The Results of Some Experiments on Convection by Surface Heating in a Simple Numerical Model for Small-Scale Processes in the Dry Atmosphere

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Abstract

The model described in this paper represents the flow of dry air in one horizontal dimension of domain length 64 km and the vertical dimension of height 16 km (the height of the tropopause in tropical latitudes). The model equations are forecast equations for the density, the wind, the vertical velocity and the temperature. The horizontal and vertical resolutions are 2 km and 500 m respectively. Predictions through time are by the leapfrog method over two time steps. Four experiments tested the convective circulations and temperature perturbations produced by a heat-island in the middle of the domain with and without cooling each side of the heated interval and the effects of using Euler and Matsuno starting steps. All four experiments gave similar results.
Keywords: Numerical Atmospheric Model/ Finite Difference/ Euler starting step/ Matsuno starting step

1 Introduction

Numerical weather prediction (NWP) models are techniques used to predict the future state of the weather by solving a set of equations which govern the behavior of the atmosphere. A simple numerical model for studying small scale processes in the dry tropical atmosphere is described in this paper [1, 2, 3]. The model equations are derived from the fundamental system of partial differential equations of computational fluid dynamics [4, 5]. The numerical model has two dimensions with length 64 km and height 16 km (from the surface to the tropopause in the tropics). The finite difference approximations to the differential equations are integrated by the leapfrog method. Experiments have been done for a simple heat-island case to investigate the difference between the results obtained starting with a simple Euler step and a Matsuno (backward Euler) step.

2 Differential Equations for the Model

The differential equations for the model are forecast equations for the density, the wind, the vertical velocity and the temperature. Pressure eliminated using the ideal gas equation. The Coriolis force is omitted because the model is for small-scale processes near the equator.

The density forecast equation is derived from the conservation of mass:

\[ \frac{\partial \rho}{\partial t} = -u \frac{\partial \rho}{\partial x} - w \frac{\partial \rho}{\partial z} - \rho \left( \frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} \right) , \]

where \( \rho \) is density, \( t \) is time, \( u \) is the horizontal velocity and \( w \) is the vertical velocity.

The horizontal velocity forecast equation, also called the wind forecast equation, is derived from the horizontal momentum equation:

\[ \frac{\partial u}{\partial t} = -u \frac{\partial u}{\partial x} - w \frac{\partial u}{\partial z} - R \frac{\partial T}{\partial x} - \frac{RT}{\rho} \frac{\partial \rho}{\partial x} + \frac{F_x}{\rho} , \]

where \( R \) is the gas constant for air, \( T \) is the temperature, \( F_x \) is the horizontal friction at the Earth’s surface, given by

\[ \frac{F_x}{\rho} = -\frac{C_D u |u|}{\Delta z} , \]

\[ C_D = \left[ k \ln \left( \frac{0.5 \Delta z}{\Delta x} \right) \right]^2 , \]
where \( k = 0.40 \) (von Karman’s constant), \( z_0 \) is the roughness length of the surface and \( C_D \) is the drag coefficient.

The vertical velocity forecast equation is derived from the vertical momentum equation:

\[
\frac{\partial w}{\partial t} = -u \frac{\partial w}{\partial x} - w \frac{\partial w}{\partial z} - R \frac{\partial T}{\partial z} - \frac{RT}{\rho} \frac{\partial \rho}{\partial z} - g,
\]

(3)

where \( g \) is the acceleration of gravity.

The temperature forecast equation is derived from the energy equation. The energy of a fluid is the sum of the internal (thermal) energy and the kinetic (mechanical) energy. In the model we use only the internal (thermal) energy. Then we can write:

\[
\frac{\partial T}{\partial t} = -u \frac{\partial T}{\partial x} - w \frac{\partial T}{\partial z} - \frac{RT}{C_v} \left[ \frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} \right] + \frac{Q}{C_v},
\]

(4)

where \( Q \) is the heating rate per unit mass and \( C_v \) is the specific heat of the air at constant volume.

