A Note on Proper Affine Symmetry in Bianchi Types $\text{VI}_0$ and $\text{VII}_0$ Space-Times

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Abstract
A study proper affine symmetry in the most general form of the Bianchi types $\text{VI}_0$ and $\text{VII}_0$ space-times is given by using holonomy and decomposability, the rank of the $6 \times 6$ Riemann matrix and direct integration techniques. It is shown that the very special classes of the above space-times admit proper affine symmetry.

Keywords: Affine vector fields, holonomy and decomposability, direct integration technique.

1 INTRODUCTION
In this paper we will explore all the possibilities when the Bianchi types $\text{VI}_0$ and $\text{VII}_0$ space-times admit proper affine symmetry. We use holonomy and decomposability, the rank of the $6 \times 6$ Riemann matrix and direct integration techniques to study proper affine symmetry in the above space-times. Throughout
$M$ represents a four dimensional, connected, Hausdorff space-time manifold with Lorentz metric $g$ of signature ($-, +, +, +$). The curvature tensor associated with $g_{ab}$ through the Levi-Civita connection, is denoted in component form by $R^a_{bcd}$. The usual covariant, partial and Lie derivatives are denoted by a semicolon, a comma and the symbol $L$, respectively. Round and square brackets denote the usual symmetrization and skew-symmetrization, respectively. Here, $M$ is assumed non-flat in the sense that the curvature tensor does not vanish over any non-empty open subset of $M$.

A vector field $X$ on $M$ is called an affine vector field if it satisfies
\[ X_{a;bc} = R_{abcd} X^d, \tag{1} \]
where $R_{abcd} = g_{af} R^f_{bd} = g_{af} (\Gamma^f_{bd,c} - \Gamma^f_{bc,d} + \Gamma^f_{ce} \Gamma^e_{bd} - \Gamma^f_{ed} \Gamma^e_{bc})$. If one decomposes $X_{ab}$ on $M$ into its symmetric and skew-symmetric parts
\[ X_{a,b} = \frac{1}{2} H_{ab} + G_{ab}, \quad (H_{ab} (\equiv X_{a;b} + X_{b;a}) = H_{ba}, \ G_{ab} = -G_{ba}) \tag{2} \]
then equation (1) is equivalent to
\[ (i) \ H_{abc} = 0 \quad (ii) \ G_{ab;e} = R_{abcd} X^d \quad (iii) \ G_{ab;e} X^e = 0. \tag{3} \]
The proof of the above equation (1) implies (3) or equations (3) implies (1) can be found in [2, 3]. If $H_{ab} = 2cg_{ab}, c \in R$, then the vector field $X$ is called homothetic (and Killing if $c = 0$). The vector field $X$ is said to be proper affine if it is not homothetic vector field and also $X$ is said to be proper homothetic vector field if it is not Killing vector field on $M$ [2]. Define the subspace $S_p$ of the tangent space $T_pM$ to $M$ at $p$ as those $k \in T_pM$ satisfying
\[ R_{abcd} k^d = 0. \tag{4} \]

### 2 Affine Vector Fields

Suppose that $M$ is a simple connected space-time. Then the holonomy group of $M$ is a connected Lie subgroup of the identity component of the Lorentz group and is thus characterized by its subalgebra in the Lorentz algebra. These have been labeled into fifteen types $R_1 - R_{15}$ [1]. It follows from [2] that the only such space-times which could admit proper affine vector fields are those which admit nowhere zero covariantly constant second order symmetric tensor field $H_{ab}$. This forces the holonomy type to be either $R_2, R_3, R_4, R_6, R_7, R_8, R_{10}, R_{11}$ or $R_{13}$ [2]. A study of the affine vector fields for the above holonomy type can be found in [2]. It follows from [4] that the rank of the $6 \times 6$ Riemann matrix of the above space-times which have holonomy type $R_2, R_3, R_4, R_6, R_7,$
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R₈, R₁₀, R₁₁ or R₁₃ is at most three. Hence for studying affine vector fields we are interested in those cases when the rank of the 6×6 Riemann matrix is less than or equal to three.

3 Main Results

Consider Bianchi types VI₀ and VII₀ space-times in usual coordinate system (t, x, y, z) (labeled by (x⁰, x¹, x², x³), respectively) with line element [5]

\[ ds^2 = -dt^2 + \left(A f^2(z) + B h^2(z)\right)dx^2 + \left(A h^2(z) + B f^2(z)\right)dy^2 + 2\left(A + B\right)f(z)h(z)dz, \]

where A, B and C are nowhere zero functions of t only. For \( f(z) = \cosh z, h(z) = \sinh z \) or \( f(z) = \cos z, h(z) = \sin z \) the above space-time (5) becomes Bianchi type VI₀ or VII₀, respectively. The above space-time admits three linearly independent Killing vector fields which are

\[ \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, -y \frac{\partial}{\partial x} - x \frac{\partial}{\partial y} + \frac{\partial}{\partial z}. \]

