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On a Fractional Stochastic Landau-Ginzburg Equation

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Abstract

The aim of this paper is to investigate a fractional stochastic Landau-Ginzburg equation for modelling superconductivity from an approximation approach by the fact that a fractional Brownian motion of Liouville form can be approximated by semimartingales in L^2-space.The existence and uniqueness of the solution are proved and its explicit form is found as well.

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1 Introduction

It is well known that the Landau-Ginzburg theory is a mathematical theory used to model superconductivity [4]. This theory examines the macroscopic properties of a superconductor with the aid of general thermodynamic arguments. Landau and Ginzburg established that the free energy of a superconductor near the semiconducting transition can be expressed in terms of an order parameter, this parameter describes how deep into the superconducting phase the system is. The module X of this parameter depending on time t can be considered as the system state satisfying the stochastic Landau-Ginzburg equation of the form

$$dX_t = (-X_t^3 + bX_t)dt + \sigma X_t dW_t\,, \tag{1.1.1}$$

where W_t is a standard Brownian motion, b and σ are some constants.

One observes that the model (1.1.1) does not exactly reflect the superconducting state of the system because the fact that the state X_t at each time t can have a long-time influence upon the system while the solution of (1.1.1) is only a Markov process that is of no-memory. So one can introduce a new model as follows:

$$dX_t = (-X_t^3 + bX_t)dt + \sigma X_t dW_t^H\,, \tag{1.1.2}$$

where the fractional Brownian motion W_t^H with the Hurst index $H, (0 < H < 1)$ is a centered Gaussian process of covariance function $R(s,t)$ given by

$$R(s,t) = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H}).$$

It is known also the the process W_t^H can be represented in the form

$$W_t^H = \frac{1}{\Gamma(H+\frac{1}{2})}\big[U_t + \int_0^t (t-s)^{H-\frac{1}{2}} dW_s\big]\,,$$

where Γ stands for the Gamma function, and $U_t = \int_{-\infty}^{0} \big((t-s)^\alpha - (-s)^\alpha\big) dW_s$ is a process with absolutely continuous paths. One notes that the long-range dependence property focuses at the term $B_t := \int_0^t (t-s)^{H-\frac{1}{2}} dW_s$ that is called the fractional Brownian motion of Liouville form (see [5, 2, 1]).

And in this paper we consider the fractional stochastic Landau-Ginzburg equation given by

$$dX_t = (-X_t^3 + bX_t)dt + \sigma X_t dB_t\,, \tag{1.1.3}$$

where $B_t = \int_0^t (t-s)^\alpha dW_s, \alpha = H - \frac{1}{2}$ and $b = \alpha + \frac{\sigma^2}{2}$.

The solution whose existence is assured as shown later expresses a long-memory state of the superconductivity of the system.

2 An approximation method

Our method is based on a result on approximation of the fractional process $B_t = \int_0^t (t-s)^\alpha dW_s$ by semimartingales given in [1, 6] that we recall below:

For every $\varepsilon > 0$, we define:

$$B_t^\varepsilon = \int_0^t (t-s+\varepsilon)^\alpha dW_s\,, \quad \alpha = H - \frac{1}{2} \in (-\frac{1}{2}, \frac{1}{2})\,. \tag{2.2.1}$$

Then we have

Theorem 2.1. I. *The process $\{B_t^\varepsilon, t \geq 0\}$ is a semimartingale with following decomposition*

$$B_t^\varepsilon = \alpha \int_0^t \varphi_\varepsilon(s) ds + \varepsilon^\alpha W_t\,, \tag{2.2.2}$$

where $\varphi_\varepsilon(t) = \int_0^t (t-s+\varepsilon)^{\alpha-1} dW_s\,.$

II. *The process B_t^ε converges to B_t in $L^2(\Omega)$ when ε tends 0. This convergence is uniform with respect to $t \in [0,T]$.*

Proof. Refer to [6]. □

Next, let us consider the following fractional differential equation in a complete probability space $(\Omega, \mathcal{F}, P)$

$$\begin{cases} dX_t = \big(-X_t^3 + b\,X_t\big)\,dt + \sigma X_t\,dB_t \\ X_t|_{t=0} = X_0 \quad , \quad t \in [0,T] \end{cases} \tag{2.2.3}$$