3 The Numerical Model

The domain in the vertical direction from the surface up to 16 km is divided into equal intervals \( \Delta z = 500 \text{ m} \), and the horizontal direction from 0 km on the left side to 64 km on the right side is divided into equal intervals \( \Delta x = 2 \text{ km} \). The interior area of the domain is \( 64 \times 16 \text{ km}^2 \) with \( 32 \times 32 \) cells. The cells are labeled \((i,k)\) with \( i = 1 \) to 32 in the \( x \) direction and \( k = 1 \) to 32 in the \( z \) direction. The numerical approximations for the terms in the forecast equations (1), (2), (3) and (4) use the notation in Fig. 1, on the Arakawa and Lamb grid type C.

3.1 Numerical Approximations

Changes in the model variables with time are calculated by the leapfrog method [6]. The model uses three time steps separated in time by the small interval \( \Delta t \).

\[
\frac{\partial \phi}{\partial t} \approx \frac{\phi^{n+1} - \phi^{n-1}}{2\Delta t} = F(\phi^n).
\]

(5)

where \( \phi \) is a vector with four components, namely the density, horizontal velocity, vertical velocity and temperature, and \( F \) represents numerical approximations to the right hand sides of equations (1) - (4).

The numerical approximations for the terms on the right side of the density forecast equation (1) are (see Figure 1) as follows:
Figure 1: Location of the model variables in a cell for the finite difference approximations.

\[
\frac{\partial \rho}{\partial x} \approx \frac{(u_{i,k} + u_{i+1,k})(\rho_{i+1,k} - \rho_{i-1,k})}{4\Delta x}.
\]

At the Earth’s surface:

\[
\frac{\partial \rho}{\partial z} \approx \frac{(w_{i,k} + w_{i,k+1})(-3\rho_{i,k} + 4\rho_{i,k+1} - \rho_{i,k+2})}{4\Delta z},
\]

at intermediate heights:

\[
\frac{\partial \rho}{\partial z} \approx \frac{(w_{i,k} + w_{i,k+1})(\rho_{i,k+1} - \rho_{i,k-1})}{4\Delta z},
\]

and at the tropopause:

\[
\frac{\partial \rho}{\partial z} \approx \frac{(w_{i,k} + w_{i,k+1})(3\rho_{i,k} - 4\rho_{i,k-1} + \rho_{i,k-2})}{4\Delta z},
\]

\[
\frac{\partial u}{\partial x} \approx \frac{\rho_{i,k}(u_{i+1,k} - u_{i,k})}{\Delta x},
\]

\[
\frac{\partial w}{\partial z} \approx \frac{\rho_{i,k}(w_{i,k+1} - w_{i,k})}{\Delta z}.
\]

The numerical approximations for the terms on the right side of the horizontal velocity forecast equation (2) are as follows:

\[
\frac{\partial u}{\partial x} \approx \frac{u_{i,k}(u_{i+1,k} - u_{i-1,k})}{2\Delta x}.
\]

At the Earth’s surface:

\[
\frac{\partial u}{\partial z} \approx \frac{(w_{i,k} + w_{i+1,k})(-3u_{i,k} + 4u_{i,k+1} - u_{i,k+2})}{8\Delta z},
\]
Results of some experiments

at intermediate heights:

\[ w \frac{\partial u}{\partial z} \approx \frac{(w_{i,k} + w_{i-1,k} + w_{i,k+1} + w_{i-1,k+1})(u_{i,k+1} - u_{i,k-1})}{8\Delta z}, \]

and at the tropopause:

\[ w \frac{\partial u}{\partial z} \approx \frac{(w_{i,k} + w_{i-1,k})(3u_{i,k} - 4u_{i,k-1} + u_{i,k-2})}{8\Delta z}, \]

\[ R \frac{\partial T}{\partial x} \approx \frac{R(T_{i,k} - T_{i-1,k})}{\Delta x}, \]

\[ RT \frac{\partial \rho}{\partial x} \approx \frac{R(T_{i,k} + T_{i,k-1})(\rho_{i,k} - \rho_{i,k-1})}{(\rho_{i,k} + \rho_{i-1,k})\Delta x}, \]

\[ \frac{F_x}{\rho} \approx \frac{(C_D)_{i,1}u_{i,k}u_{i,k}}{\Delta z}. \]

The drag coefficient in the surface layer \((k = 1)\) is given by:

\[ (C_D)_{i,k} \approx \left[ \frac{k}{\ln \left( \frac{0.5\Delta z}{z_0} \right)} \right]^2. \]

