Eigenstructure of the Discrete Laplacian on the Equilateral Triangle: The Dirichlet & Neumann Problems

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Abstract

Lamé’s formulas for the eigenvalues and eigenfunctions of the continuous Laplacian on an equilateral triangle under Dirichlet and Neumann boundary conditions are herein extended to the discrete Laplacian.

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1 Introduction

The eigenvalues and eigenfunctions of the continuous Laplacian on an equilateral triangle under Dirichlet and Neumann boundary conditions were first presented by G. Lamé [3, 4, 5] and then further explored by F. Pockels [17]. However, Lamé did not provide a complete derivation of his formulas but rather simply stated them and then proceeded to show that they satisfied the relevant equation and associated boundary conditions. A complete elementary derivation of his formulas was presented in [7] for the Dirichlet problem and in [8] for the Neumann problem.

It is the express purpose of the present work to provide the discrete counterparts to Lamé’s formulas. The motivation for the approach to be taken is as follows. It is well known that, for Dirichlet and Neumann boundary conditions, the eigenvectors of the discrete Laplacian on a square are simply the restriction of the eigenfunctions of the continuous Laplacian to the discrete set of grid points [1, pp. 281-285]. The corresponding eigenvalues are then calculated by applying the discrete Laplacian operator to these eigenvectors.
This naturally suggests the conjecture that the eigenvectors of the discrete Laplacian on an equilateral triangle under Dirichlet and Neumann boundary conditions are none other than the restriction of Lamé’s eigenfunctions to the discrete set of grid points. The truth of this conjecture can be established by simply applying the discrete Laplacian operator to Lamé’s formulas. The corresponding expression for the discrete eigenvalues is a direct byproduct of this procedure. Straightforward Taylor series expansions then yield a relation between the discrete and continuous spectra.

2 The Dirichlet & Neumann Eigenproblems for the Equilateral Triangle

\[ \Delta T(x, y) + k^2 T(x, y) = 0, \quad (x, y) \in \tau; \quad T(x, y) = 0, \quad \frac{\partial T}{\partial \nu} = 0, \quad (x, y) \in \partial \tau \quad (1) \]

where \( \Delta \) is the two-dimensional Laplacian, \( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \), and \( \tau \) is the equilateral triangle with outward pointing normal \( \nu \) shown in Figure 1. Remarkably, he was able to show that the eigenfunctions satisfying Equation (1) could be expressed in terms of combinations of sines and cosines, which are typically the province of rectangular geometries.

Lamé later encountered the same eigenproblem when considering the vibrational modes of an elastic membrane stretched over an equilateral triangle [5]. The identical problem occurs also in acoustic ducts with soft/hard walls and in the propagation of transverse magnetic/electric (TM/TE- or E/H-) modes in electromagnetic waveguides [2].
Motivated by his earlier work in crystallography, Lamé made the following observations the cornerstone of his work on heat transfer in right prisms.

**Theorem 1 (Fundamental Theorem)** Suppose that \( T(x, y) \) can be represented by the trigonometric series

\[
T(x, y) = \sum_i A_i \sin (\lambda_i x + \mu_i y + \alpha_i) + B_i \cos (\lambda_i x + \mu_i y + \beta_i) \quad (2)
\]

with \( \lambda_i^2 + \mu_i^2 = k^2 \), then

- \( T(x, y) \) is antisymmetric about any line along which it vanishes.
- \( T(x, y) \) is symmetric about any line along which its normal derivative, \( \frac{\partial T}{\partial \nu} \), vanishes.

**Proof:** See [7]. \( \Box \)

McCartin contributed the following fundamental observations.

**Lemma 1 (Fundamental Lemma)** Suppose that \( u(x, y) \) can be represented by the finite trigonometric series given by Equation (2), then

- If \( T(x, y) \) vanishes along a line segment \( L' \) then it vanishes along the entire line \( L \) containing \( L' \).
- If \( \frac{\partial T}{\partial \nu}(x, y) \) vanishes along a line segment \( L' \) then it vanishes along the entire line \( L \) containing \( L' \).

