)k)$$

(4)

We know that

$$E(T^n) = \left[ -\frac{d^n}{ds^n} L^*(s) \right]_{s=0}, \quad r = 1, 2$$

(5)
Using (4) in (5) and (6) and on simplification it can be shown that

\[ E(T) = \frac{a}{1 - \phi_A(c)\phi_B(c)} \]  

(7)

and

\[ V(T) = \frac{a^2}{(1 - \phi_A(c)\phi_B(c))^2} \]  

(8)

(7) and (8) give the mean and variance of the time to recruitment for the present model.

4. Special Case

Case (i): Assume that \(X_A^i\) and \(X_B^i\) follow Poisson distribution with parameter \(\alpha_1\) and \(\alpha_2\) respectively.

Then \(\phi_A(c) = \sum_{k=0}^{c} \frac{e^{-\alpha_1}(\alpha_1)^k}{k!}\) and \(\phi_B(c) = \sum_{k=0}^{c} \frac{e^{-\alpha_2}(\alpha_2)^k}{k!}\)  

(9)

Using (9) in (7) and (8)

\[ E(T) = \frac{a}{1 - \sum_{k=0}^{c} \frac{e^{-\alpha_1}(\alpha_1)^k}{k!} \sum_{k=0}^{c} \frac{e^{-\alpha_2}(\alpha_2)^k}{k!}} = \frac{b}{1 - R} \left[ \frac{1}{1 - \sum_{k=0}^{c} \frac{e^{-\alpha_1}(\alpha_1)^k}{k!} \sum_{k=0}^{c} \frac{e^{-\alpha_2}(\alpha_2)^k}{k!}} \right] \]  

(10)

and

\[ V(T) = \left( \frac{b}{1 - R} \right)^2 \left[ \frac{1}{1 - \sum_{k=0}^{c} \frac{e^{-\alpha_1}(\alpha_1)^k}{k!} \sum_{k=0}^{c} \frac{e^{-\alpha_2}(\alpha_2)^k}{k!}} \right]^2 \]  

(11)

(10) and (11) give the mean and variance of the time to recruitment for case (i).

Case (ii): Assume that \(X_A^i\) and \(X_B^i\) follow geometric distribution with parameter \(\beta_1\) and \(\beta_2\) respectively.

Then \(\phi_A(c) = \sum_{k=1}^{c} \beta_1(1 - \beta_1)^{k-1}\) and \(\phi_B(c) = \sum_{k=1}^{c} \beta_2(1 - \beta_2)^{k-1}\)  

(12)

Using (12) in (7) and (8)

\[ E(T) = \frac{b}{1 - R} \left[ \frac{1}{(1 - \beta_1)^c + (1 - \beta_2)^c - (1 - \beta_1)^c(1 - \beta_2)^c} \right] \]  

(13)
and
\[ V(T) = \left( \frac{b}{1 - R} \right)^2 \left[ \frac{1}{(1 - \beta_1)^c + (1 - \beta_2)^c - (1 - \beta_1)^c(1 - \beta_2)^c} \right]^2 \] \tag{14}

(13) and (14) give the mean and variance of the time to recruitment for case(ii).

**Case(iii):** Assume that \( X_i^A \) and \( X_i^B \) follow exponential distribution with parameter \( \lambda \) and \( \mu \) respectively.

Then \( \phi_A(c) = 1 - e^{-\lambda c} \) and \( \phi_B(c) = 1 - e^{-\mu c} \) \tag{15}

Using (15) in (7) and (8)

\[ E(T) = \frac{b}{1 - R} \left[ \frac{1}{e^{-\lambda c} + e^{-\mu c} - e^c - (\lambda + \mu) c} \right] \] \tag{16}

and

\[ V(T) = \left( \frac{b}{1 - R} \right)^2 \left[ \frac{1}{e^{-\lambda c} + e^{-\mu c} - e^c - (\lambda + \mu) c} \right]^2 \] \tag{17}

(16) and (17) give the mean and variance of the time to recruitment for case(iii).

5. **Numerical Illustrations and Conclusions**

The influence of parameters on the performance measures namely the mean and variance of the time to recruitment are studied numerically.

In the following table these performance measures are calculated by varying the parameters \( R, \alpha_1 \) and \( \alpha_2 \) one at a time and keeping the parameters \( c \) and \( b \) fixed.

Table 1: Effect of \( R, \alpha_1 \) and \( \alpha_2 \) on performance measures

\( (c=90, b=0.7) \)
From the numerical values computed, the following observations are made:

1. As $R$ increases, the correlation between the inter-decision times increases and hence the mean and variance of the time to recruitment increases.

2. As $\alpha_1$ increases, the mean numbers of exits in grade A increases and hence the mean and variance of the time to recruitment increase.

3. As $\alpha_2$ increases, the mean number of exits in grade B increases and hence the mean and variance of the time to recruitment increase.

In the following table the performance measures are calculated by varying the parameters $R$, $\beta_1$ and $\beta_2$ one at a time and keeping the parameters $c$ and $b$ fixed.

