Fixed Point Theorems in Dislocated

Quasi-Metric Space

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Abstract

In this paper we have proved fixed point theorems for continuous contraction mappings in dislocated quasi-metric space. Also we obtain a common fixed point theorem for a pairs of mappings in dislocated metric space.

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1 Introduction

Banach [1922] proved fixed point theorem for contraction mappings in complete metric space. It is well-known as a Banach fixed point theorem. It has many applications in various branches of mathematics such as differential equation, integral equation etc. Since then, many authors have been studying many contractions and proved fixed point theorems.


The object of this note is to prove some fixed point theorems for continuous contraction mappings defined by Dass and Gupta [2] and Rhoades [4], in dislocated quasi-metric spaces.
2 Preliminaries

Definition 2.1[5] Let $X$ be a nonempty set and let $d: X \times X \to [0, \infty)$ be a function satisfying following conditions:

(i) $d(x, y) = d(y, x) = 0$, implies $x = y$,

(ii) $d(x, y) \leq d(x, z) + d(z, y)$, for all $x, y, z \in X$.

Then $d$ is called a dislocated quasi-metric on $X$. If $d$ satisfies $d(x, y) = d(y, x)$, then it is called dislocated metric.

Definition 2.2[5] A sequence $\{x_n\}$ in dq-metric space (dislocated quasi-metric space) $(X, d)$ is called Cauchy sequence if for given $\varepsilon > 0$, $\exists n_0 \in \mathbb{N}$, such that $\forall m, n \geq n_0$, implies $d(x_m, x_n) < \varepsilon$ or $d(x_n, x_m) < \varepsilon$ i.e. $\min\{d(x_m, x_n), d(x_n, x_m)\} < \varepsilon$.

Definition 2.3[5] A sequence $\{x_n\}$ dislocated quasi-converges to $x$ if

$$\lim_{n \to \infty} d(x_n, x) = \lim_{n \to \infty} d(x, x_n) = 0$$

In this case $x$ is called a dq-limit of $\{x_n\}$ and we write $x_n \to x$.

Lemma 2.4[5] dq-limits in a dq-metric space are unique.

Definition 2.5[5] A dq-metric space $(X, d)$ is called complete if every Cauchy sequence in it is a dq-convergent.

Definition 2.6[5] Let $(X, d_1)$ and $(Y, d_2)$ be dq-metric spaces and let $f: X \to Y$ be a function. Then $f$ is continuous to $x_0 \in X$, if for each sequence $\{x_n\}$ which is $d_1$-q convergent to $x_0$, the sequence $\{f(x_n)\}$ is $d_2$-q convergent to $f(x_0)$ in $Y$.

Definition 2.7[5] Let $(X, d)$ be a dq-metric space. A map $T: X \to X$ is called contraction if there exists $0 \leq \lambda < 1$ such that

$$d(Tx, Ty) \leq \lambda d(x, y)$$

for all $x, y \in X$.

Theorem 2.8[5] Let $(X, d)$ be a dq-metric space and let $T: X \to X$ be a continuous contraction mapping. Then $T$ has unique fixed point.
3 Main results

Theorem 3.1 Let \((X,d)\) be a complete dq-metric space and let \(T : X \to X\) be a continuous mapping satisfying the following condition

\[
d(Tx,Ty) \leq \alpha \frac{d(y,Ty)(1 + d(x,Tx))}{1 + d(x,y)} + \beta d(x,y) \quad (3.1)
\]

for all \(x, y \in X\), \(\alpha > 0, \beta > 0, \alpha + \beta < 1\). Then \(T\) has unique fixed point.

Proof. Let \(\{x_n\}\) be a sequence in \(X\), defined as follows.

Let \(x_0 \in X\), \(T(x_0) = x_1\), \(T(x_1) = x_2\), \ldots, \(T(x_n) = x_{n+1}\), \ldots.

Consider

\[
d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n) \leq \alpha \frac{d(x_n, Tx_n)(1 + d(x_{n-1}, Tx_{n-1}))}{1 + d(x_{n-1}, x_n)} + \beta d(x_{n-1}, x_n)
\]

\[
\leq \alpha \frac{d(x_n, x_{n+1})(1 + d(x_{n-1}, x_n))}{1 + d(x_{n-1}, x_n)} + \beta d(x_{n-1}, x_n)
\]

Therefore,

\[
d(x_n, x_{n+1}) \leq \frac{\beta}{1 - \alpha} d(x_{n-1}, x_n)
\]

\[= \lambda d(x_{n-1}, x_n)
\]

where \(\lambda = \frac{\beta}{1 - \alpha}\) with \(0 \leq \lambda < 1\). Similarly, we will show that

\[
d(x_{n-1}, x_n) \leq \lambda d(x_{n-2}, x_{n-1})
\]

and

\[
d(x_{n-1}, x_n) \leq \lambda^2 d(x_{n-2}, x_{n-1}).
\]