The non-zero components of the Riemann tensor are

\[ R_{0101} = \frac{1}{4} \left[ \left(2A\dot{A} - A^2\right)f^2(z) + \left(2B\dot{B} - B^2\right)h^2(z)\right] = \alpha_1, \]

\[ R_{0102} = \frac{1}{4} \left[ \left(2A\dot{A} - A^2\right) + \left(2B\dot{B} - B^2\right)\right] f(z)h(z) = \alpha_7, \]

\[ R_{0113} = R_{0221} = \frac{1}{4} ABC \left[A^2\dot{B}C + \dot{A}B^2C + 2AB\dot{C}(A + B) - 3ABC(A + B)\right]f(z)h(z) = \alpha_8, \]

\[ R_{0202} = \frac{1}{4} \left[ \left(2A\dot{A} - A^2\right) + \left(2B\dot{B} - B^2\right)\right] f(z)h(z) = \alpha_7, \]

\[ R_{0213} = \frac{1}{4} \left[ (A + B)\left(\frac{\dot{A}}{A} + \frac{\dot{C}}{C}\right) - 2(\dot{A} + \dot{B})\right]h^2(z) + \left((A + B)\left(\frac{\dot{B}}{B} + \frac{\dot{C}}{C}\right) - 2(\dot{A} + \dot{B})\right)f^2(z) \]

\[ = \alpha_9, \]

\[ R_{0123} = \frac{1}{4} \left[ \left(2A\dot{A} - A^2\right)h^2(z) + \left(2B\dot{B} - B^2\right)f^2(z)\right] = \alpha_2, \]

\[ R_{0211} = \frac{1}{4} \left[ (A + B)\left(\frac{\dot{A}}{A} + \frac{\dot{C}}{C}\right) - 2(\dot{A} + \dot{B})\right]f^2(z) + \left((A + B)\left(\frac{\dot{B}}{B} + \frac{\dot{C}}{C}\right) - 2(\dot{A} + \dot{B})\right)h^2(z) \]

\[ = \alpha_{10}, \]

\[ R_{0303} = \frac{1}{4} \left[ 2C\ddot{C} - \dot{C}^2\right] = \alpha_3, \]

\[ R_{0312} = \frac{1}{4} \left[ (A + B)\left(\frac{\dot{A}}{A} + \frac{\dot{C}}{C}\right) - 2(\dot{A} + \dot{B})\right]f^2(z) + \left((A + B)\left(\frac{\dot{B}}{B} + \frac{\dot{C}}{C}\right) - 2(\dot{A} + \dot{B})\right)h^2(z) \]

\[ = \alpha_{11}, \]

\[ R_{1212} = \frac{1}{4} \left[ (A + B)^2 + \dot{A}\dot{B}\dot{C}\right]f^2(z) + \left(\dot{A}h^2(z) + Bf^2(z)\right) = \alpha_4, \]

\[ R_{1313} = \frac{1}{4} \left[ (A^3 - AB)(2A + 3B - \dot{A}\dot{C})f^2(z) - \left(AB(3A + 2B - \dot{B}\dot{C}) - B^3\right)h^2(z)\right] = \alpha_5, \]

\[ R_{1223} = \frac{1}{4} \left[ (A^2 - B^2) + B^2(B - 5A) + AB\dot{C}(\dot{A} + \dot{B})\right]f(z)h(z) = \alpha_{12}, \]

\[ R_{2323} = \frac{1}{4} \left[ (A^3 - AB)(2A + 3B - \dot{A}\dot{C})h^2(z) - \left(AB(3A + 2B - \dot{B}\dot{C}) - B^3\right)f^2(z)\right] = \alpha_6. \]
Writing the curvature tensor with components \( R_{abcd} \) at \( p \) as a \( 6 \times 6 \) symmetric matrix \([6]\)

\[
R_{abcd} = \begin{pmatrix}
\alpha_1 & \alpha_7 & 0 & 0 & \alpha_8 & \alpha_9 \\
\alpha_7 & \alpha_2 & 0 & 0 & \alpha_{10} & \alpha_8 \\
0 & 0 & \alpha_3 & \alpha_{11} & 0 & 0 \\
0 & 0 & \alpha_{11} & \alpha_4 & 0 & 0 \\
\alpha_8 & \alpha_{10} & 0 & 0 & \alpha_5 & \alpha_{12} \\
\alpha_9 & \alpha_8 & 0 & 0 & \alpha_{12} & \alpha_6
\end{pmatrix}.
\]

(7)

As mentioned in section 2, the space-times which can admit proper affine vector fields have holonomy type \( R_2, R_3, R_4, R_6, R_7, R_8, R_{10}, R_{11} \) or \( R_{13} \) and the rank of the \( 6 \times 6 \) Riemann matrix is at most three. Therefore we are only interested in those cases when the rank of the \( 6 \times 6 \) Riemann matrix is less than or equal to three. Hence there exist following two possibilities when the rank of the \( 6 \times 6 \) symmetric matrix is less or equal to three which are:

(A) Rank=3, \( \alpha_1 = \alpha_2 = \alpha_5 = \alpha_7 = \alpha_8 = \alpha_9 = \alpha_{10} = \alpha_{11} = 0, \alpha_4 \neq 0, \alpha_5 \neq 0, \alpha_6 \neq 0 \) and \( \alpha_{12} \neq 0 \).