The solution of (2.2.3) is a stochastic process such that

$$X_t = X_0 + \int_0^t \big(-X_s^3 + b\,X_s\big)\,ds + \sigma \int_0^t X_s\,dB_s \tag{2.2.4}$$

where the fractional stochastic integral $\int_0^t X_s\,dB_s$ will be defined as the L^2-limit of $\int_0^t X_s\,dB_t^\varepsilon$ when $\varepsilon \to 0$, if it exists. The initial value X_0 is a measurable random variable independent of $\{B_t : 0 \leq t \leq T\}$. Our approximation approach to solving (2.2.3) can be describes as follows:

For every $\varepsilon > 0$ we investigate a corresponding approximation equation to (2.2.3)

$$\begin{cases} dX_t^\varepsilon = \big(-(X_t^\varepsilon)^3 + b\, X_t^\varepsilon \big)\, dt + \sigma X_t^\varepsilon\, dB_t^\varepsilon \\ X_t^\varepsilon|_{t=0} = X_0 \end{cases} \tag{2.2.5}$$

where B_t^ε is defined as in (2.2.2). Suppose now that there exists a solution X_t^ε of (2.2.5), then the fact $B_t^\varepsilon \rightrightarrows B_t$ implies that the solution of equation (2.2.3) will be limit in $L^2(\Omega)$ of the solution of (2.2.5) when $\varepsilon \to 0$. Indeed, from (2.2.3) and (2.2.5) we get

$$|X_t^\varepsilon - X_t| \le \int_0^t |b(X_t^\varepsilon) - b(X_t)| ds + \sigma \Big| \int_0^t X_s^\varepsilon\, dB_s^\varepsilon - \int_0^t X_s\, dB_s^\varepsilon \Big| + \sigma \Big| \int_0^t X\, dB_s^\varepsilon - \int_0^t X_s\, dB_s \Big|$$

where $b(x) = -x^3 + bx$. Now by using a localized method it is enough to consider $|X_t^\varepsilon| \le N$ and $|X_t| \le N$ a.s. for some N, and then there exists a positive constant L_N such that

$$|b(X_t^\varepsilon) - b(X_t)| \le L_N |X_t^\varepsilon - X_t|.$$

Therefore, the inequality $(a+b+c)^2 \le 3(a^2+b^2+c^2)$ and an Itô integration lead us to the following estimation

$$\begin{aligned} E|X_t^\varepsilon - X_t|^2 &\le 3L_N^2 \int_0^t E|X_s^\varepsilon - X_s|^2 ds + 3\sigma^2 \int_0^t |X_s^\varepsilon - X_s|^2 d[B^\varepsilon]_s \\ &\quad + 3\sigma^2 E \Big| \int_0^t X\, dB_s^\varepsilon - \int_0^t X_s\, dB_s \Big|^2 \\ &= 3(L_N^2 + \sigma^2 \varepsilon^{2\alpha}) \int_0^t E|X_s^\varepsilon - X_s|^2 ds + c(t,\varepsilon), \end{aligned}$$

where $c(t,\varepsilon) = 3\sigma^2 E|\int_0^t X\, dB_s^\varepsilon - \int_0^t X_s\, dB_s|^2 \to 0$ as $\varepsilon \to 0$ by definition of the fractional stochastic integral. It follows from the latest inequality and from an application of the Gronwall's lemma that

$$E|X_t^\varepsilon - X_t|^2 \le c(t,\varepsilon)\, e^{3(L_N^2 + \sigma^2 \varepsilon^{2\alpha})t}.$$

As a consequence, $X_t^\varepsilon \to X_t$ in $L^2(\Omega)$ when ε tends to zero.

3 Main Results

From (2.2.2) we can rewrite the equations (2.2.5) as follows

$$\begin{cases} dX_t^\varepsilon = \big(-(X_t^\varepsilon)^3 + b\, X_t^\varepsilon + \sigma\alpha\, \varphi^\varepsilon(t) X_t^\varepsilon \big)\, dt + \sigma\varepsilon^\alpha\, X_t^\varepsilon\, dW_t \\ X_t^\varepsilon|_{t=0} = X_0 \end{cases} \tag{3.3.1}$$