The numerical approximations for the terms on the right side of the vertical velocity forecast equation (3) are as follows:

\[ u \frac{\partial w}{\partial x} \approx \frac{(u_{i,k} + u_{i,k-1} + u_{i+1,k} + u_{i+1,k-1})(w_{i+1,k} - w_{i-1,k})}{8\Delta x}, \]

\[ w \frac{\partial w}{\partial z} \approx \frac{w_{i,k}(w_{i,k+1} - w_{i,k-1})}{2\Delta z}, \]

\[ R \frac{\partial T}{\partial z} \approx \frac{R(T_{i,k} - T_{i,k-1})}{\Delta z}, \]

\[ RT \frac{\partial \rho}{\partial z} \approx \frac{R(T_{i,k} + T_{i,k-1})(\rho_{i,k} - \rho_{i,k-1})}{(\rho_{i,k} + \rho_{i-1,k})\Delta z}. \]

The numerical approximations for the terms on the right side of the temperature forecast equation (4) are as follows:

\[ u \frac{\partial T}{\partial x} \approx \frac{(u_{i,k} + u_{i+1,k})(T_{i+1,k} - T_{i-1,k})}{4\Delta x}. \]

At the Earth’s surface:

\[ w \frac{\partial T}{\partial z} \approx \frac{(w_{i,k} + w_{i,k+1})(-3T_{i,k} + 4T_{i,k+1} - T_{i,k+2})}{4\Delta z}, \]

at intermediate heights:

\[ w \frac{\partial T}{\partial z} \approx \frac{(w_{i,k} + w_{i,k+1})(T_{i,k+1} - T_{i,k-1})}{4\Delta z}. \]
and at the tropopause:

\[ \frac{w}{\Delta z} \approx \frac{(w_{i,k} + w_{i,k+1})(3T_{i,k} - 4T_{i,k-1} + T_{i,k-2})}{4\Delta z}, \]

\[ \frac{RT}{C_v} \left( \frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} \right) \approx \frac{RT_{i,k}}{C_v} \left( \frac{u_{i+1,k} - u_{i,k}}{\Delta x} + \frac{w_{i,k+1} - w_{i,k}}{\Delta z} \right), \]

\[ \frac{Q}{C_v} \approx \frac{Q_{i,k}}{C_v}. \]

### 3.2 Initial and Boundary Conditions

The initial values of the model variables in each cell are functions of the height of the cell above the Earth’s surface but are constant along the horizontal rows of cells. The initial horizontal velocities \( u_{i,k} \) are all zero. The initial vertical velocities \( w_{i,k} \) are zero at all times on the bottom and top boundaries of the domain. The initial densities \( \rho_{i,k} \) are calculated from the initial temperatures and pressures by the ideal gas equation. The initial temperature \( T_0 \) in the domain are approximations to the annual mean upper air temperatures at Bangkok given by

\[ T_0 = 302 - 0.00675z \]

in kelvins, where \( z \) is the height in meters. The initial pressure \( (p_0)_{i,1} \) in the bottom row of cells is set equal to the annual mean value at Bangkok according to the equation

\[ (p_0)_{i,1} = 100900 - 11.4z \]

in pascals, where \( z \) is the height in meters. Then, we can find the initial densities \( (\rho_0)_{i,1} \) in the bottom row from

\[ (\rho_0)_{i,1} = \frac{(p_0)_{i,1}}{R(T_0)_{i,1}} \]

and the initial densities in the upper air can be calculated assuming the hydrostatic condition by

\[ (\rho_0)_{i,k+1} = (\rho_0)_{i,k} \left( \frac{(T_0)_{i,k} - G}{(T_0)_{i,k+1} + G} \right), \]

where \( G = g\Delta z/2R \).

From a set of starting values \( \phi^1 \) of the model variables in the domain, we calculate a set of values predicted by a single Euler step,

\[ \phi^2 = \phi^1 + \Delta t F(\phi^1). \]

We can also calculate corrected values of the model variables after one Euler step to start the model by the Matsuno (Euler backward) method,

\[ \phi^2 = \phi^1 + \Delta t F \left( \phi^1 + \Delta t F(\phi^1) \right). \]
After calculating $\phi^2$ by the Euler method or the Matsuno method $\phi^3$ is calculated from $\phi^1$ and $\phi^2$ by the leapfrog method.