(B) Rank=1, \( \alpha_4 \neq 0 \), and

\( \alpha_1 = \alpha_2 = \alpha_3 = \alpha_5 = \alpha_6 = \alpha_7 = \alpha_8 = \alpha_9 = \alpha_{10} = \alpha_{11} = \alpha_{12} = 0. \)

We will consider each case in turn.

**Case (A):**

In this case we have \( \alpha_4 \neq 0, \alpha_5 \neq 0, \alpha_6 \neq 0, \alpha_{12} \neq 0, \alpha_1 = \alpha_2 = \alpha_3 = \alpha_5 = \alpha_6 = \alpha_7 = \alpha_8 = \alpha_9 = \alpha_{10} = \alpha_{11} = 0 \), the rank of the \( 6 \times 6 \) Riemann matrix is three and there exists a unique (up to a multiple) no where zero time like vector field \( t_a = t_a \) solution of equation (4) and \( t_{\mu \nu} \neq 0 \). From the above constraints we have \( B(t) = dA(t), C(t) = eA(t) \) and \( A(t) = (at + b)^2 \), where \( a, b, d, e \in R(d, e > 0) \). The line element in this case takes the form

\[
\frac{ds^2}{2} = -dt^2 + (at + b)^2 \left[ (f^2(z) + e h^2(z))dx^2 + 2(1 + d)f(z)h(z)dx dy + e dz^2 \right] + (8)
\]

Substituting the above information into the affine equations one find that

\[
X^0 = c_4 t + c_5, \quad X^i = -c_1 y + c_2, \quad X^2 = -c_1 x + c_3, \quad X^3 = c_1,
\]

(9)

where \( c_1, c_2, c_3, c_4, c_5 \in R. \) One can write the above equation (9) subtracting Killing vector fields as

\[
X = (c_4 t + c_5, 0, 0, 0).
\]

(10)

Clearly in this case one can easily see that the above space-times (8) admit proper affine symmetry. It is important to note that the constants \( a \) and \( b \) can not be zero simultaneously.
Case (B):
In this case we have
\[ \alpha_1 = \alpha_2 = \alpha_3 = \alpha_5 = \alpha_6 = \alpha_7 = \alpha_8 = \alpha_9 = \alpha_{10} = \alpha_{11} = \alpha_{12} = 0, \quad \alpha_4 \neq 0 \]
and the rank of the $6 \times 6$ Riemann matrix is one. Here, there exist two linear independent solutions $t_a = t_{\alpha}$ and $z_a = z_{\alpha}$ of equation (4). The vector field $t_a$ is covariantly constant whereas $z_a$ is not covariantly constant. From the above constraints we have $A(t) = B(t) = C(t) = (t + q)^2$, where $q \in \mathbb{R}$. The line element takes the form
\[ ds^2 = -dt^2 + (t + q)^2 \left[ \left( f^2 (z) + h^2 (z) \right) dx^2 + \left( h^2 (z) + f^2 (z) \right) dy^2 + \right] \]
\[ + 4 f(z)h(z)dx
dy + dz^2 \] \hspace{1cm} (11)
Affine vector fields in this case
\[ X^0 = c_4 t + c_5 z + c_6, \quad X^1 = -c_1 y + c_2, \quad X^2 = -c_1 x + c_3, \quad X^3 = c_1, \]
where $c_1, c_2, c_3, c_4, c_5, c_6 \in \mathbb{R}$. One can write the above equation (12) subtracting Killing vector fields as
\[ X = (c_4 t + c_5 z + c_6, 0, 0, 0). \] \hspace{1cm} (13)
In this case clearly the above space-times (11) admit proper affine symmetry.

SUMMARY

In this paper an attempt is made to explore all the possibilities when the Bianchi types $\text{VI}_0$ and $\text{VII}_0$ space-times admit proper affine vector fields. A different approach is adopted to study proper affine vector fields of the above space-times by using holonomy and decomposability, the rank of the $6 \times 6$ Riemann matrix and direct integration techniques. From the above study we obtain the following results:

(i) We obtain the space-time (8) that admits proper affine vector fields when the rank of the $6 \times 6$ Riemann matrix is three and there exists a unique nowhere zero independent timelike vector field, which is the solution of equation (4) and is not covariantly constant (for details see case A).

(ii) The space-time (11) is obtained, which admits proper affine vector fields (see case B) when the rank of the $6 \times 6$ Riemann matrix is one and there exist two independent solutions of equation (4) but only one independent covariantly constant vector field.

References


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