The stochastic process $X_t^\varepsilon \varphi_t^\varepsilon$ is not bounded. However, the existence of the solution can be proved as in Theorem 3.1 below and we can establish the uniqueness of the solution of equation (2.2.5) or (3.3.1) by introducing the sequence of stopping times

$$\tau_M = \inf\{t \in [0,T] : \int_0^t (\varphi_s^\varepsilon)^2 ds > M\} \wedge T\,,$$

and considering the sequence of corresponding stopped equations

$$dX_{t\wedge\tau_M}^\varepsilon = \big(-(X_{t\wedge\tau_M}^\varepsilon)^3 + b\, X_{t\wedge\tau_M}^\varepsilon + \sigma\alpha\, \varphi^\varepsilon(t) X_{t\wedge\tau_M}^\varepsilon \big)\, dt + \sigma\varepsilon^\alpha\, X_{t\wedge\tau_M}^\varepsilon\, dW_t. \tag{3.3.2}$$

We can verify the coefficients of (3.3.2) satisfy the local Lipschitz condition. Hence, the uniqueness of the solution is assured (see, for instance, [3]).

Theorem 3.1. *The solution of equation (2.2.5) can be explicitly given by*

$$X_t^\varepsilon = e^{(b-\frac{1}{2}\sigma^2\varepsilon^{2\alpha})t+\sigma B_t^\varepsilon} \left(X_0^{-2} + 2\int_0^t e^{2\left((b-\frac{1}{2}\sigma^2\varepsilon^{2\alpha})s+\sigma B_s^\varepsilon\right)} ds \right)^{\frac{-1}{2}}.$$

Proof. Put

$$Y_t = e^{-\sigma\varepsilon^\alpha W_t}.$$

According to the Itô formula we have:

$$dY_t = Y_t\big(\frac{1}{2}\sigma^2\varepsilon^{2\alpha}dt - \sigma\varepsilon^\alpha dW_t\big). \tag{3.3.3}$$

We consider $Z_t = X_t^\varepsilon Y_t$ and then an application of the integration-by-part formula gives us

$$\begin{aligned} dZ_t &= X_t^\varepsilon dY_t + Y_t dX_t^\varepsilon - \sigma^2\varepsilon^{2\alpha} X_t^\varepsilon Y_t\, dt \\ &= \Big\{ -e^{2\sigma\varepsilon^\alpha W_t}(Z_t)^3 + \big(b + \sigma\alpha\, \varphi^\varepsilon(t) - \frac{1}{2}\sigma^2\, \varepsilon^{2\alpha}\big) Z_t \Big\} dt. \end{aligned} \tag{3.3.4}$$

This is an ordinary Bernouilli equation of the form:

$$Z'_t = P(t)Z_t^3 + Q(t)Z_t$$

and the solution Z_t is given by

$$Z_t = e^{\int_0^t Q(u)du}\left(Z_0 - 2\int_0^t P(s)e^{2\int_0^s Q(u)du} ds\right)^{\frac{-1}{2}}$$

where $P(t) = -e^{2\sigma\varepsilon^\alpha W_t}$, $Q(t) = b - \frac{1}{2}\sigma^2\varepsilon^{2\alpha} + \sigma\alpha\,\varphi^\varepsilon(t)$
and $\int_0^t Q(u)du = (b - \frac{1}{2}\sigma^2\varepsilon^{2\alpha})t + \sigma\alpha\, I(t)\,, I(t) = \int_0^t \varphi^\varepsilon(s)ds\,.$
Hence, the solution Z_t of equation (3.3.4) can be expressed as

$$Z_t = e^{(b-\frac{1}{2}\sigma^2\varepsilon^{2\alpha})t+\sigma\alpha\, I(t)}\left(Z_0 + 2\int_0^t e^{2\left((b-\frac{1}{2}\sigma^2\varepsilon^{2\alpha})s+\sigma\alpha\, I(s)+\sigma\varepsilon^\alpha W(s)\right)} ds\right)^{\frac{-1}{2}}.$$

Combining the latest expression and $B_t^\varepsilon = \alpha I(t) + \varepsilon^\alpha W_t$ we obtain the solution of the approximation equation (2.2.5):

$$X_t^\varepsilon = e^{(b-\frac{1}{2}\sigma^2\varepsilon^{2\alpha})t+\sigma B_t^\varepsilon}\left(X_0^{-2} + 2\int_0^t e^{2\left((b-\frac{1}{2}\sigma^2\varepsilon^{2\alpha})s+\sigma B_s^\varepsilon\right)} ds\right)^{\frac{-1}{2}}.$$

The proof is thus complete. □

Theorem 3.2. *Suppose that X_0 is a random variable such that $X_0 > 0$ a.s and $E[X_0^2] < \infty$. If $H > \frac{1}{2}$ then the stochastic process X_t^* defined by*

$$X_t^* = e^{bt+\sigma B_t}\left(X_0^{-2} + 2\int_0^t e^{2\,(bs+\sigma B_s)} ds\right)^{\frac{-1}{2}} \qquad (3.3.5)$$

is the limit in $L^2(\Omega)$ of X_t^ε. This limit is uniform with respect to $t \in [0,T]$.