**Proof:** See [14]. \( \Box \)

The Fundamental Theorem when combined with the Fundamental Lemma has the following immediate consequences.
Corollary 1 (Fundamental Corollary) With $T(x, y)$ as defined by Equation (2),

- If $T = 0$ along the boundary of a polygon then $T = 0$ along the boundaries of the family of congruent and symmetrically placed polygons obtained by reflection about its sides.

- If $\frac{\partial T}{\partial \nu} = 0$ along the boundary of a polygon then $\frac{\partial T}{\partial \nu} = 0$ along the boundaries of the likewise defined family of polygons.

4 Triangular Coordinate System

![Figure 3: Triangular Coordinate System](image)

Define the **triangular coordinates** $(u, v, w)$ of a point $P$ (Figure 3) by

\[ u = r - y, \]
\[ v = \frac{\sqrt{3}}{2} \cdot (x - \frac{h}{2}) + \frac{1}{2} \cdot (y - r), \tag{3} \]
\[ w = \frac{\sqrt{3}}{2} \cdot (\frac{h}{2} - x) + \frac{1}{2} \cdot (y - r), \]

where $r = h/(2\sqrt{3})$ is the inradius of the triangle. The coordinates $u, v, w$ may be described as the distances of the triangle center to the projections of the point onto the altitudes, measured positively toward a side and negatively toward a vertex.

Note that the triangular coordinates satisfy the relation

\[ u + v + w = 0. \tag{4} \]

Moreover, the center of the triangle has coordinates $(0, 0, 0)$ and the three sides of the triangle are given by $u = r$, $v = r$, and $w = r$, thus simplifying the application of boundary conditions.
5 Continuous Eigenstructure

Before proceeding any further, we will decompose the sought after eigenfunction into parts symmetric and antisymmetric about the altitude $v = w$ (see Figure 4)

$$T(u, v, w) = T_s(u, v, w) + T_a(u, v, w),$$

where

$$T_s(u, v, w) = \frac{T(u, v, w) + T(u, w, v)}{2}; \quad T_a(u, v, w) = \frac{T(u, v, w) - T(u, w, v)}{2},$$

henceforth to be dubbed a symmetric/antisymmetric mode, respectively. We next take up the determination of $T_s$ and $T_a$ separately.

5.1 Dirichlet Modes

The Dirichlet eigenvalues are given by [7]

$$k^2 = \frac{2}{27} \left( \frac{\pi}{r} \right)^2 [l^2 + m^2 + n^2] = \frac{4}{27} \left( \frac{\pi}{r} \right)^2 [m^2 + mn + n^2],$$

where

$$l + m + n = 0,$$

The corresponding symmetric mode is given by

$$\Phi_{s}^{m,n} = \frac{1}{2} \left\{ \sin \left[ \frac{2\pi}{9r} (lu + mv + nw + 3lr) \right] + \sin \left[ \frac{2\pi}{9r} (nu + mv + lw + 3nr) \right] + \sin \left[ \frac{2\pi}{9r} (mu + nv + lw + 3mr) \right] + \sin \left[ \frac{2\pi}{9r} (mu + lv + nw + 3mr) \right] + \sin \left[ \frac{2\pi}{9r} (nu + lv + mw + 3nr) \right] + \sin \left[ \frac{2\pi}{9r} (lu + nv + mw + 3lr) \right] \right\},$$

Figure 4: Modal Line of Symmetry/Antisymmetry
which will vanish identically if and only if any one of \( l, m, n \) is equal to zero.