Thus

\[
d(x_n, x_{n+1}) \leq \lambda^2 d(x_1, x_0)
\]

Since \(0 \leq \lambda < 1\), as \(n \to \infty\), \(\lambda^n \to 0\). Hence \(\{x_n\}\) is a dq-sequence in the complete dislocated quasi-metric space \(X\). Thus \(\{x_n\}\) dislocated quasi-converges to some \(t_0\). Since \(T\) is continuous, we have
Thus \( T(t_0) = \lim_{n \to \infty} T(x_n) = \lim_{n \to \infty} x_{n+1} = t_0 \).

Uniqueness: Let \( x \) be a fixed point of \( T \). Then by given condition, we have
\[
d(x, x) = d(Tx, Tx) \leq (\alpha + \beta)d(x, x)
\]
Which gives \( d(x, x) = 0 \), since \( 0 \leq (\alpha + \beta) < 1 \) and \( d(x, x) \geq 0 \). Thus \( d(x, x) = 0 \), if \( x \) is fixed point of \( T \).

Let \( x, y \in X \) be fixed point of \( T \). That is, \( Tx = x, Ty = y \).
Then by condition (3.1),
\[
d(x, y) = d(Tx, Ty) \leq \beta d(x, y)
\]
which gives \( d(x, y) = 0 \), since \( 0 \leq \beta < 1 \) and \( d(x, y) \geq 0 \). Similarly \( d(y, x) = 0 \) and hence \( x = y \).
Thus fixed point of \( T \) is unique.

Remark: In Theorem 3.1 if we put \( \alpha = 0 \) we obtain Theorem 2.8.

**Theorem 3.2** Let \((X, d)\) be a complete dq-metric space ant let \( T : X \to X \) be a continuous mapping satisfying the follows condition
\[
d(Tx, Ty) \leq \alpha d(x, Ty) + \beta d(y, Tx) + \gamma d(x, y) \quad (3.2)
\]
where \( \alpha, \beta, \gamma \) are nonnegative, which may depends on both \( x \) and \( y \), such that \( \sup \{2\alpha + 2\beta + \gamma : x, y \in X \} < 1 \). Then \( T \) has unique fixed point.

**Proof.** Let \( \{x_n\} \) be a sequence in \( X \), defined as follows.
Let \( x_0 \in X \), \( T(x_0) = x_1 \), \( T(x_1) = x_2, \ldots, T(x_n) = x_{n+1}, \ldots \).
Consider
\[
d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n) = \alpha d(x_{n-1}, Tx_n) + \beta d(x_n, Tx_{n-1}) + \gamma d(x_{n-1}, x_n)
\]
\[
\leq \alpha d(x_{n-1}, x_n) + \beta d(x_n, x_{n+1}) + \gamma d(x_{n-1}, x_n)
\]
\[
\leq \alpha d(x_{n-1}, x_n) + \alpha d(x_n, x_{n+1}) + \beta d(x_{n-1}, x_n) + \beta d(x_n, x_{n+1}) + \gamma d(x_{n-1}, x_n)
\]

Therefore,

$$d(x_n, x_{n+1}) \leq \frac{\alpha + \beta + \gamma}{1 - (\alpha + \beta)} = \lambda d(x_{n-1}, x_n)$$

where $\lambda = \frac{\alpha + \beta + \gamma}{1 - (\alpha + \beta)}$. Similarly, we have $d(x_{n-1}, x_n) \leq \lambda d(x_{n-2}, x_{n-1})$. In this way, we get

$$d(x_n, x_{n+1}) \leq \lambda^n d(x_0, x_1).$$

Since $0 \leq \lambda < 1$, so for $n \to \infty$, we have $d(x_n, x_{n+1}) \to 0$. Similarly we show that $d(x_{n+1}, x_n) \to 0$. Hence $\{x_n\}$ is a Cauchy sequence in the complete dislocated quasi-metric space $X$, so there is a point $t_0 \in X$, such that $x_n \to t_0$. Since $T$ is continuous, we have

$$T(t_0) = \lim_{n \to \infty} T(x_n) = \lim_{n \to \infty} x_{n+1} = t_0.$$

Thus $T(t_0) = t_0$. Thus $T$ has a fixed point.

**Uniqueness:** Let $x$ be a fixed point of $T$. Then by given condition, we have

$$d(x, x) = d(Tx, Tx) \leq (\alpha + \beta + \gamma)d(x, x)$$

which is true only if $d(x, x) = 0$, since $0 \leq (\alpha + \beta + \gamma) < 1$ and $d(x, x) = 0$. Thus $d(x, x) = 0$ if $x$ is fixed point of $T$.