<table>
<thead>
<tr>
<th>$R$</th>
<th>$\alpha_1$</th>
<th>$\alpha_2$</th>
<th>$E(W)$</th>
<th>$V(W)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-0.4</td>
<td>0.2</td>
<td>1</td>
<td>0.5647</td>
<td>0.3189</td>
</tr>
<tr>
<td>-0.2</td>
<td>0.2</td>
<td>1</td>
<td>0.6588</td>
<td>0.4340</td>
</tr>
<tr>
<td>0</td>
<td>0.2</td>
<td>1</td>
<td>0.7906</td>
<td>0.6250</td>
</tr>
<tr>
<td>0.2</td>
<td>0.2</td>
<td>1</td>
<td>0.9882</td>
<td>0.9766</td>
</tr>
<tr>
<td>0.4</td>
<td>0.2</td>
<td>1</td>
<td>1.3176</td>
<td>1.7362</td>
</tr>
<tr>
<td>0.4</td>
<td>0.4</td>
<td>1</td>
<td>1.4738</td>
<td>2.1721</td>
</tr>
<tr>
<td>0.4</td>
<td>0.6</td>
<td>1</td>
<td>1.6321</td>
<td>2.6639</td>
</tr>
<tr>
<td>0.4</td>
<td>0.8</td>
<td>1</td>
<td>1.7896</td>
<td>3.2027</td>
</tr>
<tr>
<td>0.4</td>
<td>1</td>
<td>1</td>
<td>1.9427</td>
<td>3.7752</td>
</tr>
<tr>
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<td>1</td>
<td>1.2</td>
<td>2.0897</td>
<td>4.3669</td>
</tr>
<tr>
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<td>1</td>
<td>1.4</td>
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<td>1</td>
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<td>5.5433</td>
</tr>
<tr>
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<td>1</td>
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<td>2.4697</td>
<td>6.0995</td>
</tr>
<tr>
<td>0.4</td>
<td>1</td>
<td>2</td>
<td>2.5730</td>
<td>6.6204</td>
</tr>
</tbody>
</table>

Table 2: Effect of $R$, $\beta_1$ and $\beta_2$ on performance measures

(c=90,b=0.7)
Stochastic model

From the numerical values computed, the following observations are made:

1. As $R$ increases, the average inter-decision time increases and hence the mean and variance of the time to recruitment increases.

2. As $\beta_1$ increases, the mean loss of manpower in grade A decreases and hence the mean and variance of the time to recruitment increase.

3. As $\beta_2$ decreases, the mean loss of manpower in grade B increases and hence the mean and variance of the time to recruitment decrease.

In the following table the performance measures are calculated by varying the parameters $\lambda$ and $\mu$ one at a time and keeping the parameters $c$ and $b$ fixed.

**Table 2: Effect of $R$, $\lambda$ and $\mu$ on performance measures**

(c=90, b=0.7)

<table>
<thead>
<tr>
<th>$R$</th>
<th>$\beta_1$</th>
<th>$\beta_2$</th>
<th>$E(W)$</th>
<th>$V(W)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-0.4</td>
<td>0.2</td>
<td>0.9</td>
<td>2.6355e+008</td>
<td>6.9461e+016</td>
</tr>
<tr>
<td>-0.2</td>
<td>0.2</td>
<td>0.9</td>
<td>3.0748e+008</td>
<td>9.4544e+016</td>
</tr>
<tr>
<td>0</td>
<td>0.2</td>
<td>0.9</td>
<td>3.6898e+008</td>
<td>1.3614e+017</td>
</tr>
<tr>
<td>0.2</td>
<td>0.2</td>
<td>0.9</td>
<td>4.6122e+008</td>
<td>2.1272e+017</td>
</tr>
<tr>
<td>0.4</td>
<td>0.2</td>
<td>0.9</td>
<td>6.1496e+008</td>
<td>3.7818e+017</td>
</tr>
<tr>
<td>0.4</td>
<td>0.4</td>
<td>0.9</td>
<td>1.0798e+020</td>
<td>1.1659e+040</td>
</tr>
<tr>
<td>0.4</td>
<td>0.6</td>
<td>0.9</td>
<td>7.6129e+035</td>
<td>5.7956e+071</td>
</tr>
<tr>
<td>0.4</td>
<td>0.8</td>
<td>0.9</td>
<td>9.4243e+062</td>
<td>8.8817e+125</td>
</tr>
<tr>
<td>0.4</td>
<td>0.8</td>
<td>0.7</td>
<td>1.3367e+047</td>
<td>1.7868e+094</td>
</tr>
<tr>
<td>0.4</td>
<td>0.8</td>
<td>0.5</td>
<td>1.4443e+027</td>
<td>2.0859e+054</td>
</tr>
<tr>
<td>0.4</td>
<td>0.8</td>
<td>0.3</td>
<td>1.0189e+014</td>
<td>1.0381e+028</td>
</tr>
<tr>
<td>0.4</td>
<td>0.8</td>
<td>0.1</td>
<td>1.5315e+004</td>
<td>2.3455e+008</td>
</tr>
</tbody>
</table>
From the above table, we observe the following:

1. As $R$ increases, the correlation between the inter-decision times increases and hence the mean and variance of the time to recruitment increases.

2. As $\lambda$ increases, the mean numbers of exits in grade A decreases and hence the mean and variance of the time to recruitment increase.

3. As $\mu$ increases, the mean number of exits in grade B decreases and hence the mean and variance of the time to recruitment increase.

References


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