The corresponding antisymmetric mode is given by

\[
\Phi_{m,n}^a = \frac{1}{2} \left\{ \cos\left[ \frac{2\pi}{9r} (lu + mv + nw + 3lr) \right] - \cos\left[ \frac{2\pi}{9r} (nu + mv + lw + 3nr) \right] \\
+ \cos\left[ \frac{2\pi}{9r} (mu + nv + lw + 3mr) \right] - \cos\left[ \frac{2\pi}{9r} (mu + lv + nw + 3mr) \right] \\
+ \cos\left[ \frac{2\pi}{9r} (nu + lv + mw + 3nr) \right] - \cos\left[ \frac{2\pi}{9r} (lu + nv + mw + 3lr) \right] \right\},
\]

which may be identically zero.

### 5.2 Neumann Modes

The Neumann eigenvalues are given by [8]

\[
k^2 = \frac{2}{27} \left( \frac{\pi}{r} \right)^2 \left[ l^2 + m^2 + n^2 \right] = \frac{4}{27} \left( \frac{\pi}{r} \right)^2 [m^2 + mn + n^2],
\]

where

\[
l + m + n = 0,
\]

The corresponding symmetric mode is given by

\[
\Psi_{m,n}^s = \frac{1}{2} \left\{ \cos\left[ \frac{2\pi}{9r} (lu + mv + nw + 3lr) \right] + \cos\left[ \frac{2\pi}{9r} (nu + mv + lw + 3nr) \right] \\
+ \cos\left[ \frac{2\pi}{9r} (mu + nv + lw + 3mr) \right] + \cos\left[ \frac{2\pi}{9r} (mu + lv + nw + 3mr) \right] \\
+ \cos\left[ \frac{2\pi}{9r} (nu + lv + mw + 3nr) \right] + \cos\left[ \frac{2\pi}{9r} (lu + nv + mw + 3lr) \right] \right\},
\]

which never vanishes identically.

The corresponding antisymmetric mode is given by

\[
\Psi_{m,n}^a = \frac{1}{2} \left\{ \sin\left[ \frac{2\pi}{9r} (lu + mv + nw + 3lr) \right] - \sin\left[ \frac{2\pi}{9r} (nu + mv + lw + 3nr) \right] \\
+ \sin\left[ \frac{2\pi}{9r} (mu + nv + lw + 3mr) \right] - \sin\left[ \frac{2\pi}{9r} (mu + lv + nw + 3mr) \right] \\
+ \sin\left[ \frac{2\pi}{9r} (nu + lv + mw + 3nr) \right] - \sin\left[ \frac{2\pi}{9r} (lu + nv + mw + 3lr) \right] \right\},
\]

which may be identically zero.
5.3 Modal Properties

It is established in [7, 8] that \( T_{s,m,n}^s \) and \( T_{a,m,n}^a \) are invariant under a cyclic permutation of \((l, m, n)\), while \( T_{s,m}^{m,n} = T_{s,n}^{m,n} \) and \( T_{a,n}^{m,n} = -T_{a,m}^{m,n} \) (which are essentially the same since modes are only determined up to a nonzero constant factor). Thus, we need only consider \( n \geq m \). Moreover, since \( T_{s,a}^{m,n}, T_{s,a}^{-n,m+n}, T_{s,a}^{-m,n-m}, T_{s,a}^{-m,n+n} \) and \( T_{s,a}^{-n,-m} \) all produce equivalent modes, we may also neglect negative \( m \) and \( n \).

Furthermore, for the Dirichlet modes, if either \( m \) or \( n \) is equal to zero then both \( T_{s,m,n}^s \) and \( T_{a,m,n}^a \) vanish identically. Hence, we need only consider the collection \( \{T_{s,m,n}^s; T_{a,m,n}^a, n \geq m > 0\} \) for the Dirichlet modes and the collection \( \{T_{s,m,n}^s; T_{a,m,n}^a, n \geq m \geq 0\} \) for the Neumann modes.

We may pare this collection further through the following observations due to Lamé [4].

**Theorem 2 (Modal Properties)**

1. For the Dirichlet problem, \( T_{s,m,n}^s \) vanishes identically if and only if at least one of \( l, m, n \) is equal to zero while, for the Neumann problem, \( T_{s,m,n}^s \) never vanishes identically.