Let $x, y$ be fixed point of $T$. That is, $Tx = x, Ty = y$. then by given condition, we have

$$d(x, y) = d(Tx, Ty) \leq \alpha d(x, y) + \beta d(y, Ty) + \gamma d(x, x)$$

$$= \alpha d(x, y) + \beta d(y, x) + \gamma d(x, y)$$

$$= (\alpha + \gamma)d(x, y) + \beta d(y, x)$$

Similarly we have

$$d(y, x) \leq (\alpha + \gamma)d(y, x) + \beta d(x, y)$$

Hence $|d(x, y) - d(y, x)| \leq [(\alpha + \gamma) - \beta]|d(x, y) - d(y, x)|$, which implies

$$d(x, y) = d(y, x), \text{ since } 0 \leq [(\alpha + \gamma) - \beta] < 1. \text{ Again from (3.2)}$$

$$d(x, y) \leq (\alpha + \beta + \gamma)d(x, y), \text{ which gives } d(x, y) = 0, \text{ since } 0 \leq (\alpha + \beta + \gamma) < 1.$$
Further \( d(x, y) = d(y, x) = 0 \) gives \( x = y \). Hence fixed point is unique. Hence the proof is completed.

**Theorem 3.3** Let \((X, d)\) be a complete dislocated metric space. Let \(f, g : X \to X\) be continuous mappings satisfying:

\[
d(fx, gy) \leq h \max \left\{ d(x, y), d(x, fx), d(y, gy), \frac{d(x, gy) + d(y, fx)}{2} \right\}
\]

for all \( x, y \in X \) and \( 0 < h < 1 \). Then \( f \) and \( g \) have common fixed point.

**Proof.** Let \( x_0 \in X \). Define the sequence \( \{x_n\} \) by

\[ x_1 = f(x_0), x_2 = g(x_1), \ldots, x_{2n} = g(x_{2n-1}), x_{2n+1} = f(x_{2n}), \ldots \]  

Consider

\[
d(x_{2n+1}, x_{2n+2}) = d(fx_{2n}, gx_{2n+1})
\]

\[
\leq h \max \left\{ d(x_{2n}, x_{2n+1}), d(x_{2n}, fx_{2n}), d(x_{2n+1}, gx_{2n+1}), \frac{d(x_{2n}, gx_{2n+1}) + d(x_{2n+1}, fx_{2n})}{2} \right\}
\]

\[
= h \max \left\{ d(x_{2n}, x_{2n+1}), d(x_{2n}, x_{2n+1}), d(x_{2n}, x_{2n+2}), \frac{d(x_{2n}, x_{2n+2}) + d(x_{2n+1}, x_{2n+1})}{2} \right\}
\]

\[
= hd(x_{2n}, x_{2n+1}).
\]

Therefore

\[
d(x_{2n+1}, x_{2n+2}) \leq hd(x_{2n}, x_{2n+1})
\]

Similarly

\[
d(x_{2n}, x_{2n+1}) \leq hd(x_{2n-1}, x_{2n})
\]

and so

\[
d(x_{2n+1}, x_{2n+2}) \leq h^2d(x_{2n-1}, x_{2n})
\]

In this way we have

\[
d(x_{2n+1}, x_{2n+2}) \leq h^n d(x_0, x_1)
\]

since \( 0 < h < 1 \), as \( h^{2n} \to 0 \) as \( n \to \infty \). Thus \( \{x_n\} \) is a Cauchy sequence in a complete dislocated metric space \( X \). There exists a point \( u \in X \) such that \( x_n \to u \).
Therefore the subsequences \( \{f_{2n}\} \to u \) and \( \{g_{2n+1}\} \to u \). Since \( f \) and \( g \) are continuous function, so we have \( fu = u \) and \( gu = u \).

**Uniqueness of common fixed point:** Let \( u, v \) be a common fixed point of \( f \) and \( g \). Then
\[
d(u,v) \leq d(fu,gv)
\]
\[
\leq h \max \left\{ \frac{d(u,v),d(u,fu),d(v,gv)+d(v,gu)}{2} \right\}
\]
\[
= h \max \left\{ \frac{d(u,v),d(u,u),d(v,v)}{2} \right\}
\]
Replacing \( v \) by \( u \), we get \( d(u,u) \leq hd(u,u) \). Since \( 0 < h < 1 \), we have \( d(u,u) = 0 \). Similarly we have \( d(v,v) = 0 \).

In this way we have \( d(u,v) \leq hd(u,v) \). Since \( 0 < h < 1 \), we have \( d(u,v) = 0 \). Similarly, we have \( d(v,u) = 0 \) and so \( u = v \). Hence the proof is completed.

**References**


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