Proof. Put $\theta_\varepsilon(t) = e^{(b-\frac{1}{2}\sigma^2\varepsilon^{2\alpha})t+\sigma B_t^\varepsilon}$ and $\theta(t) = e^{bt+\sigma B_t}$ then it is clear that for each $m \geq 1$ there exists a finite constant $M_m > 0$ such that $E[\theta_\varepsilon^m(t)] \leq M_m$, $E[\theta^m(t)] \leq M_m$ for every $t \in [0,T]$. Indeed,

$$E[\theta^m(t)] = e^{mbt}E[e^{m\,\sigma B_t}] = e^{mbt+\frac{1}{4H}m^2\sigma^2 t^{2H}} < \infty,$$

and

$$E[\theta_\varepsilon^m(t)] = e^{m(b-\frac{1}{2}\sigma^2\varepsilon^{2\alpha})t+\frac{1}{4H}m^2\sigma^2[(t+\varepsilon)^{2H}-\varepsilon^{2H}]} < \infty\,.$$

Moreover, applying the Hölder inequality we have following estimates for any $m, k \geq 1$:

$$E[\theta_\varepsilon^m(t)\,\theta^k(t)] \leq \ (E|\theta_\varepsilon(t)|^{2m})^{\frac{1}{2}}\,(E|\theta(t)|^{2k})^{\frac{1}{2}}$$

So there exists a finite constant $M_{m,k} > 0$ such that

$$E[\theta_\varepsilon^m(t)\,\theta^k(t)] \leq M_{m,k}\ \forall\ t \in [0,T]\,. \tag{3.3.6}$$

We now can prove that $\theta_\varepsilon(t) \xrightarrow{L^2} \theta(t)$ uniformly with respect to $t \in [0,T]$, i.e:

$$\lim_{\varepsilon\to 0}\ \sup_{0\leq t\leq T} \|\theta_\varepsilon(t) - \theta(t)\|_2 = 0\,. \tag{3.3.7}$$

Indeed, we see that

$$\begin{aligned}\|\theta_\varepsilon(t) - \theta(t)\|_2 &\leq \|\theta(t)\|_4 \|\exp\big(-\frac{1}{2}\sigma^2\varepsilon^{2\alpha}t + \sigma(B_t^\varepsilon - \mathrm{B}_t)\big) - 1\|_4 \\ &\leq M_4 \|\exp\big(-\frac{1}{2}\sigma^2\varepsilon^{2\alpha}t + \sigma(B_t^\varepsilon - \mathrm{B}_t)\big) - 1\|_4 \end{aligned} \tag{3.3.8}$$

Using the relation $e^x - 1 = x + o(x)$, we obtain

$$\begin{aligned}\|\exp\big(-\frac{1}{2}\sigma^2\varepsilon^{2\alpha}t + \sigma(B_t^\varepsilon - \mathrm{B}_t)\big) - 1\|_4 &\leq \| - \frac{1}{2}\sigma^2\varepsilon^{2\alpha}t + \sigma(B_t^\varepsilon - \mathrm{B}_t)\|_4 + \|o(...)\|_4 \\ &\leq \frac{1}{2}\sigma^2\varepsilon^{2\alpha}T + \|\sigma(B_t^\varepsilon - \mathrm{B}_t)\|_4 + \|o(...)\|_4 \end{aligned} \tag{3.3.9}$$

and thus (3.3.7) follows from Corollary 2.2.

We have also that $\int_0^t \theta_\varepsilon^2(s)ds \xrightarrow{L^2} \int_0^t \theta^2(s)ds$ uniformly with respect to $t \in [0,T]$. Indeed, we have the following estimate:

$$E\left|\int_0^t \theta_\varepsilon^2(s)ds - \int_0^t \theta^2(s)ds\right|^2 \leq t\int_0^t E|\theta_\varepsilon^2(t) - \theta^2(t)|^2 ds\ \forall\ t \in [0,T]\,. \tag{3.3.10}$$

Once again, an application of the Hölder inequality yields for every $t \in [0,T]$

$$E|\theta_\varepsilon^2(t) - \theta^2(t)|^2 = E[|\theta_\varepsilon(t) - \theta(t)|\,A_\varepsilon(t)] \leq \|\theta_\varepsilon(t) - \theta(t)\|_2\,\|A_\varepsilon(t)\|_2 \tag{3.3.11}$$

where $A_\varepsilon(t) = |\theta_\varepsilon(t) - \theta(t)|\big(\theta_\varepsilon(t) + \theta(t)\big)^2$.