2. For the Dirichlet problem, \( T_{a,m,n}^a \) vanishes identically if and only if either at least one of \( l, m, n \) is equal to zero or if two of them are equal while, for the Neumann problem, \( T_{a,m,n}^a \) vanishes identically if and only if two of \( l, m, n \) are equal.

**Proof:** See [7, 8].

Hence, our system of Dirichlet eigenfunctions is

\[
\{\Phi_{s,m,n}^s (n \geq m > 0); \Phi_{a,m,n}^a (n > m > 0)\}
\]

and our system of Neumann eigenfunctions is

\[
\{\Psi_{s,m,n}^s (n \geq m \geq 0); \Psi_{a,m,n}^a (n > m \geq 0)\}.
\]

It can be shown that, for both the Dirichlet [7] and Neumann [8] problems, the above collections form complete orthonormal sets of eigenfunctions. Furthermore, these same references contain a wealth of information concerning nodal/antinodal lines, rotational symmetry and other modal features as well as give full consideration to the eigenstructure of related geometries. Also, the treatment of multiple eigenvalues (“modal degeneracy”) has been fully explored [9].
6 Discrete Eigenstructure

Armed with the above analysis of the eigenstructure of the continuous Laplacian, we now analyze the eigenstructure of the discrete Laplacian on the equilateral triangle. Specifically, we will consider the discrete triangular lattice of points denoted by solid dots in Figure 5. We will employ the Control Region Approximation [6] to discretize the Laplacian on this lattice.

6.1 Discrete Dirichlet Eigenstructure

Consider the shaded hexagonal control region, $D$, surrounding the interior lattice point $C$ in Figure 6 and, using Equation (3), define

\[
\phi_C = \phi(u, v, w),
\]

\[
\phi_E = \phi(u, v + \hat{h}\sqrt{3}, w - \hat{h}\sqrt{3}),
\]

\[
\phi_{NE} = \phi(u - \hat{h}\sqrt{3}, v + \hat{h}\sqrt{3}, w),
\]

\[
\phi_{NW} = \phi(u - \hat{h}\sqrt{3}, v, w + \hat{h}\sqrt{3}),
\]

\[
\phi_W = \phi(u, v - \hat{h}\sqrt{3}, w + \hat{h}\sqrt{3}),
\]

\[
\phi_{SW} = \phi(u + \hat{h}\sqrt{3}, v - \hat{h}\sqrt{3}, w),
\]

\[
\phi_{SE} = \phi(u + \hat{h}\sqrt{3}, v, w - \hat{h}\sqrt{3}),
\]
where $\hat{h} := h/N$ is the edge length of the component triangles comprising the discrete lattice.

Integrating Equation (1) over the control region $D$ and applying the Divergence Theorem produces

$$\oint_{\partial D} \frac{\partial \phi}{\partial \nu} d\sigma + k^2 \int_D \phi \, dA = 0,$$

(22)

where $\sigma$ denotes arc length around the periphery of $D$.

Approximating the integrals in Equation (22) as described in [6] and denoting the eigenvalue of the resulting discrete operator by $\hat{k}$ yields

$$(\phi_E + \phi_{NE} + \phi_{NW} + \phi_W + \phi_{SW} + \phi_{SE} - 6\phi_C) \cdot \frac{1}{\sqrt{3}} + \hat{k}^2 \phi_C \cdot \hat{h}^2 \frac{\sqrt{3}}{2} = 0,$$

(23)

which may be rearranged as

$$\mathcal{L}[\phi] := \frac{2}{3\hat{h}^2} \cdot (\phi_E + \phi_{NE} + \phi_{NW} + \phi_W + \phi_{SW} + \phi_{SE} - 6\phi_C) = -\hat{k}^2 \phi_C.$$

(24)