At the lateral boundaries on the left and right hand sides of the domain the model variables are held constant. The vertical velocities $w$ are zero at all times on the bottom and top boundaries of the domain.

### 3.3 Smoothing in Time and Relaxation at the Boundaries

The leapfrog method has a tendency to give a space dependence in a checkerboard pattern between the even and odd time steps. A smoothing operation is applied to the model variables $\rho$, $u$, $w$ and $T$ to prevent this. The smoothed value is

$$\alpha\phi^1 + (1 - 2\alpha)\phi^2 + \alpha\phi^3,$$

where $\alpha$ is a smoothing factor with default value 0.05.

Fixed boundary values produce reflections of waves at the boundary. To reduce these reflections predicted values $\phi^*$ of the model variables next to the boundary are replaced after each time step by

$$\phi^2 = \frac{1}{2}(\bar{\phi} + \phi^*),$$

where $\bar{\phi}$ is the boundary value. This is the relaxation method of Jones et al. as quoted in [7].

### 4 Experiments

Four different experiments have been done to study the modeling of convection in dry air and effect of the Euler and Matsuno starting conditions. In the interval at the surface of the Earth of length 16 km in the middle of the domain the heating rate was 500 Wm$^{-2}$. The time steps $\Delta t$ in all these experiments were 0.2s.

In the first two experiments there was no other heating. In the third and fourth experiments it was assumed that cooling at 166.66 Wm$^{-2}$ occurred over the remaining part of the bottom boundary, of total length 48 km (24 km each side of the middle section), so that the net heat input over the bottom boundary was zero. The surface was assumed to be smooth, with roughness length $z_0 = 0.0001$ m.

The first and third experiments started with a simple Euler step. The remaining two experiments started with a Matsuno step.
5 Results

Figures 2-5 show qualitatively the air velocity vectors and the temperature perturbations $T - T_0$, at 5, 10 and 20 minutes after the start of the model run. Plus and minus symbols show the sign of the temperature perturbations. The velocities plotted in each cell have components

\[
\left[ \frac{u_{i,k} + u_{i+1,k}}{2}, \frac{w_{i,k} + w_{i,k+1}}{2} \right].
\]

The whole domain is shown on the left side of these figures. The middle part of the domain is shown from 16 km to 50 km, and from the Earth’s surface to a height 4 km on the right side.

<table>
<thead>
<tr>
<th>Table 1: Results for experiments 1 and 2: Surface heating in the middle and no surface cooling.</th>
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<tbody>
<tr>
<td>time (s)</td>
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<tr>
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<th>Table 2: Results for experiments 3 and 4: Surface heating in the middle and surface cooling at the sides.</th>
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In the first kilometer above the heated interval there is a positive temperature perturbation and rising air due to its buoyancy in the surrounding cooler
air. Above 1 km adiabatic cooling of the rising air produces a small negative temperature perturbation which reaches about 10 km after about 10 min. Outside the heated interval the air near the surface descends and recirculates into the rising air column. The effect of the surface cooling outside the heated interval is very small.

Tables 1 and 2 compare the maximum speeds and temperature perturbations in the four experiments. These results show that the starting methods have a small effect on the maximum speeds, but no effect on the maximum temperature perturbations. In contrast, the surface cooling has a small effect on the maximum temperature perturbations, but no effect on the maximum speeds.
Figure 2: Air movement and temperature perturbation contours after 5 min, 10 min and 20 min in Experiment 1 (Surface heating only with Euler start).
Figure 3: Air movement and temperature perturbation contours after 5 min, 10 min and 20 min in Experiment 2 (Surface heating only with Matuno start).
Figure 4: Air movement and temperature perturbation contours after 5 min, 10 min and 20 min in Experiment 3 (Surface heating and cooling with Euler start).
Figure 5: Air movement and temperature perturbation contours after 5 min, 10 min and 20 min in Experiment 4 (Surface heating and cooling with Matsumo start).
6 Conclusion

Our results have shown the air movements and the temperature perturbations in a set of numerical heat-island experiments. We also show that there is only a small difference between the models using Euler starting and Matsuno starting steps. Therefore the Euler starting method will be used in future work because it is simpler.

References


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