Using inequalities of the form (3.3.6) we see that there exists a finite constant $M_3 > 0$ such that

$$\|A_\varepsilon(t)\|_2 \leq M_3\ \forall\ t \in [0,T]\,. \tag{3.3.12}$$

It follows from (3.3.10),(3.3.11) and (3.3.12) that

$$E\Big|\int_0^t \theta_\varepsilon^2(s)ds-\int_0^t \theta^2(s)ds\Big|^2 \le M_3\, t^2 \sup_{0\le t\le T}\|\theta_\varepsilon(t)-\theta(t)\|_2$$
$$\le M_3\, T^2 \sup_{0\le t\le T}\|\theta_\varepsilon(t)-\theta(t)\|_2 \;\forall\; t\in[0,T]\,. \tag{3.3.13}$$

The latest inequality assures that

$$\sup_{0\le t\le T}\|\int_0^t \theta_\varepsilon^2(s)ds-\int_0^t \theta^2(s)ds\|_2 \to 0 \;\text{ as }\; \varepsilon\to 0\,.$$

Put $\eta_\varepsilon(t) = X_0^{-2}+2\int_0^t \theta_\varepsilon^2(s)ds$ and $\eta(t) = X_0^{-2}+2\int_0^t \theta^2(s)ds$. From results above we can see that $\eta_\varepsilon(t) \xrightarrow{L^2} \eta(t)$ uniformly with respect to $t\in[0,T]$. Next we will show that

$$\eta_\varepsilon^{\frac{-1}{2}}(t) \xrightarrow{L^2} \eta^{\frac{-1}{2}}(t) \text{ uniformly with respect to } t\in[0,T]\,. \tag{3.3.14}$$

Indeed, we have $\eta_\varepsilon(t)\ge X_0^{-2}$ and $\eta(t)\ge X_0^{-2}$ a.s for every $t\in[0,T]$.
The theorem of finite increments applied to the function $g(x)=x^{\frac{-1}{2}}$ yields

$$\|\eta_\varepsilon^{\frac{-1}{2}}(t)-\eta^{\frac{-1}{2}}(t)\|_2 \le \frac{1}{2}\|X_0^3(\eta_\varepsilon(t)-\eta(t))\|_2\,.$$

By an argument analogous to the previous one, we get

$$\|\eta_\varepsilon^{\frac{-1}{2}}(t)-\eta^{\frac{-1}{2}}(t)\|_2 \le M\,\|\eta_\varepsilon(t)-\eta(t)\|_2 \;\forall\; t\in[0,T]\,.$$

where $M>0$ is a finite constant. And (3.3.14) follows from this estimate.

As a consequence we have following assertion

$$X_t^\varepsilon = \theta_\varepsilon(t)\eta_\varepsilon^{\frac{-1}{2}}(t) \xrightarrow{L^2} \theta(t)\eta^{\frac{-1}{2}}(t) = X^*(t)\,.$$

The proof of theorem is thus complete. □

Now as proved in Section 2, the process X_t^* is exactly solution of the equation (2.2.3). Then we have

Corollary 3.3. *The solution of the fractional Ginzburg-Landau equation*

$$dX_t = (-X_t^3+(\alpha+\frac{\sigma^2}{2})X_t)\,dt+\sigma X_t\,dB_t\,,$$

is unique and given by

$$X_t = \frac{X_0\, e^{\sigma B_t + (\alpha + \frac{\sigma^2}{2})t}}{\big(1 + 2X_0^2 \int\limits_0^t e^{\sigma B_s + (\alpha + \frac{\sigma^2}{2})s}\big)^{\frac{1}{2}}} .$$

Proof. The uniqueness of the solution can be seen as follows: If $X_t^{*,1}$ and $X_t^{*,2}$ are two limits of X_t^{ε} in L^2, then

$$\|X_t^{*,1} - X_t^{*,2}\| \le \|X_t^{\varepsilon} - X_t^{*,1}\| + \|X_t^{\varepsilon} - X_t^{*,2}\| \;\to\; 0$$

as $\varepsilon \to 0$. □

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