Next, denote by $\vec{\phi}$ the vector obtained by evaluating either Equation (9) or Equation (10) on the discrete lattice. This vector automatically satisfies the Dirichlet boundary condition on $\partial \tau$. Applying the interior discrete operator $\mathcal{L}$, defined by Equation (24), to $\vec{\phi}$ and invoking appropriate trigonometric identities leads directly to

$$\hat{k}^2 = \frac{4}{3\hat{h}^2} \cdot \left\{ 3 - \cos \left[ \frac{\sqrt{3} \pi \hat{h}}{9r} \cdot (m - l) \right] - \cos \left[ \frac{\sqrt{3} \pi \hat{h}}{9r} \cdot (n - l) \right] - \cos \left[ \frac{\sqrt{3} \pi \hat{h}}{9r} \cdot (m - n) \right] \right\}.$$

(25)
Now, applying Taylor series and the identity \( l + m + n = 0 \) establishes an important relationship between the Dirichlet spectra of the discrete and continuous Laplacians on the equilateral triangle:

\[
\hat{k}_{m,n}^2 = \hat{k}_{m,n}^2 - \left(\frac{\pi^2}{27r^2}\right)^2 \left(m^4 + 2m^3n + 3m^2n^2 + 2mn^3 + n^4\right) \cdot \hat{h}^2 + O(\hat{h}^4). \tag{26}
\]

The corresponding system of linearly independent Dirichlet eigenvectors is

\[
\{\phi_{m,n}^s \mid N \geq 3 : N - 2 \geq n \geq 1, \min(n, N - 1 - n) \geq m \geq 1\};
\phi_{m,n}^a \mid N \geq 4 : N - 2 \geq n \geq 2, \min(n - 1, N - 1 - n) \geq m \geq 1\}. \tag{27}
\]

### 6.2 Discrete Neumann Eigenstructure

Let \( \vec{\psi} \) denote the vector obtained by evaluating either Equation (13) or Equation (14) on the discrete lattice. Applying the interior discrete operator \( \mathcal{L} \), defined by

\[
\mathcal{L}[\psi] := \frac{2}{3h^2} \cdot (\psi_E + \psi_{NE} + \psi_{NW} + \psi_W + \psi_{SW} + \psi_{SE} - 6\psi_C) = -\hat{k}^2\psi_C, \tag{28}
\]

to \( \vec{\psi} \) and invoking appropriate trigonometric identities once again leads directly to Equation (25) for \( \hat{k}^2 \).

However, this vector does not automatically satisfy the Neumann boundary condition on \( \partial \tau \). Unlike the case of the Dirichlet boundary condition, the Neumann boundary condition requires special treatment which we next provide.

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Figure 7: Edge Control Regions
Consider the shaded control regions of Figure 7 surrounding the lattice points along the edges of $\tau$, exclusive of the vertices. Along the bottom/right/left edges, application of the Control Region Approximation [6] yields

$$\mathcal{L}_1[\psi] := \frac{2}{3h^2} \cdot (\psi_E + 2\psi_{NE} + 2\psi_{NW} + \psi_W - 6\psi_C) = -\hat{k}^2 \psi_C,$$

(29)

$$\mathcal{L}_2[\psi] := \frac{2}{3h^2} \cdot (\psi_{NW} + 2\psi_W + 2\psi_{SW} + \psi_{SE} - 6\psi_C) = -\hat{k}^2 \psi_C,$$

(30)

$$\mathcal{L}_3[\psi] := \frac{2}{3h^2} \cdot (2\psi_E + \psi_{NE} + \psi_{SW} + 2\psi_{SE} - 6\psi_C) = -\hat{k}^2 \psi_C,$$

(31)

respectively.

However, by the Fundamental Corollary, we have the following symmetry relations along the bottom/right/left edges:

$$\psi_{SE} = \psi_{NE}; \quad \psi_{SW} = \psi_{NW},$$

(32)

$$\psi_{E} = \psi_{SW}; \quad \psi_{NE} = \psi_{W},$$

(33)

$$\psi_{NW} = \psi_{E}; \quad \psi_{W} = \psi_{E},$$

(34)

respectively.

Substitution of Equation (32/33/34) into Equation (28) (already known to lead to Equation (25) for $\hat{k}^2$) produces Equation (29/30/31), respectively. Thus, $\vec{\psi}$ satisfies the Neumann boundary condition along the edges of $\tau$, exclusive of the vertices of $\tau$.

Next, consider the shaded control regions of Figure 8 surrounding the three lattice points at the vertices of $\tau$. At the right/top/left vertex, application of
the Control Region Approximation [6] yields

\[ \mathcal{L}_{1,2}[\psi] := \frac{2}{3h^2} \cdot (3\psi_{NW} + 3\psi_W - 6\psi_C) = -\hat{k}^2 \psi_C, \]  

(35)

\[ \mathcal{L}_{2,3}[\psi] := \frac{2}{3h^2} \cdot (3\psi_{SW} + 3\psi_{SE} - 6\psi_C) = -\hat{k}^2 \psi_C, \]  

(36)

\[ \mathcal{L}_{3,1}[\psi] := \frac{2}{3h^2} \cdot (3\psi_E + 3\psi_{NE} - 6\psi_C) = -\hat{k}^2 \psi_C, \]  

(37)

respectively.

However, by the Fundamental Corollary, we have the following symmetry relations at the right/top/left vertex:

\[ \psi_E = \psi_{SW} = \psi_{NW}; \quad \psi_{SE} = \psi_{NE} = \psi_W, \]  

(38)

\[ \psi_{NE} = \psi_W = \psi_{SE}; \quad \psi_{NW} = \psi_E = \psi_{SW}, \]  

(39)

\[ \psi_W = \psi_{SE} = \psi_{NE}; \quad \psi_{SW} = \psi_{NW} = \psi_E, \]  

(40)

respectively.

Substitution of Equation (38/39/40) into Equation (28) (already known to lead to Equation (25) for \( \hat{k}^2 \)) produces Equation (35/36/37), respectively. Thus, \( \vec{\psi} \) also satisfies the Neumann boundary condition at the vertices of \( \tau \).

Hence, Equation (25) provides the spectrum of the discrete Laplacian under both Dirichlet and Neumann boundary conditions. Consequently, Equation (26) also provides the relationship between the Neumann spectra of the discrete and continuous Laplacians on the equilateral triangle.

The corresponding system of linearly independent Neumann eigenvectors is, for \( N \geq 1 \),

\[ \{ \vec{\psi}^{m,n}_s \mid N \geq n \geq 0, \min(n, N-n) \geq m \geq 0 \}; \]

\[ \vec{\psi}^{m,n}_a \mid N \geq n \geq 1, \min(n-1, N-n) \geq m \geq 0 \}. \]  

(41)

7 Conclusion

In the foregoing, Lamé’s formulas for the eigenvalues and eigenfunctions of the continuous Laplacian on an equilateral triangle under Dirichlet and Neumann boundary conditions [15] were extended to the discrete Laplacian. This was accomplished by establishing that the discrete eigenvectors are none other than the restriction of the continuous eigenfunctions to the discrete lattice of Figure 5. In turn, this led directly to an explicit expression for the discrete eigenvalues and an associated relationship between them and their continuous counterparts.
Unfortunately, these interesting results do not extend to the Robin boundary condition [16]. (Recall that the Robin boundary condition subsumes as special cases the radiation [10], the absorbing [11] and the impedance [13] boundary conditions.) It is impossible to define a discrete boundary operator with the usual stencil that is consistent with the Robin boundary condition and which leads to the same expression upon application to the restriction of the continuous eigenfunctions to the discrete lattice as that obtained by applying the discrete interior operator, $L$, as was possible for the Dirichlet and Neumann boundary conditions. (Try it!)

The situation is even more acute in that the same situation obtains for even the one-dimensional Sturm-Liouville boundary value problem [12]. I.e., the device employed in the present paper works admirably for the Dirichlet and Neumann boundary conditions yet fails miserably for the Robin boundary condition. This makes it less surprising that difficulties arise in the two-dimensional boundary value problem considered in the present paper.

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